Monografías Matemáticas *García de Galdeano* **42**, 25–34 (2019)

Adaptive augmented mixed FEM for the Oseen problem with mixed boundary conditions

Tomás P. Barrios, José Manuel Cascón and María González

Abstract. We present an adaptive augmented dual-mixed method for the Oseen problem with mixed boundary conditions in the pseudostress-velocity variables. The new variational formulation and the corresponding Galerkin scheme are well-posed for appropriate values of the stabilization parameters. We provide the rate of convergence when each row of the pseudostress is approximated by Raviart-Thomas elements and the velocity is approximated by continuous piecewise polynomials. Moreover, we give an a posteriori error indicator and show the performance of the corresponding adaptive algorithm through a numerical example.

Keywords: Incompressible flow, Oseen, mixed finite element, stabilization, a posteriori error estimates.

AMS classification: 65N30, 65N12, 65N15.

§1. Introduction

The problem of computing the flow of a viscous and incompressible fluid at small Reynolds numbers is described by the Oseen equations. In the recent paper [4], we introduced a new augmented variational formulation for this problem in the pseudostress-velocity variables under homogeneous Dirichlet boundary conditions for the velocity, and developed a simple a posteriori error analysis.

Now, we propose a related method for the case when mixed boundary conditions are considered. We remark that the new method is not an extension of the one proposed in [4] since here the Dirichlet boundary condition is imposed weakly.

The paper is organized as follows. In Section 2 we describe a new augmented dual-mixed variational formulation for the Oseen problem in the pseudostress-velocity variables with mixed boundary conditions. Then, in Section 3 we analyze the stabilized mixed finite element method. In Section 4 we present an a posteriori error indicator that is reliable and locally efficient. Finally, numerical experiments are reported in Section 5.

§2. The augmented dual-mixed variational formulation

Assume that the fluid at hand occupies the region Ω, a polygonal domain in $\mathbb{R}^2$ with boundary Γ. We assume that $\Gamma = \Gamma_D \cup \Gamma_N$, where Γ_D is a closed part of Γ with positive measure and $\Gamma_N = \Gamma \setminus \Gamma_D$. Let $\nu > 0$ be the kinematic viscosity of the fluid, and let $\mathbf{a} \neq \mathbf{0}$ denote the advective velocity. We assume that $\mathbf{a}$ is solenoidal in Ω. Let $\mathbf{f}$ be an external body force, and denote by $\mathbf{u}_D$ a prescribed velocity on Γ_D and by $\mathbf{g}$ the Neumann data.

We consider the following Oseen problem: find the velocity field $\mathbf{u}$ and the pressure p such that

$$\begin{cases} -\nu\Delta\mathbf{u} + \mathbf{a}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega\,, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_D & \text{on } \Gamma_D\,, \\ -p\,\mathbf{n} + \nu\dfrac{\partial\mathbf{u}}{\partial\mathbf{n}} = \mathbf{g} & \text{on } \Gamma_N\,, \end{cases} \tag{1}$$

where $\mathbf{n}$ is the unit outward normal to Γ_N.

Let $\mathbf{I}$ be the identity matrix in $\mathbb{R}^{2\times 2}$ and denote by $\boldsymbol{\sigma} := \nu\nabla\mathbf{u} - p\,\mathbf{I}$ the pseudostress. Proceeding similarly as in [4], problem (1) can be stated equivalently in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$, and the pressure can be recovered as $p = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma})$.

For simplicity, we consider the following decomposition of $\boldsymbol{\sigma}$: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\mathbf{0}} + \boldsymbol{\sigma}_{\mathbf{g}}$, with $\boldsymbol{\sigma}_{\mathbf{0}}\mathbf{n} = \mathbf{0}$ and $\boldsymbol{\sigma}_{\mathbf{g}}\mathbf{n} = \mathbf{g}$ on Γ_N. Moreover, given a tensor $\boldsymbol{\tau}$, we denote by $\boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}$ the *deviator* of $\boldsymbol{\tau}$. Then, problem (1) is equivalent to the following problem:

$$\begin{cases} -\mathbf{div}(\boldsymbol{\sigma}_{\mathbf{0}}) + \mathbf{a}\cdot\nabla\mathbf{u} = \tilde{\mathbf{f}} & \text{in } \Omega\,, \\ \dfrac{1}{\nu}\,\boldsymbol{\sigma}_{\mathbf{0}}^{\mathrm{d}} = \nabla\mathbf{u} + \boldsymbol{\zeta} & \text{in } \Omega\,, \\ \mathbf{u} = \mathbf{u}_D & \text{on } \Gamma_D\,, \\ \boldsymbol{\sigma}_{\mathbf{0}}\mathbf{n} = \mathbf{0} & \text{on } \Gamma_N\,, \end{cases} \tag{2}$$

where $\tilde{\mathbf{f}} := \mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_{\mathbf{g}})$ and $\boldsymbol{\zeta} := -\frac{1}{\nu}\boldsymbol{\sigma}_{\mathbf{g}}^{\mathrm{d}}$.

