Monografías Matemáticas *García de Galdeano* **41**, 21–30 (2018)

Blow up of the solutions to a linear elliptic system involving Schrödinger operators

Bénédicte Alziary and Jacqueline Fleckinger

Abstract. We show how the solutions to a 2×2 linear system involving Schrödinger operators blow up as the parameter μ tends to some critical value which is the principal eigenvalue of the system; here the potential is continuous positive with superquadratic growth and the square matrix of the system is with constant coefficients and may have a double eigenvalue.

Keywords: Maximum Principle, Antimaximum Principle, Elliptic Equation and Systems, Cooperative and Non-cooperative Systems, Principle Eigenvalue.

AMS classification: 35P, 35J10.

§1. Introduction

We study here the behavior of the solutions to a 2×2 system (considered in its variational formulation):

$$(S) \qquad LU := (-\Delta + q(x))U = AU + \mu U + F(x) \; in \; \mathbb{R}^N,$$

$$U(x)_{|x| \to \infty} \to 0$$

where q is a continuous positive potential tending to $+\infty$ at infinity with superquadratic growth; U is a column vector with components u_1 and u_2 and A is a 2×2 square matrix with constant coefficients. F is a column vector with components f_1 and f_2.
Such systems have been intensively studied mainly for $\mu = 0$ and for A with 2 distinct eigenvalues; here we consider also the case of a double eigenvalue. In both cases, we show the blow up of solutions as μ tends to some critical value ν which is the principal eigenvalue of System (S). This extends to systems involving Schrödinger operators defined on $\mathbb{R}^N$ earlier results valid for systems involving the classical Laplacian defined on smooth bounded domains with Dirichlet boundary conditions.
This paper is organized as follows: In Section 2 we recall known results for one equation. In Section 3 we consider first the case where A has two different eigenvalues and then we study the case of a double eigenvalue.

§2. The equation

We shortly recall the case of one equation

$$(E) \qquad Lu := (-\Delta + q(x))u \; = \; \sigma u + f(x) \in \mathbb{R}^N,$$

$$\lim_{|x| \Rightarrow +\infty} u(x) = 0.$$

σ is a real parameter.

Hypotheses
(H_q) q is a positive continuous potential tending to $+\infty$ at infinity.
(H_f) $f \in L^2(\mathbb{R}^N)$, $f \geq 0$ and $f > 0$ on some subset with positive Lebesgue measure.
It is well knwon that if (H_q) is satisfied, L possesses an infinity of eigenvalues tending to $+\infty$: $0 < \lambda_1 < \lambda_2 \leq \dots$.

Notation: (Λ, ϕ) Denote by Λ the smallest eigenvalue of L; it is positive and simple and denote by ϕ the associated eigenfunction, positive and with L^2-norm $||\phi|| = 1$.

It is classical [9, 11] that if $f > 0$ and $\sigma < \Lambda$ the positivity is improved, or in other words, the maximum principle **(MP)** is satisfied:

$$(MP) \qquad f \geq 0, \not\equiv 0 \ \Rightarrow \ u > 0.$$

Lately, for potentials growing fast enough (faster than the harmonic oscillator), another notion has been introduced [2, 3, 5, 6] which improves the maximum (or antimaximum principle): the "groundstate positivity" (**GSP**) (resp. " negativity" (**GSN**)) which means that there exists $k > 0$ such that

$$u > k\phi \text{ (GSP) (resp. } u < -k\phi \text{ (GSN))} \qquad .$$

We also say shortly "fundamenal positivity" or" negativity", or also "ϕ-positivity" or "negativity".
The first steps in this direction use a radial potential. Here we consider a small perturbation of a radial one as in [5].

