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STABILITY OF DOMAIN WALLS IN FERROMAGNETIC RINGS

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Abstract. In this work we consider a one-dimensional model of ferromagnetic ring taking into account curvature and anisotropy effects. We describe all the planar static solutions representing domain walls and we study their stability.

Keywords: ferromagnetism, Landau-Lifshitz equation, stability, domain walls,... .
AMS classification: 35K55, 35Q60.

§1. Introduction

Ferromagnetic materials are permanent magnets characterized by a spontaneous magnetization [1, 4]. In ferromagnetic nanowires, the wire axis is a preferential axis of magnetization, and one observes formation of domains (zone in which the magnetization is oriented along the wire) separated by domain walls (zones of magnetization switching). This property plays an important role for applications in data storage or logic devices (see [10] and [2]).

In this paper, we deal with ferromagnetic rings and we study the influence of their shape on their performances for data storage. In particular, the main criterium is the number of stable configurations possibly stored by the device.

First we recall the three-dimensional model of the ferromagnetic materials (see [7, 9]): we denote by $\Omega \subset \mathbb{R}^3$ the ferromagnetic domain and by $\mathbf{m}(\mathbf{t}, \mathbf{x})$ the distribution of the magnetization at time t and at point $x \in \Omega$. We suppose that the material is saturated, *i.e.* the norm of $\mathbf{m}$ constant equals to $\mathbf{m}_s$. The magnetic induction $\mathbf{b}$ and the magnetic field $\mathbf{h}$ are linked by the constitutive relation $\mathbf{b} = \mathbf{h} + \overline{\mathbf{m}}$, where $\overline{\mathbf{m}}$ is the extension of $\mathbf{m}$ by zero outside Ω. The variation of $\mathbf{m}$ satisfies the following Landau-Lifshitz equation:

$$\frac{\partial \mathbf{m}}{\partial \mathbf{t}} = -\gamma \mathbf{m} \times \mathbf{h}_{\mathbf{eff}} - \frac{\alpha\gamma}{\mathbf{m}_s} \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\mathbf{eff}}),$$

where γ is the gyromagnetic ratio, α is the damping coefficient, and $\mathbf{h}_{\mathbf{eff}}$ is the effective field given by:

$$\mathbf{h}_{\mathbf{eff}}(\mathbf{m}) = \frac{A}{\mu_0 \mathbf{m}_s^2} \Delta \mathbf{m} + \mathbf{h}_{\mathbf{d}}(\mathbf{m}),$$

where A is the exchange constant, μ_0 the permeability of the vacuum and $\mathbf{h}_{\mathbf{d}}$ the demagnetizing field obtained by solving Maxwell-Faraday equation:

$$\operatorname{curl} \mathbf{h}_{\mathbf{d}}(\mathbf{m}) = 0, \qquad \operatorname{div}(\mathbf{h}_{\mathbf{d}}(\mathbf{m}) + \overline{\mathbf{m}}) = 0, \quad \text{in } \mathbb{R}^3.$$

We consider a ferromagnetic ring $\Omega_\eta \subset \mathbb{R}^3$ obtained by rotation around the z axis of the ellipse contained in the plane $x = 0$ of equation $\dfrac{(y-R)^2}{a^2} + \dfrac{z^2}{b^2} < \eta^2$, where η is a small parameter.

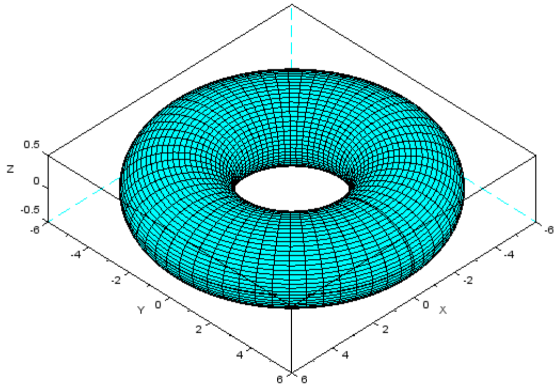

Figure 1: Ferromagnetic Ring.