Throughout this paper, we will use the standard notations for Sobolev spaces and norms. In particular, we denote by $H(\mathbf{div},\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2\}$ and $\mathbf{H_0} := \{\boldsymbol{\tau} \in H(\mathbf{div},\Omega) : \boldsymbol{\tau}\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_N\}$.

Let us define now the bilinear forms $a : \mathbf{H_0}\times\mathbf{H_0} \to \mathbb{R}$, $b : [H^1(\Omega)]^2 \times \mathbf{H_0} \to \mathbb{R}$ and $c : [H^1(\Omega)]^2 \times [H^1(\Omega)]^2 \to \mathbb{R}$ as follows:

$$a(\boldsymbol{\sigma},\boldsymbol{\tau}) := \frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}\,, \qquad b(\mathbf{u},\boldsymbol{\tau}) := \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau})\,, \qquad c(\mathbf{u},\mathbf{v}) := \int_\Omega (\mathbf{a}\cdot\nabla\mathbf{u})\cdot\mathbf{v}\,,$$

for any $\boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbf{H}_0$ and $\mathbf{u},\mathbf{v} \in [H^1(\Omega)]^2$.

We also define the linear functionals $l : [L^2(\Omega)]^2 \to \mathbb{R}$ and $m : \mathbf{H_0} \to \mathbb{R}$ as follows:

$$l(\mathbf{v}) := -\int_\Omega \tilde{\mathbf{f}}\cdot\mathbf{v}, \quad \forall\, \mathbf{v} \in [L^2(\Omega)]^2\,,$$

$$m(\boldsymbol{\tau}) := \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau} + \int_\Gamma \mathbf{u}_D\cdot\boldsymbol{\tau}\mathbf{n}, \quad \forall\, \boldsymbol{\tau} \in \mathbf{H_0}\,.$$

Then, we have the following dual-mixed variational formulation of problem (2): find $(\boldsymbol{\sigma}_0,\mathbf{u}) \in \mathbf{H_0}\times[H^1(\Omega)]^2$ such that

$$\begin{cases} a(\boldsymbol{\sigma}_0,\boldsymbol{\tau}) + b(\mathbf{u},\boldsymbol{\tau}) = m(\boldsymbol{\tau})\,, & \forall\,\boldsymbol{\tau} \in \mathbf{H_0}\,, \\ b(\mathbf{v},\boldsymbol{\sigma}_0) - c(\mathbf{u},\mathbf{v}) = l(\mathbf{v})\,, & \forall\,\mathbf{v} \in [H^1(\Omega)]^2\,. \end{cases} \tag{3}$$

We remark that the variational formulation (3) exhibits a generalized saddle-point structure, with a non-symmetric bilinear form $c(\cdot,\cdot)$. According to [5], to ensure that problem (3) has a unique solution, we require, among other conditions, that the bilinear form $a(\cdot,\cdot)$ be coercive on $\mathbf{H_0}$. However, it is well-known that $a(\cdot,\cdot)$ is coercive in the divergence free subspace of $\mathbf{H_0}$ (see, for instance, the proof of Theorem 2.3 in [6]) but not on $\mathbf{H_0}$. We also require that the bilinear form $b(\cdot,\cdot)$ satisfies an inf-sup condition in $\mathbf{H_0} \times [H^1(\Omega)]^2$. These facts motivated us to consider an augmented formulation of problem (2).