The potential q We define first a class $\mathcal{P}$ of radial potentials:

$$\mathcal{P} := \{Q \in C(\mathbb{R}_+, (0, \infty)) / \ \exists R_0 > 0, Q' > 0 \, a.e. \, on \, [R_0, \infty), \int_{R_0}^{\infty} Q(r)^{-1/2} < \infty\}. \qquad (1)$$

The last inequality holds if Q is growing sufficiently fast ($> r^2$). Now we give results of GSP or GSN for a potential q which is a small perturbation of Q; we assume:
(H'_q) q satisfies (H_q) and there exists two functions Q_1 and Q_2 in $\mathcal{P}$, and two positive constants R_0 and C_0 such that

$$Q_1(|x|) \leq q(x) \leq Q_2(|x|) \leq C_0 Q_1(|x|), \ \forall x \in \mathbb{R}^N, \qquad (2)$$

$$\int_{R_0}^{\infty} (Q_2(s) - Q_1(s)) \int_{R_0}^{s} exp(- \int_{r}^{s} [Q_1(t)^{1/2} + Q_2(t)^{1/2}] dt) dr ds < \infty. \qquad (3)$$

Denoting by Φ_1 (resp. Φ_2) the groundstate of $L_1 := -\Delta + Q_1$ (resp. $L_2 = -\Delta + Q_2$), Corollary 3.3 in [5] says that all these groundstates are "comparable" that is there exists constants $0 < k_1 \leq k_2 \leq \infty$ such that $k_1\phi \leq \Phi_1, \Phi_2 \leq k_2\phi$.

Theorem 1. *(GSP) [5] If (H'_q) and (H_f) are satisfied, then, for $\sigma < \Lambda$, there is a unique solution u to (E) which is positive, and there exists a constant $c > 0$, such that*

$$u > c\phi. \tag{4}$$

Moreover, if also $f \leq C\phi$ with some constant $C > 0$, then

$$u \leq \frac{C}{\Lambda - \sigma}\phi. \tag{5}$$

Remark 1. This holds also if we only assume $f \in L^2$ and $f^1 := \int f\phi > 0$

The space $\mathcal{X}$: It is convenient for several results to introduce the space of "groundstate bounded functions":

$$\mathcal{X} := \{h \in L^2(\mathbb{R}^N) : \; h/\phi \in L^\infty(\mathbb{R}^N)\}, \tag{6}$$

equipped with the norm $||h||_{\mathcal{X}} = ess\sup_{\mathbb{R}^n}(|h|/\phi)$.

Hypothesis (H'_f) We consider now functions f which are such that

(H'_f): $f \in \mathcal{X}$ and $f^1 := \int f\phi > 0$.

For a potential satisfying (H'_q) and a function $f \in \mathcal{X}$, there is also a result of "groundstate negativity" **(GSN)** for (E); it is an extension of the antimaximum principle, introduced by Clément and Peletier in 1978 [8] for the Laplacian when the parameter σ crosses Λ.

Theorem 2. *(GSN) [5] Assume (H'_q) and (H'_f) are satisfied; then there exists $\delta(f) > 0$ and a positive constant $c' > 0$ such that for all $\sigma \in (\Lambda, \Lambda + \delta)$,*

$$u \leq -c'\phi. \tag{7}$$

Theorem 3. *Assume (H'_q) and (H'_f) are satisfied. Then there exists $\delta > 0$, independant of σ, such that for $\Lambda - \delta < \sigma < \Lambda$ there exists positive constants k' and K', depending on f and δ such that*

$$0 < \frac{k'}{\Lambda - \sigma}\phi < u < \frac{K'}{\Lambda - \sigma}\phi. \tag{8}$$

If $\Lambda < \sigma < \Lambda + \delta$, there exists positive constants k" and K", depending on f and δ such that

$$\frac{k"}{\Lambda - \sigma}\phi < u < \frac{K"}{\Lambda - \sigma}\phi < 0. \tag{9}$$

This result extends earlier one in [10] and a close result is Theorem 2.03 in [7]. It shows in particular that $u \in \mathcal{X}$ and $|u| \to \infty$ as $|\nu - \mu| \to 0$.