When η tends to zero, the ring tends to the circle contained in the plane $z = 0$ of equation $x^2 + y^2 = R^2$. We parametrize this circle by $\theta \mapsto (R\cos\theta, R\sin\theta, 0)$. As it is established in [3] by asymptotic process, we obtain the following one-dimensional limit model: writing the magnetic moment as $\mathbf{m}(\mathbf{t}, R\cos\theta, R\sin\theta, 0) = m_s\,\mathcal{M}(\frac{\gamma \mathbf{m}_s}{\lambda}\mathbf{t}, \theta)$, where the parameter λ is given by $\lambda = \dfrac{A}{\mu_0 R^2 m_s^2}$, the new unknown $\mathcal{M}$ satisfies the renormalized saturation constraint $|\mathcal{M}| = 1$ and verifies:

$$\begin{cases} \mathcal{M} : (t,\theta) \mapsto \mathcal{M}(t,\theta) \in S^2 \subset \mathbb{R}^3, \quad 2\pi\text{-periodic in the variable } \theta, \\[2ex] \dfrac{\partial \mathcal{M}}{\partial t} = -\mathcal{M} \times \mathcal{H}_{\mathbf{eff}}(\mathcal{M}) - \alpha \mathcal{M} \times (\mathcal{M} \times \mathcal{H}_{\mathbf{eff}}(\mathcal{M})), \\[2ex] \mathcal{H}_{\mathbf{eff}}(\mathcal{M}) = \partial_{\theta\theta}\mathcal{M} + \dfrac{1}{\lambda}\mathcal{H}_{\mathbf{d}}(\mathcal{M}), \\[2ex] \mathcal{H}_{\mathbf{d}}(\mathcal{M})(\theta) = -\dfrac{b}{a+b}\,\langle \mathcal{M}(\theta) | \mathbf{e_r}(\theta) \rangle\, \mathbf{e_r}(\theta) - \dfrac{a}{a+b}\mathcal{M}_3(\theta)\mathbf{e_3}, \end{cases} \tag{1}$$

with

$$\mathbf{e_r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \quad \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We remark in particular that the limit demagnetizing operator $\mathcal{H}_d$ is local in the one-dimensional model. We introduce the rotation R_σ given by

$$R_\sigma = \begin{pmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 1. Equation (1) is invariant by translation-rotation: if we denote by $\mathcal{M}$ a solution of (1), we consider $\mathcal{M}^\sigma$ defined by:

$$\mathcal{M}^\sigma(t,\theta) = R_\sigma(\mathcal{M}(t,\theta-\sigma)).$$

Since $\mathbf{e}_r(\theta) = R_\sigma(\mathbf{e}_r(\theta-\sigma))$, we have:

$$\mathcal{H}_d(\mathcal{M}^\sigma)(t,\theta) = R_\sigma(\mathcal{H}_d(\mathcal{M})(t,\theta-\sigma)).$$

In addition,

$$\partial_{\theta\theta}\mathcal{M}^\sigma(t,\theta) = R_\sigma(\partial_{\theta\theta}\mathcal{M}(t,\theta-\sigma)) \text{ and } \frac{\partial\mathcal{M}^\sigma}{\partial t}(t,\theta) = R_\sigma(\frac{\partial\mathcal{M}}{\partial t}(t,\theta-\sigma)).$$

Therefore $\mathcal{M}^\sigma$ is also solution for (1).