Combining ideas from [4] and [9], we subtract the second equation in (3) from the first one and then, add the following least-squares type terms, that arise from the equilibrium and constitutive equations in (2) and from the Dirichlet boundary condition:

$$\kappa_1 \int_\Omega (\mathbf{div}(\boldsymbol{\sigma_0}) - \mathbf{a}\cdot\nabla\mathbf{u}) \cdot (\mathbf{div}(\boldsymbol{\tau}) + \mathbf{a}\cdot\nabla\mathbf{v}) \;=\; -\kappa_1 \int_\Omega \tilde{\mathbf{f}} \cdot (\mathbf{div}(\boldsymbol{\tau}) + \mathbf{a}\cdot\nabla\mathbf{v})$$

$$\kappa_2 \int_\Omega (\nabla\mathbf{u} - \frac{1}{\nu}\boldsymbol{\sigma}_\mathbf{0}^{\mathrm{d}}) \,:\, (\nabla\mathbf{v} + \frac{1}{\nu}\boldsymbol{\tau}^{\mathrm{d}}) \;=\; -\kappa_2 \int_\Omega \boldsymbol{\zeta} : (\nabla\mathbf{v} + \frac{1}{\nu}\boldsymbol{\tau}^{\mathrm{d}}),$$

and

$$\kappa_3 \int_{\Gamma_D} \mathbf{u}\cdot\mathbf{v} = \kappa_3 \int_{\Gamma_D} \mathbf{u}_D \cdot \mathbf{v}$$

where $(\boldsymbol{\sigma_0}, \mathbf{u}) \in \mathbf{H_0}\times[H^1(\Omega)]^2$ is a solution of (2) and $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H_0}\times[H^1(\Omega)]^2$ is a test function. The parameters κ_1, κ_2 and κ_3 are positive constants to be chosen so that the augmented bilinear form

$$A((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) := \frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} \,:\, \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) - \int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{v} + \int_\Omega (\mathbf{a}\cdot\nabla\mathbf{u})\cdot\mathbf{v}$$

$$+\, \kappa_1 \int_\Omega (\mathbf{div}(\boldsymbol{\sigma}) - \mathbf{a}\cdot\nabla\mathbf{u})\cdot(\mathbf{div}(\boldsymbol{\tau}) + \mathbf{a}\cdot\nabla\mathbf{v}) + \kappa_2 \int_\Omega (\nabla\mathbf{u} - \frac{1}{\nu}\boldsymbol{\sigma}^{\mathrm{d}}) : (\nabla\mathbf{v} + \frac{1}{\nu}\boldsymbol{\tau}^{\mathrm{d}}) + \kappa_3 \int_{\Gamma_D} \mathbf{u}\cdot\mathbf{v}$$

be coercive in the whole space $\mathbf{H_0} \times [H^1(\Omega)]^2$.

Let us define the linear functional $F : \mathbf{H_0} \times [H^1(\Omega)]^2 \to \mathbb{R}$ by

$$F(\boldsymbol{\tau},\mathbf{v}) := \int_\Omega \boldsymbol{\zeta} : \boldsymbol{\tau} + \int_{\Gamma_D} \mathbf{u}_D \cdot \boldsymbol{\tau}\mathbf{n} + \int_\Omega \tilde{\mathbf{f}}\cdot\mathbf{v} - \kappa_1 \int_\Omega \tilde{\mathbf{f}}\cdot(\mathbf{div}(\boldsymbol{\tau}) + \mathbf{a}\cdot\nabla\mathbf{v})$$

$$-\, \kappa_2 \int_\Omega \boldsymbol{\zeta} : (\nabla\mathbf{v} + \frac{1}{\nu}\boldsymbol{\tau}^{\mathrm{d}}) + \kappa_3 \int_{\Gamma_D} \mathbf{u}_D\cdot\mathbf{v}, \qquad \forall\, (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H_0}\times[H^1(\Omega)]^2 .$$

Then, the augmented variational formulation of problem (2) reads: find $(\boldsymbol{\sigma_0},\mathbf{u}) \in \mathbf{H_0} \times [H^1(\Omega)]^2$ such that

$$A((\boldsymbol{\sigma_0},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) = F(\boldsymbol{\tau},\mathbf{v}), \qquad \forall\, (\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H_0}\times[H^1(\Omega)]^2 . \tag{4}$$

Remark 1. In case of homogeneous Dirichlet boundary conditions, that is, when $\Gamma_D = \Gamma$, $\Gamma_N = \emptyset$ and $\mathbf{u}_D = \mathbf{0}$ on Γ, we obtain the same linear functional F as in [4]. However, the variational formulation is not equivalent, since here we look for $\mathbf{u} \in [H^1(\Omega)]^2$.