Proof: Decompose u and f on ϕ and its orthogonal:

$$u = u^1\phi + u^\perp \,; \; f = f^1\phi + f^\perp. \tag{10}$$

We derive from (E): $Lu = \sigma u + f$ that

$$Lu^\perp = \sigma u^\perp + f^\perp \tag{11}$$

$$Lu^1\phi = \Lambda u^1\phi = \sigma u^1\phi + f^1\phi. \tag{12}$$

We notice that, since q is smooth, so is u. Also, since $f \in \mathcal{X}$, $f^{\perp}$, u and $u^{\perp}$ are also in $\mathcal{X}$ and hence are bounded. Choose $\sigma < \Lambda$ and assume (H'_f). We derive from Equation (11) (by [4] Thm 3.2) that : $\|u^{\perp}\|_{\mathcal{X}} < K_1$. Therefore $|u^{\perp}|$ is bounded by some $cste.\phi > 0$.
From Equation (12) we derive

$$u^1 = \frac{f^1}{(\Lambda - \sigma)} \rightarrow \pm\infty\, as\, (\Lambda - \sigma) \rightarrow 0. \tag{13}$$

Take δ small enough and $\sigma \in (\Lambda - \delta, \Lambda)$. Since $u = u^1\phi + u^{\perp}$, then

$$0 < \frac{K'}{\Lambda - \sigma}\phi < u < \frac{K"}{\Lambda - \sigma}\phi.$$

For $\sigma > \Lambda$. we do exactly the same, except that the signs are changed for u^1 in (13).

§3. A 2×2 Linear system

Consider now a linear system with constant coefficients.

$$(S) \qquad LU = AU + \mu U + F(x)\ in\ \mathbb{R}^N.$$

As above, $L := -\Delta + q$ where the potential q satisfies (H'_q), and where μ is a real parameter. L can be detailed as 2 equations:

$$(S) \qquad \begin{cases} Lu_1 &= au_1 + bu_2 + \mu u_1 + f_1(x) \\ Lu_2 &= cu_1 + du_2 + \mu u_2 + f_2(x) \end{cases} \quad in\ \mathbb{R}^N,$$

$$u_1(x), u_2(x)_{|x| \rightarrow \infty} \rightarrow 0.$$

Assume

$$(H_A) \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } b > 0 \text{ and } D := (a-d)^2 + 4bc \geq 0.$$

Note that $b > 0$ does not play any role since we can always change the order of the equations. The eigenvalues of A are

$$\xi_1 = \frac{a + d + \sqrt{D}}{2} \geq \xi_2 = \frac{a + d - \sqrt{D}}{2}.$$

As far as we know, all the previous studies suppose that the largest eigenvalue ξ_1 is simple (i.e. $D = (a-d)^2 + 4bc > 0$). Here we also study, in the second subsection, the case of a double eigenvalue $\xi_1 = \xi_2$, that is $D = 0$; this implies necessarily $bc < 0$ and necessarily the matrix is not cooperative.

3.1. Case $\xi_1 > \xi_2$

This is the classical case where ξ_1 is simple. Set $\xi_1 > \xi_2$. The eigenvectors are

$$X_k = \begin{pmatrix} b \\ \xi_k - a \end{pmatrix},$$

As above, denote by (Λ, ϕ), $\phi > 0$, the principal eigenpair of the operator $L = (-\Delta + q(x))$. It is easy to see that

$$L(X_k\phi) - AX_k\phi = (\Lambda - \xi_k)X_k\phi, \; k = 1, 2$$

Set $X := X_1$. Hence

$$\nu = \Lambda - \xi_1 \tag{14}$$

is the principal eigenvalue of (S) with associated eigenvector $X\phi$. Note that the components of $X\phi$ do not change sign, but, in the case of a non cooperative matrix they are not necessarily both positive.

3.1.1. Behavior for $\mu \to \nu = \Lambda - \xi_1$.

We prove:

Theorem 4. *Assume (H'_q), $b > 0$ and $D > 0$. Assume also that f_1 and f_2 are in $\mathcal{X}$ and*

$$(a - \xi_2)f_1^1 + bf_2^1 > 0. \tag{15}$$

Then, there exists $\delta > 0$, independant of μ, such that if $\nu - \delta < \mu < \nu$, there exists a positive constant γ depending only on F and Matrix A such that

For cooperative systems

$$c > 0 \Rightarrow u_1, u_2 \geq \frac{\gamma}{\nu - \mu}\phi > 0, \tag{16}$$

For non-cooperative systems

$$d > a \Rightarrow u_1, u_2 \geq \frac{\gamma}{\nu - \mu}\phi > 0, \tag{17}$$

$$a > d \Rightarrow u_1, -u_2 \geq \frac{\gamma}{\nu - \mu}\phi > 0. \tag{18}$$

If $\nu < \mu < \nu + \delta$, the sign are reversed.