We denote by $\mathbf{M} = \begin{pmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \\ \mathbf{M}_3 \end{pmatrix}$ the vector of the coordinates of $\mathcal{M}(\mathbf{t},\theta)$ in the frame $(\mathbf{e_r},\mathbf{e}_\theta,\mathbf{e_3})$:

$$\mathcal{M}(\mathbf{t},\theta) = \mathbf{M}_1(t,\theta)\mathbf{e_r}(\theta) + \mathbf{M}_2(t,\theta)\mathbf{e}_\theta(\theta) + \mathbf{M}_3(t,\theta)\mathbf{e_3}.$$

Rewriting equation (1) in the mobile frame $(\mathbf{e_r}(\theta),\mathbf{e}_\theta(\theta),\mathbf{e_3})$ with these coordinates, we obtain the following model:

$$\begin{cases} \mathbf{M} : (t,\theta) \mapsto \mathbf{M}(\mathbf{t},\theta) \in S^2 \quad 2\pi - \text{periodic in } \theta, \\ \\ \dfrac{\partial\mathbf{M}}{\partial t} = -\mathbf{M}\times\mathbf{H_{eff}}(\mathbf{M}) - \alpha\mathbf{M}\times(\mathbf{M}\times\mathbf{H_{eff}}(\mathbf{M})), \\ \\ \mathbf{H_{eff}}(\mathbf{M}) = \partial_{\theta\theta}\mathbf{M} + 2\mathbf{e_3}\times\partial_\theta\mathbf{M} - \mathbf{M}_1\mathbf{e_1} - \mathbf{M}_2\mathbf{e_2} + \dfrac{1}{\lambda}\mathbf{H_d}(\mathbf{M}), \\ \\ \mathbf{H_d}(\mathbf{M}) = -\dfrac{1}{a+b}(b\mathbf{M}_1\mathbf{e_1} + a\mathbf{M}_3\mathbf{e_3}). \end{cases} \tag{2}$$

Remark 2. From the invariance by rotation-translation of (1), we obtain that (2) is invariant by translation, *i.e.* if $\mathbf{M}$ satisfies (2), then for all $\sigma \in \mathbb{R}$, $(t,\theta) \mapsto \mathbf{M}(t,\theta-\sigma)$ is also solution for (2).

We focus on static planar solutions $\mathbf{M}^0$ for Equation (2) taking their values in the plane $z = 0$, that is on the form

$$\mathbf{M^0} = (\cos u(\theta), \sin u(\theta), 0), \tag{3}$$

where $u \in H^1_{loc}(\mathbb{R};\mathbb{R})$ satisfies:

$$\exists k \in \mathbb{Z},\ \forall \theta \in \mathbb{R}, u(\theta + 2\pi) = u(\theta) + 2k\pi, \tag{4}$$

in order to ensure that $\mathbf{M}^0$ is 2π-periodic. We denote by $\mathcal{M}^0$ the corresponding solution for Equation (1):

$$\mathcal{M}^0(\theta) = \cos u(\theta)\mathbf{e_r}(\theta) + \sin u(\theta)\mathbf{e}_\theta(\theta).$$

We remark that $k + 1$ is the winding number of $\mathcal{M}^0$ as a function from the unit circle S^1 into itself. As already said, we take care about Domain Walls. Since the wire direction is an easy axis of magnetization, we call domain a point in which the magnetization $\mathcal{M}^0$ is tangent to the ring, and we call Domain Wall (or magnetization switching) a point separating two consecutive domains in which the magnetization is orthogonal to the ring. We remark that by periodicity argument, the number of switchings in even. The key point for applications is to address the stability of the configurations in order to fix the number of switchings. As we will see after, the number of switching for a configuration u is equal to $2|k|$ in Formula (4). In the following section, we will describe all the static planar configurations, and we will study their stability in Section 3.

Remark 3. We can construct static solutions M^0 of (2) taking their values in the plane $y = 0$, that is on the form $M^0 = (\cos u(\theta), 0, \sin u(\theta))$ (for example $M^0 \equiv \mathbf{e}_3$). From the physical point of view, since $a > b$, we can prove that these solutions are unstable. The existence of static solutions of (2) which do not take their values either in the plane $z = 0$ or in the plane $y = 0$ remains an open problem.