In what follows, we assume that $\mathbf{a} \in [L^\infty(\Omega)]^2$, $\mathbf{a}\cdot\mathbf{n} \geq 0$ on Γ_N, and $\mathbf{f} \in [L^2(\Omega)]^2$. Moreover, we assume that

$$0 < \kappa_1 < \frac{\kappa_2}{2\,\|\mathbf{a}\|^2_{[L^\infty(\Omega)]^2}}\,, \qquad 0 < \kappa_2 < \nu\,, \qquad \text{and} \qquad \kappa_3 > \frac{1}{2}\,\|\mathbf{a}\cdot\mathbf{n}\|_{L^\infty(\Omega)}\,.$$

Then, there exists $C_{\mathtt{ell}} > 0$ such that

$$A((\boldsymbol{\tau},\mathbf{v}),(\boldsymbol{\tau},\mathbf{v})) \geq C_{\mathtt{ell}}\,\|(\boldsymbol{\tau},\mathbf{v})\|^2_{\mathbf{H_0}\times[H^1(\Omega)]^2}\,, \quad \forall\,(\boldsymbol{\tau},\mathbf{v}) \in \mathbf{H_0}\times[H^1(\Omega)]^2\,,$$

with

$$C_{\mathtt{ell}} = \min\Big(\frac{1}{\nu}\,(1-\frac{\kappa_2}{\nu})c_1, \frac{\kappa_1}{2}c_1, \frac{\kappa_1}{2}, (\kappa_2 - 2\,\kappa_1\,\|\mathbf{a}\|^2_{[L^\infty(\Omega)]^2})\,c_2, (\kappa_3 - \frac{1}{2}\|\mathbf{a}\cdot\mathbf{n}\|_{L^\infty(\Omega)})\,c_2\Big)\,,$$

where c_1 and c_2 are the positive constants in Lemma 3.1 in [2] and in Lemma 3.3 in [8], respectively.

Theorem 1. *Under the previous hypotheses, problem* (4) *has a unique solution* $(\boldsymbol{\sigma}_\mathbf{0},\mathbf{u}) \in \mathbf{H_0}\times[H^1(\Omega)]^2$ *and*

$$\|(\boldsymbol{\sigma}_\mathbf{0},\mathbf{u})\|_{\mathbf{H_0}\times[H^1(\Omega)]^2} \;\leq\; C^{-1}_{\mathtt{ell}}\,M\,(\|\mathbf{f}\|_{[L^2(\Omega)]^2} \;+\; \|\mathbf{u}_D\|_{[H^{1/2}(\Gamma_D)]^2} \;+\; \|\boldsymbol{\sigma}_\mathbf{g}\|_{H(\mathbf{div};\Omega)})\,,$$

where $M := \max(1 \;+\; \kappa_1\,(1 \;+\; \sqrt{2}\,\|\mathbf{a}\|_{[L^\infty(\Omega)]^2}), \frac{1}{\nu}(1+\kappa_2(1+\frac{1}{\nu})), 1+\kappa_3)$.

Proof. It follows from the Lax-Milgram Lemma. □

§3. Augmented mixed finite element method

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular meshes of $\bar{\Omega}$ made up of triangles. We denote by h_T the diameter of an element $T \in \mathcal{T}_h$ and define $h := \max_{T\in\mathcal{T}_h} h_T$.

Let $\mathbf{H}_{\mathbf{0},h}$ and V_h be any finite element subspaces of $\mathbf{H_0}$ and $[H^1(\Omega)]^2$, respectively. Then, the Galerkin scheme associated to problem (4) reads: find $(\boldsymbol{\sigma}_{\mathbf{0},h},\mathbf{u}_h) \in \mathbf{H}_{\mathbf{0},h}\times V_h$ such that

$$A((\boldsymbol{\sigma}_{\mathbf{0},h},\mathbf{u}_h),(\boldsymbol{\tau}_h,\mathbf{v}_h)) = F(\boldsymbol{\tau}_h,\mathbf{v}_h)\,, \qquad \forall\,(\boldsymbol{\tau}_h,\mathbf{v}_h) \in \mathbf{H}_{\mathbf{0},h}\times V_h\,. \tag{5}$$

Under the same hypotheses as for the continuous problem (4), problem (5) has a unique solution $(\boldsymbol{\sigma}_{\mathbf{0},h},\mathbf{u}_h) \in \mathbf{H}_{\mathbf{0},h}\times V_h$. Moreover, there exists a constant $C_{\mathtt{Cea}} > 0$, independent of h, such that

$$\|(\boldsymbol{\sigma}_\mathbf{0}-\boldsymbol{\sigma}_{\mathbf{0},h},\mathbf{u}-\mathbf{u}_h)\|_{\mathbf{H_0}\times[H^1(\Omega)]^2} \;\leq\; C_{\mathtt{Cea}} \inf_{(\boldsymbol{\tau}_h,\mathbf{v}_h)\in\mathbf{H}_{\mathbf{0},h}\times V_h} \|(\boldsymbol{\sigma}_\mathbf{0}-\boldsymbol{\tau}_h,\mathbf{u}-\mathbf{v}_h)\|_{\mathbf{H_0}\times[H^1(\Omega)]^2}\,. \tag{6}$$

