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GENERALIZED FRACTIONAL DIFFERENTIAL EQUATIONS WITH ORDER VARYING IN TIME IN COMPLEX BANACH SPACES: ANALYTIC AND NUMERICAL ASYMPTOTIC BEHAVIOR

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Abstract. The asymptotic behavior of the solution of generalized fractional order integral equations with order varying in time arising in image processing is investigated in this work. It is shown here that the asymptotic behavior is extended from the corresponding property for the *scalar* abstract equation $u(t) = \partial_t^{-\alpha(t)} Au(t) + f(t), \quad 0 \le t \le T$, for a given $\alpha : [0, T] \to (1, 2)$, f defined in $0 \le t \le T$, $A : \mathcal{D}(A) \subset X \to X$ a bounded operator, and X a Banach space. It is also proved that a first order time discretization inherits the behavior of the continuous solution.

Keywords: fractional integrals, variable order, Banach spaces, convolution quadrature.

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§1. Introduction

One of the most interesting properties in time dependent partial differential equations based models for image processing is the asymptotic behavior of the analytic solution as time goes to infinity, but an even more important issue is if the time discretization inherits this behavior. In fact, the asymptotic behavior allows us to predict the diffusion level of the solution as the scale parameter t grows up, or in image processing terminology, this allows one to predict the degree of blurring acting on the image as time tends to infinity.

The asymptotic behavior of most of local models related to image processing has been extensively investigated, on the contrary what happens with nonlocal models. The memory effect in nonlocal equations makes in many cases the study of the asymptotic behavior more difficult if compared to the local models, but in spite of this the study has been carried out for general Volterra equations [8], and in a particular and very well known kind of nonlocal models as they are the linear integro–differential equations of fractional order [1]. This behavior has been already experienced in practical instances related to image processing (see e.g. the pioneer work [3]).

Recently an extension of the integro–differential equations of fractional order in [3] consisting in replacing the constant fractional order by fractional order varying in time has been successfully applied in the framework of image filtering [2]. To the best of our knowledge up to now there was no particular results on the asymptotic behavior adapted to fractional equations with order varying in time, however a recent work solves this issue. In fact, in [5] the authors study the asymptotic behavior of such a kind of equations, and they extend the result to its time discretization. The well–posedness, and the regularity of the solution is studied in [5] as well, everything done in the abstract framework of complex Banach spaces.

The main contribution of this work is the extension of these results to the case of generalized fractional equations in the sense of [2], whose main difference is that this approach involves several varying in time integration orders in a matrix–form.

The paper is organized as follows, Section 2 is devoted to mathematical background and model formulation, in Section 3 and 4 we present the main results of this work related to the continuous and discrete solutions respectively, and finally in Section 5 we present some observations and final conclusions.

§2. Mathematical background

The present work is motivated by the nonlocal in time evolution partial differential equations based approach to image processing introduced in [2], whose formulation is given in terms of time fractional integrals with orders varying in time. In fact, let $\mathbf{u}_0$ be an initial data, standing for a $J \times J$, perturbed sampled image, $J > 0$, vector–arranged as $J^2 \times 1$ vector, and intended to be restored. The nonlocal evolutionary model proposed in [2] reads

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t A_h \mathbf{D}(t-s)\mathbf{u}(s)\,\mathrm{d}s, \qquad t > 0, \tag{1}$$

where $\mathbf{u} : [0,T] \to \mathcal{M}_{J^2\times 1}(\mathbb{R})$, stands for the original image evolved up to the time level $t > 0$, which has been vector–arranged as a column vector with J^2 entries, i.e. $\mathbf{u} = (u_j)_{1\le j\le J^2}$. Moreover, $A_h \in \mathcal{M}_{J^2\times J^2}(\mathbb{R})$ is a symmetric and negative semi–definite matrix. An example of matrix A_h is the one corresponding to the discrete Laplacian based on a second order finite difference scheme, including discrete and homogeneous Newman boundary conditions. Notice that most of classical spatial discretizations of the Laplacian give rise to one of these matrices. Finally, $\mathbf{D} : [0,T] \to \mathcal{M}_{J^2\times J^2}(\mathbb{R})$ stands for a diagonal matrix, $\mathbf{D} = \mathrm{diag}_{1\le i\le J^2}(k_j)$, where the entries $k_j(t)$, $1 \le j \le J^2$, coincide with the convolution kernels those define the fractional integral with order varying in time $\alpha_j(t)$, for each $1 \le j \le J^2$.