§2. Construction of static profiles

By a straightforward calculation, $\mathbf{M}^0$ is a static solution of (2) is and only if u satisfies (4) and the pendulum equation:

$$u'' + \frac{b}{\lambda(a+b)} \cos u \sin u = 0. \tag{5}$$

By multiplying the pendulum equation by u' and integration, we obtain that there exists a constant ρ such that for all θ,

$$(u'(\theta))^2 + \frac{b}{\lambda(a+b)} \sin^2 u(\theta) = \rho^2. \tag{6}$$

2.1. Case $k = 0$

First we look for planar static solutions $\mathcal{M}^0$ for (1) of winding number equal to one, *i.e.* we look for the solutions u of (5) such that $u(\theta + 2\pi) = u(\theta)$ (*i.e.* with $k = 0$). The periodic solutions of (5) are either the constant solutions equal to 0 modulo $\frac{\pi}{2}$ or are the non constant trajectories between the separatrix, which are the lines $p = \pm\sqrt{\frac{b}{\lambda(a+b)}} \cos u$ in the phase portrait (where we denote by (u, p) the coordinate in the phase plane, see Figure 2). By

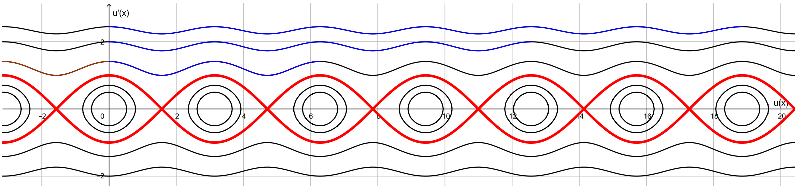

Figure 2: Phase portrait $(u(\theta), u'(\theta))$ for (5).

classical arguments, such a solution $\theta \mapsto (u(\theta), u'(\theta))$ remains in one cell C_n between the separatrix, where:

$$C_n = \left\{ (u,p) \in \mathbb{R}^2, -\pi/2 + n\pi < u < \pi/2 + n\pi \text{ with } |p| < \sqrt{\frac{b}{\lambda(a+b)}} |\cos u| \right\}.$$

We first look for the 2π-periodic solutions in C_0. By translation in the variable θ, we can assume that $u(0) \in]0, \frac{\pi}{2}[$ and $u'(0) = 0$. For $\gamma \in]0, \frac{\pi}{2}[$, we denote by u_γ the solution of (5) such that $u_\gamma(0) = \gamma$ and $u'(0) = 0$. We have:

$$\forall \theta \in \mathbb{R}, \quad (u_\gamma'(\theta))^2 + \frac{b}{\lambda(a+b)} \sin^2 u_\gamma(\theta) = \frac{b}{\lambda(a+b)} \sin^2 \gamma.$$

By classical calculation, the period $L(\gamma)$ of this solution is given by:

$$L(\gamma) = 4\sqrt{\frac{\lambda(a+b)}{b}} \int_0^\gamma \frac{du}{\sqrt{\sin^2 \gamma - \sin^2 u}}.$$

The function u_γ satisfies $u_\gamma(0) = u_\gamma(2\pi)$ if and only if there exists $n \in \mathbb{N}^*$ such that $nL(\gamma) = 2\pi$. The function L is continuous and non decreasing. In addition, we have $\lim\limits_{\gamma \to \frac{\pi}{2}} L(\gamma) = +\infty$, and $\lim\limits_{\gamma \to 0} L(\gamma) = 2\pi\sqrt{\frac{\lambda(a+b)}{b}}$.

Therefore, if $\dfrac{b}{\lambda(a+b)} \leq 1$, for all $\gamma \in]0, \frac{\pi}{2}[$, $L(\gamma) > 2\pi$, so there is no 2π-periodic solution of this type.