In order to establish a rate of convergence result, we consider specific finite element subspaces $\mathbf{H}_{\mathbf{0},h}$ and V_h. Hereafter, given $T \in \mathcal{T}_h$ and an integer $l \geq 0$, we denote by $\mathcal{P}_l(T)$ the space of polynomials of total degree at most l on T and, given an integer $r \geq 0$, we denote by $\mathcal{RT}_r(T)$ the local Raviart-Thomas space of order $r+1$ (cf. [12]),

$$\mathcal{RT}_r(T) := [\mathcal{P}_r(T)]^2 \oplus [\mathbf{x}]\mathcal{P}_r(T) \subset [\mathcal{P}_{r+1}(T)]^2\,,$$

where $\mathbf{x}$ is a generic vector of $\mathbb{R}^2$.

Let $r \geq 0$ and $m \geq 1$. Then, we let $\mathbf{H}_{\mathbf{0},h}$ be

$$\mathbf{H}_{\mathbf{0},h} := [\mathcal{RT}_r^{\mathrm{t}}]^2 = \Big\{\boldsymbol{\tau}_h \in \mathbf{H}_0 \;:\; \boldsymbol{\tau}_h\big|_T \in [\mathcal{RT}_r(T)^{\mathrm{t}}]^2, \quad \forall\, T \in \mathcal{T}_h\Big\},$$

and define

$$V_h := [\mathcal{L}_m]^2 = \Big\{\mathbf{v}_h \in [C(\overline{\Omega})]^2 \;:\; \mathbf{v}_h\big|_T \in [\mathcal{P}_m(T)]^2, \quad \forall\, T \in \mathcal{T}_h\Big\}.$$

The corresponding rate of convergence is given in the next theorem.

Theorem 2. *Assume $\boldsymbol{\sigma}_{\mathbf{0}} \in [H^t(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}_{\mathbf{0}}) \in [H^t(\Omega)]^2$ and $\mathbf{u} \in [H^{t+1}(\Omega)]^2$. Then, there exists $C = O(C_{\mathsf{Cea}}) > 0$, independent of h, such that*

$$\|(\boldsymbol{\sigma}_{\mathbf{0}} - \boldsymbol{\sigma}_{\mathbf{0},h}, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{H}_{\mathbf{0}}\times[H^1(\Omega)]^2} \;\leq\; C\, h^{\alpha}\Big(\|\boldsymbol{\sigma}_{\mathbf{0}}\|_{[H^t(\Omega)]^{d\times d}} + \|\mathbf{div}(\boldsymbol{\sigma}_{\mathbf{0}})\|_{[H^t(\Omega)]^2} + \|\mathbf{u}\|_{[H^{t+1}(\Omega)]^2}\Big), \tag{7}$$

where $\alpha := \min\{t, m, r+1\}$.

Proof. It follows straightforwardly from inequality (6) and the approximation properties of the corresponding finite element subspaces. □

§4. A posteriori error analysis

The a posteriori error analysis of the Oseen equations is very important for the numerical solution of the stationary incompressible Navier-Stokes equations. The incompressibility condition and the presence of a non-selfadjoint operator in the momentum equations are the main difficulties to obtain a posteriori error estimates for the Oseen problem.

We let E_h be the set of all the edges induced by the triangulation $\mathcal{T}_h$ and write $E_h = E_I \cup E_{\Gamma_D} \cup E_{\Gamma_N}$, where $E_I := \{e \in E_h \;:\; e \subseteq \Omega\}$, $E_{\Gamma_D} := \{e \in E_h \;:\; e \subseteq \Gamma_D\}$ and $E_{\Gamma_N} := \{e \in E_h \;:\; e \subseteq \Gamma_N\}$. Also, for each edge $e \in E_h$, we denote by h_e the length of edge e and fix a unit normal vector $\mathbf{n}_e := (n_1, n_2)^{\mathrm{t}}$; finally, we let $\mathbf{t}_e := (-n_2, n_1)^{\mathrm{t}}$ be the corresponding fixed unit tangential vector along e.