Recall that several definitions for non integer integrals (or derivatives) with order varying in time can be found in the literature, and the convenience of using one vs. the others has been largely discussed, and basically depends on the purposes of the model. For the shortness of the presentation, we do not include such a discussion here, we just adopt the same definition as in [5] and we refer there the reader for a more precise motivation of this choice. Before recalling this definition let us denote $\mathcal{L}$ and $\mathcal{L}^{-1}$ the Laplace transform operator and the inverse Laplace transform operator, respectively. In that manner, let $\alpha : [0,T] \to (1,2)$ be a piecewise continuous function then, for $f \in L^1(0,+\infty)$, the fractional integral of order $\alpha(t)$ is defined as

$$\partial_t^{-\alpha(t)} f(t) = \int_0^t k(t-s) f(s)\,\mathrm{d}s, \qquad t > 0, \tag{2}$$

where,

$$k(t) := \mathcal{L}^{-1}(K)(t), \quad \text{and} \quad K(z) := \frac{1}{z^{z\tilde{\alpha}(z)}}, \tag{3}$$

and

$$\tilde{\alpha}(z) = \mathcal{L}(\alpha)(z), \tag{4}$$

for $z \in \mathcal{D}(K) \subset \mathbb{C}$. Simply observe that, if the fractional order turns out to be constant, then $\tilde{\alpha}(z) = \alpha/z$ for certain constant α, and the definition (2)–(4) coincides with the very well known Riemann–Liouville one [9]. We refer the reader to [5] for a deeper discussion on this matter.

The underlying idea behind the use of this model in image filtering is that the diffusion in the original image $\mathbf{u}_0$ applies pixel–by–pixel by setting different viscosity parameters (or diffusion coefficients) $\alpha_j(t)$ for each single pixel, which evolves in time according to some criteria (edge–preserving, texture–preserving, among others). This fact gives rise to the convolution kernels $k_j(t)$, $1 \leq j \leq J^2$ of the type mentioned above. This approach extends many other previous fractional approaches whose diffusion orders keep constant along the whole time interval.

§3. Main result

In this section we present the main theorem of the paper related to the continuous solution, but we previously recall the result on which this is based on.

Let $(Y, \|\cdot\|)$ be a complex Banach space, $\alpha : [0, T] \to (1, 2)$ a piecewise continuous function, and consider the abstract integral equation

$$u(t) = u_0 + \partial_t^{-\alpha(t)}(Au)(t), \quad t > 0, \tag{5}$$

where $A : \mathcal{D}(A) \subset Y \to Y$ is a linear, closed, and θ–sectorial operator in Y, $0 < \theta < \pi/2$, $u_0 \in Y$ stands for the initial data, and $\partial_t^{-\alpha(t)}$ defines the fractional integral according the definition (2)–(4).

Recall that a linear and closed operator is θ–sectorial, $0 < \theta < \pi/2$, if there exist $w \in \mathbb{R}$ and $L > 0$ such that

- The resolvent $(zI - A)^{-1}$ is analytic, and
- It satisfies

$$\|(zI - A)^{-1}\|_{Y \to Y} \leq \frac{L}{|z - w|},$$

for $z \in \mathbb{C}$, with $\mathrm{Arg}(z - w) > \pi - \theta$.

Notice that, since we are assuming that $\alpha(t)$ is piecewise continuous in $[0, T]$, $\alpha(t)$ admits Laplace transform in a complex domain $\mathrm{Re}(z) \geq C_\alpha$, for some $C_\alpha > 0$. In addition assume that there exist $1 < m < M < 2$, $C > 0$, and $0 < \varepsilon < 1$, such that, for $z \in \mathbb{C}$, $\mathrm{Re}(z) \geq C_\alpha$,

(A1) $m \leq \mathrm{Re}(z\tilde{\alpha}(z)) \leq M$, and $\dfrac{M\pi}{2} < \varepsilon(\pi - \theta)$.

(A2) $|\mathrm{Im}(z\tilde{\alpha}(z))| \leq C$, and

$$\left|\log\left(|z| \mathrm{Im}(z\tilde{\alpha}(z))\right)\right| < (1 - \varepsilon)(\pi - \theta),$$

where ε is expected to be close to 1.

Assume also that
$$0<\theta<\pi-\frac{M\pi}{2}-\max_{r\geq R}\frac{\log(r)}{r^{\varepsilon}},$$
for $R>0$ large enough.