If $\dfrac{b}{\lambda(a+b)} > 1$, let $l \in \mathbb{N}^*$ such that $l + 1 \geq \sqrt{\frac{b}{\lambda(a+b)}} > l$. Then, $\frac{2\pi}{l+1} \leq \lim\limits_{\gamma \to 0} L(\gamma) < \frac{2\pi}{l}$.
So on the one hand, by monotonicity argument, for all $n \in \{1, \ldots, l\}$, there exists only one $\gamma_n \in]0, \frac{\pi}{2}[$ such that $L(\gamma_n) = \frac{2\pi}{n}$. On the other hand, for all $\gamma \in]0, \frac{\pi}{2}[$, $L(\gamma) > \frac{2\pi}{l+1}$, so the minimal possible period of such solutions is $\frac{2\pi}{l}$. Therefore, there are exactly l 2π-periodic solutions (modulo translation in θ) in the cell C_0.
By the same arguments, we find exactly l 2π-periodic solutions in the cell C_1.

So, in the case $k = 0$, we have the following theorem:

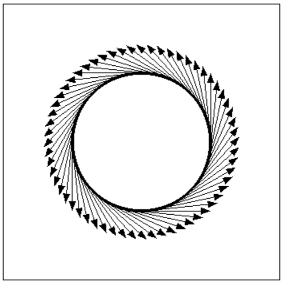

Figure 3: Profile of e_θ.

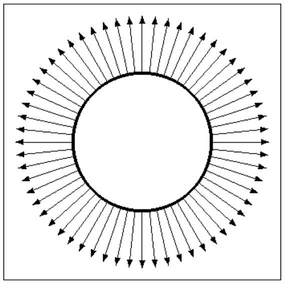

Figure 4: Profile of e_r.

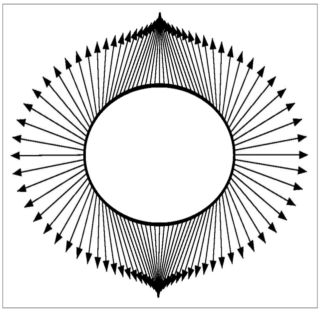

Figure 5: Solution with $l = 2$.

Theorem 1. *Let $\lambda > 0$, $a > 0$, $b > 0$. Let $l \in \mathbb{N}$ such that $l < \sqrt{\frac{b}{\lambda(a+b)}} \leq l + 1$. In addition to the solutions $\pm e_r$ and $\pm e_\theta$, Equation (1) admits $2l$ other degree-one planar static solutions modulo rotation-translation.*

2.2. Case $k \neq 0$

Now we look for planar static solutions of (1) of degree $k+1$, $k \neq 0$, *i.e.* we look for solutions u for (5) such that $u(\theta + 2\pi) = u(\theta) + 2k\pi$, with $k \neq 0$. These solutions are outside the separatrix, since the solutions inside the separatrix remain in intervals which sizes are less than π. These solutions satisfy (6) with $|\rho|^2 > \dfrac{b}{\lambda(a+b)}$.

For $k \geq 1$, we consider, for $\rho > \sqrt{\frac{b}{\lambda(a+b)}}$, the solution v_ρ of (5) such that $v_\rho(0) = 0$ and $v_\rho'(0) = \rho$. Writing (6), we obtain that v_ρ reaches the value $2k\pi$ at the point θ_ρ given by:

$$\theta_\rho = \int_0^{2k\pi} \frac{dv}{\sqrt{\rho^2 - \dfrac{b}{\lambda(a+b)}\sin^2 v}} = 4k\int_0^{\frac{\pi}{2}} \frac{dv}{\sqrt{\rho^2 - \dfrac{b}{\lambda(a+b)}\sin^2 v}}.$$

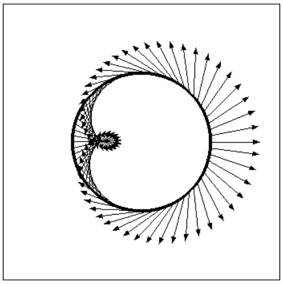

Figure 6: Solution with 2 walls (k=1).