We define the local a posteriori error indicator

$$\begin{aligned}\theta_T^2 := \|\tilde{\mathbf{f}} + \mathbf{div}(\boldsymbol{\sigma}_{0,h}) - \mathbf{a}\cdot\nabla\mathbf{u}_h\|^2_{[L^2(T)]^2} + \|\boldsymbol{\zeta} + \nabla\mathbf{u}_h - \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}}_{0,h}\|^2_{[L^2(T)]^{2\times 2}} \\ + \sum_{e\in E_{\Gamma_D}\cap\partial T} h_e\Big(\|\mathbf{u}_D - \mathbf{u}_h\|^2_{[L^2(e)]^2} + \|\nabla(\mathbf{u}_D - \mathbf{u}_h)\mathbf{t}_e\|^2_{[L^2(e)]^2}\Big)\end{aligned}$$

and the global a posteriori error indicator

$$\theta := \Big(\sum_{T\in\mathcal{T}_h} \theta_T^2\Big)^{1/2}$$

The following theorem establishes the reliability of the a posteriori error indicator.

Theorem 3. *Assume $\mathbf{u}_D \in [H^1(\Gamma_D)]^2$. Then, there exists $C_{\mathtt{rel}} > 0$, independent of h, such that*

$$\|(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{0,h}, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{H}_0 \times [H^1(\Omega)]^2} \leq C_{\mathtt{rel}}\, \theta$$

Proof. We proceed as in [4] to bound the error in terms of residuals, but use a quasi-Helmholtz decomposition [7] instead of the usual Helmholtz decomposition. □

The next theorem establishes the local efficiency of the a posteriori error indicator.

Theorem 4. *Assume $\mathbf{u}_D \in [H^1(\Gamma)]^2$ is component-piecewise polynomial on Γ_D. Then, there exists $C_{\mathtt{eff}} = C(\nu, \kappa_1, \kappa_2, , \kappa_3, \mathbf{a}) > 0$, independent of h, such that for all $T \in \mathcal{T}_h$ we have*

$$C_{\mathtt{eff}}\, \theta_T \;\leq\; \|(\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{0,h}, \mathbf{u} - \mathbf{u}_h)\|_{\mathbf{H}_0(T) \times [H^1(T)]^2} \qquad \forall\, T \in \mathcal{T}_h$$

Proof. We proceed with the first two terms of θ_T as usual. The second term is bounded using a discrete trace inequality [1, Theorem 3.10]. Finally, the last term is bounded similarly as in Lemma 3.9 in [3]. □

§5. Numerical experiments

We performed numerical experiments with the finite spaces $\mathbf{H}_{0,h}$ and V_h defined in Section 3, with $r = 0$ and $m = 1$. We implemented the standard adaptive finite element method (AFEM) based on the loop

$$\text{SOLVE} \;\rightarrow\; \text{ESTIMATE} \;\rightarrow\; \text{MARK} \;\rightarrow\; \text{REFINE}$$

(see, for instance, [11]). For the numerical experiments, we used the finite element toolbox ALBERTA [13]. This toolbox employs the Kossaczky refinement algoritm, that uses recursive bisection [10]. The corresponding linear systems are solved using MATLAB (UMFPACK).

We consider an example in which $\Omega = (0, 1) \times (0, 1)$ is the unit square, $\Gamma_N = \{0\} \times [0, 1]$ and $\Gamma_D = \Gamma \setminus \Gamma_N$. We take the kinematic viscosity $\nu = 1$ and the advective velocity $\mathbf{a} = (1, 0)$. Then, we let

$$\phi(x, y) = 10x^2y^2(1 - y)^2 \tanh\left(100(x - \frac{1}{2})\right),$$

and choose $\mathbf{f}$ and $\mathbf{u}_D$ so that the exact solution is

$$\mathbf{u} = \mathbf{curl}\ \phi = \left(\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial x}\right), \qquad p(x, y) = \exp\left(-(x - \frac{1}{2})^2\right).$$

We remark that the velocity $\mathbf{u}$ exhibits an inner layer around the line $x = \frac{1}{2}$.

In Figure 1 we show the individual errors in the velocity and the pseudostress for the uniform (U) and adaptive (A) refinements with respect to the number of degrees of freedom (DOFs). We can observe that the adaptive refinement performs better than the uniform refinement. In Figure 2 we show the total error and the estimator vs. the DOFs for the uniform and adaptive refinements. In this case, we can observe that the estimator fits the total error. Accordingly, in this example the efficiency indices are almost one for both refinements (see Figure 3).