Under these assumptions, equation (5) can be written in terms of the Laplace transform as
$$U(z)=\frac{H(z)}{z}(H(z)I-A)^{-1}u_0, \tag{6}$$
where
$$H(z):=z^{z\tilde{\alpha}(z)}, \quad \text{and} \quad U(z)=\mathcal{L}(u)(z),$$
for $\mathrm{Re}(z)\geq C_\alpha$. Therefore there exists an evolution operator $E(t)$, $t>0$, such that the mild solution of (5) can be written as
$$u(t)=E(t)u_0, \quad t>0. \tag{7}$$
In addition the evolution operator $E(t)$ can be expressed by means of the Bromwich formula as
$$E(t)=\frac{1}{2\pi\,\mathrm{i}}\int_\Gamma \mathrm{e}^{tz}\frac{H(z)}{z}(H(z)I-A)^{-1}\,\mathrm{d}z, \tag{8}$$
where Γ is a convenient complex path running from $-\,\mathrm{i}\infty$ to $+\,\mathrm{i}\infty$ within the analyticity domain of the resolvent of A, and positively oriented, i.e. with increasing imaginary part (see [5] for more details).

The asymptotic behavior of the solution of (5) is stated in [5, Theorem 5.1], in fact it is proved that there exists $C>0$ such that
$$\|E(t)\|_{Y\to Y}\leq\frac{CL}{1+|w|t^m}, \quad \text{as} \quad t\to+\infty. \tag{9}$$

The first contribution of this work consists of extending the asymptotic behavior of the solution of (5) to the solution of (1). To this end denote the Banach space $Y=L^1((0,T),\mathbb{R})$ normed as usual by $\|\cdot\|_{L^1}$ and denoted by simplicity as $\|\cdot\|$.

Let $(X,\|\cdot\|_X)$ be the Banach space defined by
$$X:=\prod_{j=1}^{J^2}Y \quad \text{normed by} \quad \|\mathbf{v}\|_X:=\sup_{1\leq j\leq J^2}\|v_j\|, \tag{10}$$
for $\mathbf{v}=(v_j)_{1\leq j\leq J^2}\in X$.

It is straightforward to prove that the operator $A_h\mathbf{D}(t)$ in (1), and described in Section 2, is on the one hand commutative, i.e. $A_h\mathbf{D}(t)=\mathbf{D}(t)A_h$, and on the other hand θ_0–sectorial for certain $0<\theta_0<\pi/2$, and $w\in\mathbb{R}^-$.

Assume that the diffusion coefficients $\alpha_j(t)$ involved in the definition of kernels in the matrix $\mathbf{D}$, admit Laplace transform in a complex domain $\mathrm{Re}(z)\geq C_\alpha$, for some $C_\alpha>0$, and in addition we assume (A1) and (A2) for each one. In fact assume that there exist $1<m_j<M_j<2$, $C_j>0$ and $0<\varepsilon_j<1$, for $1\leq j\leq J^2$, such that, for $z\in\mathbb{C}$, $\mathrm{Re}(z)\geq C_\alpha$,

(B1) $m_j\leq \mathrm{Re}(z\tilde{\alpha}_j(z))\leq M_j$, and $\dfrac{M_j\pi}{2}<\varepsilon_j(\pi-\theta_0)$.

(B2) $|\mathrm{Im}(z\tilde{\alpha}_j(z))| \leq C$, and

$$|\log\left(|z|\mathrm{Im}(z\tilde{\alpha}_j(z))\right)| < (1-\varepsilon_j)(\pi-\theta_0)$$

where all ε_j are expected to be close to 1.

Assume also that

(C) $0 < \theta_0 < \pi - \dfrac{\max\limits_{1\leq j\leq J^2}\{M_j\}\cdot\pi}{2} - \max\limits_{r\geq R}\dfrac{\log(r)}{r^{\varepsilon}},$

for $R > 0$ large enough, and

$$\varepsilon = \max_{1\leq j\leq J^2} \varepsilon_j. \tag{11}$$

The well–posedness, and the regularity of the solution stated in [5] can be straightforwardly extended to (1) under the assumptions (B1), (B2), and (C). Therefore, in order to not extend unnecessarily this work we will focus solely on the asymptotic behavior of the solution of (1). On the other hand, the mild solution $\mathbf{u}(t)$ of (1) can be writen as $\mathbf{u}(t) = \mathbf{E}(t)\mathbf{u}_0$ where the evolution operator $\mathbf{E}: X \to X$ admits the expression

$$\mathbf{E}(t) := \frac{1}{2\pi\,\mathrm{i}}\int_{\Gamma_0}\frac{\mathrm{e}^{tz}}{z}(I-\widetilde{\mathbf{D}}(z)A_h)^{-1}\,\mathrm{d}z, \quad t>0, \tag{12}$$

where $\widetilde{\mathbf{D}}(z)$ stands for the componentwise Laplace transform of $\mathbf{D}(t)$, and Γ_0 is once again a convenient complex path connecting $-\,\mathrm{i}\infty$ and $+\,\mathrm{i}\infty$ with increasing imaginary part.