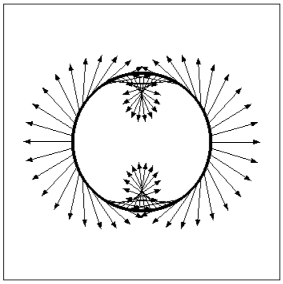

Figure 7: Solution with 4 walls (k=2).

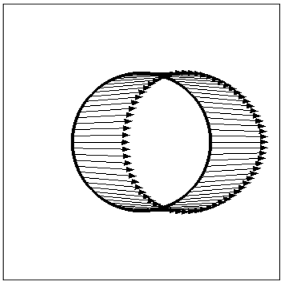

Figure 8: Solution with 2 walls (k=-1).

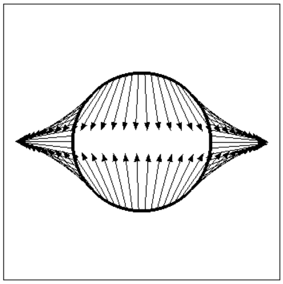

Figure 9: Solution with 4 walls (k=-2).

We remark that $\rho \mapsto \theta_\rho$ is continuous and non increasing. In addition, we have:

$$\lim_{\rho \to \sqrt{\frac{b}{\lambda(a+b)}}} \theta_k(\rho) = +\infty \text{ and } \lim_{\rho \to +\infty} \theta_k(\rho) = 0.$$

Then we deduce that for all fixed $k \geq 1$ there exist an unique $\rho \in \left] \sqrt{\frac{b}{\lambda(a+b)}}, +\infty \right[$ such that $\theta_k(\rho) = 2\pi$.

By the same way we find the same result for $k \leq -1$ with $\rho < -\sqrt{\dfrac{b}{\lambda(a+b)}}$. So, in the case $k \in \mathbb{Z}^*$ we have the following theorem:

Theorem 2. *For any fixed $k \in \mathbb{Z}^*$, Equation (1) admits a planar static solution of degree $k+1$. This solution is unique modulo translation-rotations and presents $2|k|$ walls.*

§3. Stability of wall profiles

In this part we address the stability of the solutions given in the previous part. The first difficultly comes from the saturation constraint: we must consider only perturbations satisfying this constraint. To solve this problem we use the mobile frame technique developed in [5].

3.1. Mobile frame technique

We address the stability of a static solution $\mathbf{M^0} = \begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix}$ for Equation (2), obtained either in Theorem 1 or in Theorem 2. We denote by ρ^2 the conserved quantity $(u')^2 + \frac{b}{\lambda(a+b)} \sin^2 u$ in (6). We introduce the mobile frame $(\mathbf{M}^0(\theta), \mathbf{M}^1(\theta), \mathbf{M}^2)$, where

$$\mathbf{M}^1(\theta) = \begin{pmatrix} -\sin u(\theta) \\ \cos u(\theta) \\ 0 \end{pmatrix} \text{ and } \mathbf{M}^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We describe the perturbations of $\mathbf{M}^0$ as follows:

$$\mathbf{M}(t,\theta) = r_1(t,\theta)\mathbf{M^1}(\theta) + r_2(t,\theta)\mathbf{M^2} + (1 + \nu(r(t,\theta)))\mathbf{M^0}(\theta),$$

with $\nu(r) = \sqrt{1 - r_1^2 - r_2^2} - 1$, so that the saturation constraint is satisfied. We write the Landau-Lifshitz equation (2) with this new unknown $r : \mathbb{R}^+ \times [0, 2\pi] \to \mathbb{R}^2$, and by projection onto $\mathbf{M^1}$ and $\mathbf{M^2}$, we establish as in [5] that $\mathbf{M}$ satisfies (2) if and only if r satisfied the equation:

$$\partial_t r = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} Lr + F(\theta, r, \partial_\theta r, \partial_{\theta\theta} r), \tag{7}$$