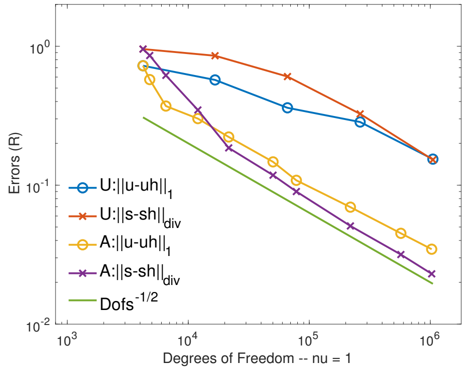

Figure 1: Individual errors in the velocity and the pseudostress.

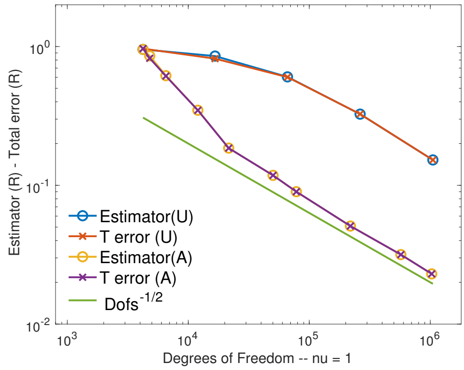

Figure 2: Total error and estimator vs. the DOFs.

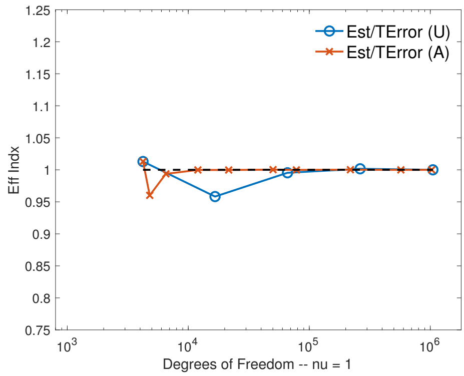

Figure 3: Efficiency indices for the uniform and adaptive refinements.

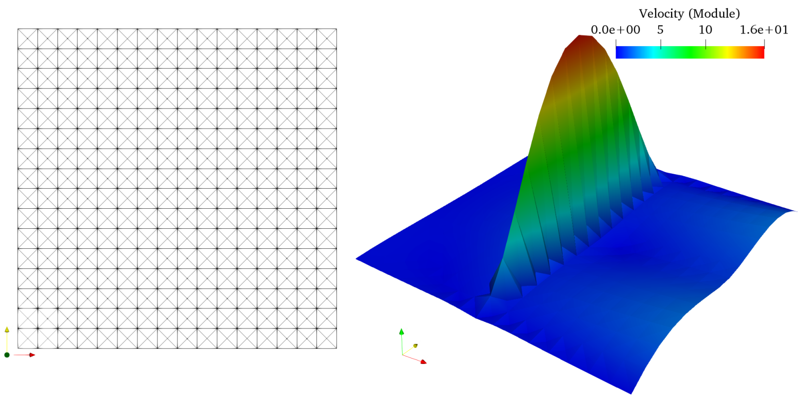

Figure 4: Initial mesh and corresponding velocity module.

In Figures 4-6 we show, respectively, the initial mesh, an intermediary mesh and the final mesh (after 8 iterations) obtained with the AFEM algorithm, together with the corresponding velocity modules. We can observe that the AFEM algorithm is able to locate the inner layer of the solution, since the refinement is essentially concentrated around the line $x = \frac{1}{2}$.

Acknowledgements

The research of T.P. Barrios is partially supported by Dirección de Investigación of the Universidad Católica de la Santísima Concepción (Chile) and through CONICYT-Chile FONDECYT project 1160578. The research of J.M. Cascón is partially supported by the Consejería de Educación of the Junta de Castilla y León, Grant SA020U16. The research of M. González is partially supported by the Spanish Ministerio de Economía y Competitividad Grant MTM2016-76497-R.

References

[1] Agmon, S. *Lectures on Elliptic Boundary Value Problems*. Princeton, 1965.

[2] Arnold, D., Douglas, J., and Gupta, C. A family of higher order mixed finite element methods for plane elasticity. *Numer. Math. 45* (1984), 1–22.

[3] Barrios, T., Bustinza, R., García, G., and González, M. A posteriori error analyses of a velocity-pseudostress formulation of the generalized Stokes problem. *Journal of Computational and Applied Mathematics 357* (2019), 349–365.

[4] Barrios, T., Cascón, J., and González, M. Augmented mixed finite element method for the Oseen problem: A priori and a posteriori error analyses. *Comput. Methods Appl. Mech. Engrg. 313* (2017), 216–238.