The theorem below represents the main contribution of this section.

Theorem 1. *Let* $\mathbf{E}(t)$ *be the evolution operator* (12) *corresponding to the mild solution of* (1) *under assumptions (B1), (B2), and (C).*

If zero does not belong to the spectrum of A_h*, then there exists* $C > 0$ *independent on* t*, such that*

$$\|\mathbf{E}(t)\|_{X\to X} \leq \frac{C}{1+|\lambda|t^m}, \quad as \quad t\to+\infty, \tag{13}$$

where $m = \min\limits_{1\leq j\leq J^2}\{m_j\}$*, and* λ *is the spectral value of* A_h *corresponding to same index as* m*.*

If zero belongs to the spectrum on A_h*, then* $\mathbf{E}(t)$ *is merely bounded, i.e. there exists* $C > 0$ *independent on* t*, such that*

$$\|\mathbf{E}(t)\|_{X\to X} \leq C, \quad t>0.$$

Proof. Since A_h stands for a symmetric and negative semi–definite matrix, there exists an orthogonal matrix P, and a diagonal matrix D_A with non positive diagonal entries such that $A_h = PD_AP^T$.

On the one hand, we can write $\mathbf{E}(t)$ as follows

$$\mathbf{E}(t) = \frac{1}{2\pi\,\mathrm{i}}\int_{\Gamma_0}\mathrm{e}^{tz}P^T\frac{\mathbf{H}(z)}{z}(\mathbf{H}(z)-D_A)^{-1}P\,\mathrm{d}z, \quad t>0,$$

where $\mathbf{H}(z) = P\widetilde{\mathbf{D}}^{-1}(z)P^T$ is a bounded operator in X along the complex path Γ_0. On the other hand

$$\|\mathbf{E}(t)\mathbf{u}_0\|_X \leq \frac{1}{2\pi}\int_{\Gamma_0}\left|\frac{\mathrm{e}^{tz}}{z}\right|\|\mathbf{H}(z)\|_{X\to X}\|(\mathbf{H}(z)-D_A)^{-1}\mathbf{u}_0\|_X\,\mathrm{d}z,$$

for $t > 0$.

Moreover, $\|\mathbf{H}(z)\|_{X\to X} = \|\widetilde{\mathbf{D}}^{-1}(z)\|_{X\to X}$, for $z \in \Gamma_0$, and the resolvent $(\mathbf{H}(z) - D_A)^{-1}$ corresponds to a system of scalar equations where the diagonal matrix D_A plays the role of the operator A in (5), and the convolution kernel associated in (5) is here replaced by a linear combination of kernels of the same type.

First of all recall that the spectrum of D_A is located in the negative real line. Therefore, in order to accomplish the bounds of the resolvent $(\mathbf{H}(z) - D_A)^{-1}$ and the term $\mathbf{H}(z)$, we make use, as in [5], of a suitable choice of the complex path Γ_0 in (12), now under the restrictions imposed by (B1), (B2), and (C). In particular, define the complex paths $\Gamma_0^{(1)}$ and $\Gamma_0^{(2)}$ respectively by

$$\gamma_0^{(1)}(\phi) := \frac{1}{t^m} + \rho_0\, \mathrm{e}^{\mathrm{i}\phi}, \qquad -\varepsilon(\pi-\theta) \le \phi \le \varepsilon(\pi-\theta),$$

and

$$\gamma_0^{(2)}(\rho) := \rho\, \mathrm{e}^{\pm\mathrm{i}\varepsilon(\pi-\theta)}, \qquad \rho \ge \rho_0,$$

where ε is defined in (11), $\pm$ in $\gamma_0^{(2)}$ represents the upper and lower branches (positive and negative imaginary parts respectively), and ρ_0 stands for the distance from the origin to the intersection point of $\gamma_0^{(1)}$ and $\gamma_0^{(2)}$. Therefore,

$$\Gamma_0 := \Gamma_0^{(1,1/m)} \cup \Gamma_0^{(2,1/m)}, \tag{14}$$

where $\Gamma_0^{(1,1/m)}$ and $\Gamma_0^{(2,1/m)}$ come parametrized by $(\gamma_0^{(1)}(\phi))^{1/m}$ and $(\gamma_0^{(2)}(\rho))^{1/m}$ respectively.