where $F(\theta, r, \partial_\theta r, \partial_{\theta\theta} r)$ is the non linear part, and with:

$$Lr = \begin{pmatrix} \mathcal{L}_1 r_1 \\ \mathcal{L}_2 r_2 \end{pmatrix}$$

with

$$\begin{aligned} \mathcal{L}_1 &= -\partial_{\theta\theta} + \frac{b}{\lambda(a+b)}(\sin^2 u - \cos^2 u), \\ \mathcal{L}_2 &= \mathcal{L}_1 + (\frac{a}{\lambda(a+b)} - \rho^2 - 2u' - 1). \end{aligned} \tag{8}$$

In addition, $\mathbf{M}$ is stable for (2) if and only if 0 is stable for (7). The positivity of $\mathcal{L}_1$ and $\mathcal{L}_2$ is crucial for the stability (see [6]). Let us study the different cases.

3.2. Stability of e_θ

The static planar solution $\mathbf{e}_\theta$ for Equation (1) corresponds to the static planar solution $\mathbf{M^0} = (0, 1, 0)$ for Equation (2) with $u = \frac{\pi}{2}$. The obtained linearization is L given by:

$$L = \begin{pmatrix} -\partial_{\theta\theta} + \dfrac{b}{\lambda(a+b)} \\ -\partial_{\theta\theta} + \left(\dfrac{a}{\lambda(a+b)} - 1\right) \end{pmatrix}.$$

As already said, we prove in [6] that if $L > 0$ then 0 is asymptoticly stable for (7). The operator $\mathcal{L}_1$ is positive. Concerning $\mathcal{L}_2$, its positiveness is related to the sign of $\dfrac{a}{\lambda(a+b)} - 1$, so we obtain the following theorem:

Theorem 3. *If $\lambda < \frac{a}{a+b}$, then $\mathbf{e}_\theta$ is asymptoticly stable. If $\lambda > \frac{a}{a+b}$ then $\mathbf{e}_\theta$ is linearly unstable.*

Remark 4. In the previous theorem, if $\lambda > \frac{a}{a+b}$, i.e. if the radius of the ring is sufficiently small, then the exchange energy of $\mathbf{e}_\theta$ becomes large and creates instability.

3.3. Instability of e_r

We study the static planar solution $\mathbf{e_r}$ of the equation (1), which corresponds to the static planar solution $\mathbf{M^0} = (1, 0, 0)$ for Equation (2), *i. e.* with $u = 0$. The obtained linearization is given by:

$$L = \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix},$$

where

$$\mathcal{L}_1 = -\partial_{\theta\theta} - \frac{b}{\lambda(a+b)} \text{ and } \mathcal{L}_2 = -\partial_{\theta\theta} + \frac{a-b}{\lambda(a+b)} - 1.$$

In particular, $\mathcal{L}_1$ admits negative eigenvalues so we have the following theorem:

Theorem 4. *Whatever $\lambda > 0$, $a > 0$ and $b > 0$, e_r is linearly unstable for Equation (1)..*

3.4. Stability of the non constant solutions

We address the stability of a non constant solution $\mathbf{M^0} = \begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix}$ for Equation (2), obtained either in Theorem 1 in the case $\frac{b}{\lambda(a+b)} > 1$, or in Theorem 2. We denote by ρ^2 the conserved quantity $(u')^2 + \frac{b}{\lambda(a+b)} \sin^2 u$ in (6). We recall that the stability for $\mathbf{M}^0$ is related to the positivity of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ given by (8).