[5] Brezzi, F., and Fortin, M. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, 1991.

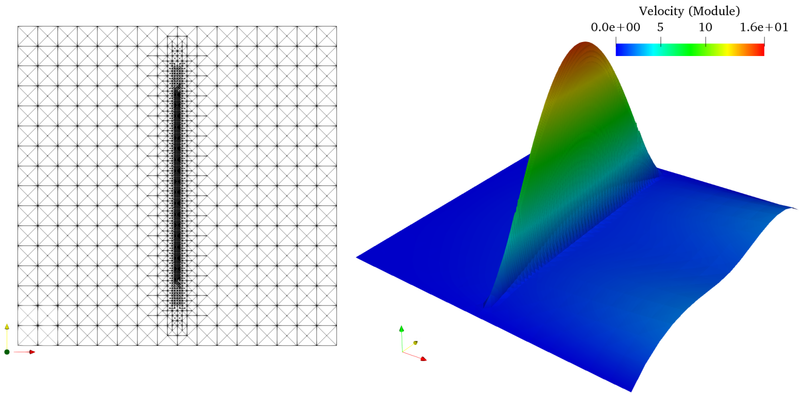

Figure 5: Mesh generated by the adaptive algorithm and velocity module at iteration 4.

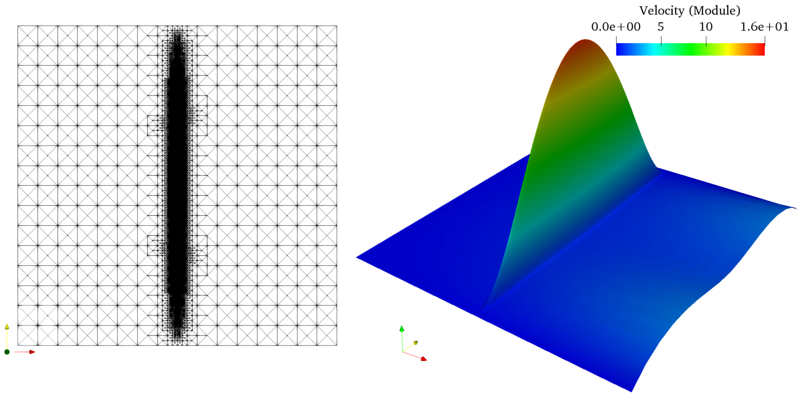

Figure 6: Mesh generated by the adaptive algorithm and velocity module at iteration 8.

[6] Cai, Z., Tong, C., Vassilevski, P., and Wang, C. Mixed finite element methods for incompressible flow: stationary Stokes equations. *Numer. Methods PDEs 26* (2010), 957–978.

[7] Cascón, J. M., Nochetto, R. H., and Siebert, K. G. Design and convergence of AFEM in H(div). *Mathematical Models and Methods in Applied Sciences 17* (2007), 1849–1881.

[8] Figueroa, L., Gatica, G. N., and Márquez, A. Augmented mixed finite element methods for the stationary Stokes equations. *SIAM Journal on Scientific Computing 31* (2008), 1082–1119.

[9] González, M. Stabilized dual-mixed method for the problem of linear elasticity with mixed boundary conditions. *Applied Mathematics Letters 30* (2014), 1–5.

[10] Kossaczký, I. A recursive approach to local mesh refinement in two and three dimensions. *J. Comput. Appl. Math. 55* (1994), 275–288.

[11] Morin, P., Nochetto, R., and Siebert, K. Convergence of adaptive finite element methods. *SIAM Review 44* (2002), 631–658.

[12] Roberts, J., and Thomas, J.-M. Mixed and hybrid methods. In *Handbook of Numerical Analysis* (Amsterdam, 1991), vol. II, North-Holland.

[13] Schmidt, A., and Siebert, K. G. Design of adaptive finite element software: The finite element toolboox ALBERTA. In *Lecture Notes in Computer Science and Engineering* (2005), vol. 42, Springer.

T. P. Barrios
Departamento de Matemática y Física Aplicadas
Universidad Católica de la Santísima Concepción
Casilla 297, Concepción, Chile
tomas@ucsc.cl

J. M. Cascón
Departamento de Economía e Historia Económica
Universidad de Salamanca
Salamanca, 37008, Spain
casbar@usal.es

M. González
Departamento de Matemáticas & CITIC
Universidade da Coruña
Campus de Elviña s/n, 15071, A Coruña, Spain
maria.gonzalez.taboada@udc.es