So, from the bounds along Γ_0 of all terms involved in the integral (12) the proof follows. □

§4. Time discretization

The time discretization considered in [5] is based on the backward Euler convolution quadrature (see [4, 6, 7]). Now we extend the formulation to the non–scalar case,

$$\mathbf{U}_n = \mathbf{u}_0 + \sum_{j=0}^{n} \mathbf{Q}_{n-j} A_h \mathbf{U}_j, \quad 0 \le n \le N, \tag{15}$$

where the $J^2 \times J^2$ quadrature weights $\{\mathbf{Q}_n\}_{n\ge 0}$, come out from the evaluation

$$\widetilde{\mathbf{D}}\left(\frac{1-\zeta}{\tau}\right) = \sum_{n=0}^{+\infty} \mathbf{Q}_n \zeta,$$

$\tau = T/N$, and $\mathbf{D}(t)$ is the matrix–valued function in (1). In order to not extend unnecessarily this presentation we refer again the reader for more details on the convolution quadratures to [6, 7], and in fact for the one based on the backward Euler method see [4].

The key point here is that the numerical solution can be written in terms of discrete evolution operators $\{\mathbf{E}_n\}_{n\ge 0}$.

$$\mathbf{U}_n = \mathbf{E}_n \mathbf{u}_0, \quad 0 \le n \le N, \tag{16}$$

and that the Bromwich formula in vectorial form allows us to write

$$\mathbf{E}_n = \frac{1}{2\pi\,\mathrm{i}} \int_{\Gamma_0} \frac{r_n(tz)}{z} (I - \widetilde{\mathbf{D}}(z) A_h)^{-1}\,\mathrm{d}z, \quad n \ge 0,$$

where Γ_0 is the complex path stated in (14), and $r_n(z) := 1/(1-z)^n$. Notice that $r_n(z)$ stands for the characteristic function of the backward Euler method.

What follows is the main results of this section.

Theorem 2. *Let* $\{\mathbf{E}_n\}_{n \ge 0}$ *be the discrete evolution operators* (16) *associated to the numerical solution* (15) *under assumptions (B1), (B2), and (C).*

If zero does not belong to the spectrum of A_h*, then there exists* $C > 0$*, independent on* t*, such that*

$$\|\mathbf{E}_n\|_{X \to X} \le \frac{C}{1 + |\lambda| t_n^m}, \quad \textit{as} \quad t \to +\infty, \tag{17}$$

where $m = \min\limits_{1 \le j \le J^2} \{m_j\}$*, and* λ *is the spectral value of* A_h *corresponding to same index as* m*.*

If zero belongs to the spectrum on A_h*, then* $\mathbf{E}$ *is merely bounded, i.e. there exists* $C > 0$ *independent on* t*, such that*

$$\|\mathbf{E}_n\|_{X \to X} \le C, \quad t > 0.$$

The proof of Theorem 2 follows the same steps as the one of the Theorem 1, now replacing the exponential e^{tz} by the rational function $r_n(z)$.

§5. Observations and final conclusions

The first to be observed is that the numerical solution inherits the behavior as t goes to infinity of the analytic solution. Observe also that the asymptotic behavior turns out to be independent of the initial data $\mathbf{u}_0$ and its regularity since the proofs of both, Theorem 1 and 2, are done merely for the continuous and discrete evolution operators respectively. In other words, the regularity of the initial data does not affect the asymptotic behavior nor of the analytic solution neither the numerical one. This fact is a crucial issue specially in the context of image processing because this proves that the blurring is the same whatever the original image one has.

Observe also that if the matrix D_A has a null eigenvalue, the evolution operator is merely bounded and the decrease is not longer guaranteed. This confirms what happens in the case of abstract infinitesimal semigroup generators when $w = 0$ (according the notation in Section 2). The reason is that the evolution operators do not longer admit analytic extension to the left hand side complex plane.

Moreover, if $\lambda \ne 0$ in Theorems 1 and 2, then the decrease of $\mathbf{u}$ is limited by the slowest decrease along all components, or in other words it is limited by the lowest diffusion along every single pixels of the image represented by $\mathbf{u}$.

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