3.4.1. Linear instability for the non constant solutions given by Theorem 1

We assume that $\rho^2 < \frac{b}{\lambda(a+b)}$. In this case, the trajectories $\theta \mapsto (u(\theta), u'(\theta))$ are between the separatrix. We remark that $\mathcal{L}_1 \cos u = (\rho^2 - \frac{b}{\lambda(a+b)}) \cos u$. So $\rho^2 - \frac{b}{\lambda(a+b)}$ is a negative eigenvalue associated to the eigenvector $\cos u$. Thus, $\mathcal{L}_1$ is not positive. Therefore we have the following Theorem:

Theorem 5. *In the case $\rho^2 < \frac{b}{\lambda(a+b)}$, the static solution M^0 is linearly unstable for (1).*

3.4.2. Linear stability for the non constant solutions given by Theorem 2

We assume now that $\rho^2 > \frac{b}{\lambda(a+b)}$. We have the following proposition:

Proposition 6. *$\mathcal{L}_1$ is a linear non negative operator. In addition $Ker\mathcal{L}_1 = \mathbb{R}u'$.*

Proof. We set $\ell_1 = \partial_\theta + \dfrac{b}{\lambda(a+b)} \dfrac{\sin u \cos u}{u'}$, then $\ell_1^* = -\partial_\theta + \dfrac{b}{\lambda(a+b)} \dfrac{\sin u \cos u}{u'}$ and we have the factorization:

$$\ell_1^* \circ \ell_1 = \mathcal{L}_1.$$

So $\mathcal{L}_1$ is a positive operator. We have also

$$\mathcal{L}_1 u' = -\left(u'' + \frac{b}{\lambda(a+b)} \cos u \sin u\right)' = 0,$$

so $u' \in Ker\mathcal{L}_1$. □

Therefore in this case, $\mathcal{L}_1$ is non negative. The existence of an order-one vanishing eigenvalue is an additional difficulty to obtain the nonlinear stability. This is due to the invariance of (2) by translation (see Remark 2), so that there exists a one-parameter family of constant solutions for (2): $\theta \mapsto \mathbf{M}^0(\theta - \sigma)$ depending of the parameter σ. By projection on the mobile frame, we obtain the existence of a one-parameter family of constant solutions for (7): $\theta \mapsto R(\sigma)(\theta)$.

In order to take into account the zero eigenvalue of L, as in [5] or [8], we rewrite r in the following new system of coordinates:

$$r(t,\theta) = R(\sigma(t))(\theta) + w(t,\theta),$$

where now the parameter σ depends on the time variable: $\sigma \in C^1(\mathbb{R}^+;\mathbb{R})$, and $w \in C^1(\mathbb{R}^+; H^2_{per})$ such that the first component w_1 of w satisfies the orthogonality condition:

$$\forall t > 0, \quad \int_0^{2\pi} w_1(t,\theta) u'(\theta) d\theta = 0.$$

In this new unknown (σ, w), we are able to separate the dynamics of w and the dynamics of σ. In particular, if $\mathcal{L}_2$ is positive, we can prove by variational estimates that $w(t)$ tends to zero in H^1 and that $\sigma(t)$ tends to a finite limit when t tends to $+\infty$. This means that $\mathbf{M}(t)$ tends to a translation of $\mathbf{M}^0$ when t tends to $+\infty$ (asymptotic stability modulo translation in the variable θ).

Now, the difficulty is to prove the study of $\mathcal{L}_2$. We prove in [6] the following Theorems:

Theorem 7. *We consider the solutions of (1) given by Theorem 2 in the case $k \le 1$.*
If $a \le b$, these solutions are unstable.
If $a > b$, if λ is large enough, these solutions are unstable.
If $a > b$, there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then there exists $k_0 > 0$ such that the solutions with $k \le k_0$ are stable and the solutions with $k > k_0$ are unstable.

Theorem 8. *We consider the solutions of (1) given by Theorem 2 in the case $k \ge -1$. If $a > b$, there exists $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$ then there exists $k_0 < 0$ such that the solutions with $k_0 \le k \le -1$ are stable and the solutions with $k < k_0$ are unstable.*

Remark 5. We remark in particular that, if $a > b$, the solution with $k = -1$ is stable whatever $\lambda > 0$. In addition, we establish that the larger the diameter of the ring, the more information it can store.

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