Remark 2. It is noticeable that for all these cases, $|u_1|, |u_2| \to +\infty$ as $|\nu - \mu| \to 0$.

These results extend Theorem 4.2 in [2].

Proof: As in [1], we use J the associated Jordan matrix (which in this case is diagonal) and P the change of basis matrix which are such that

$$A = PJP^{-1}.$$

Here

$$P = \begin{pmatrix} b & b \\ \xi_1 - a & \xi_2 - a \end{pmatrix}, \quad P^{-1} = \frac{1}{b(\xi_1 - \xi_2)} \begin{pmatrix} a - \xi_2 & b \\ \xi_1 - a & -b \end{pmatrix}. \tag{19}$$

$$J = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}.$$

Denoting $\tilde{U} = P^{-1}U$ and $\tilde{F} = P^{-1}F$, we derive from System (S) (after multiplication by $P^{-1}U$ to the left):

$$L\tilde{U} = J\tilde{U} + \mu\tilde{U} + \tilde{F}.$$

Since J is diagonal we have two independant equations:

$$L\tilde{u}_k = (\xi_k + \mu)\tilde{u}_k + \tilde{f}_k,\ k = 1\, or\, 2. \tag{20}$$

The projection on ϕ and on its orthogonal for $k = 1$ and 2 gives

$$\tilde{u}_k = (\tilde{u}_k)^1\,\phi + \tilde{u}_k^{\perp}, \quad \tilde{f}_k = (\tilde{f}_k)^1\,\phi + \tilde{f}_k^{\perp};$$

hence

$$L(\tilde{u}_k)^1\,\phi = \Lambda(\tilde{u}_k)^1\,\phi = \xi_k(\tilde{u}_k)^1\,\phi + \mu(\tilde{u}_k)^1\,\phi + (\tilde{f}_k)^1\phi, \tag{21}$$

$$L\tilde{u}_k^{\perp} = \xi_k\tilde{u}_k^{\perp} + \mu\tilde{u}_k^{\perp} + \tilde{f}_k^{\perp}. \tag{22}$$

If both f_k are in $\mathcal{X}$, f_k/ϕ are bounded and hence both $\tilde{f}_k^{\perp}/\phi$ are bounded . Therefore, by (22) both $\tilde{u}_k^{\perp}/\phi$ are also bounded since the smallest eigenvalue for L acting on $\phi^{\perp}$ is $\lambda_2 \neq< \Lambda$.
We derive from (21) that

$$(\tilde{u}_k)^1 \ = \ \frac{(\tilde{f}_k)^1}{\Lambda - \xi_k - \mu}.$$

Consider again Equation (21) for $k = 2$; obviously, $(\tilde{u}_2)^1$ stays bounded as $\mu \to \nu = \Lambda - \xi_1 \neq< \Lambda - \xi_2$ and therefore $\tilde{u}_2/\phi$ stays bounded.
For $k = 1$, $(\tilde{u}_1)^1 = \frac{(\tilde{f}_1)^1}{\nu - \mu} \to \infty$ as $\mu \to \nu = \Lambda - \xi_1$, where $(\tilde{f}_1)^1 = \frac{1}{\xi_1 - \xi_2}((a - \xi_2)f_1^1 + bf_2^1) > 0$; this is the condition (15) which appears in Theorem 4. Then, we simply apply Theorem 3 to (20) for $k = 1$ and deduce that there existes $\delta > 0$, such that, for $|\Lambda - \xi_1 - \mu| = |\nu - \mu| < \delta$, there exists a positive constant $C > 0$ such that

$$\mu < \nu \ \Rightarrow\ \tilde{u}_1 \geq \frac{C}{\nu - \mu}\phi > 0;\ \mu > \nu \ \Rightarrow\ \tilde{u}_1 \leq \frac{C}{\nu - \mu}\phi < 0.$$

If $|\mu - \nu|$ small enough

$$(\tilde{u}_1)^1 \geq \frac{K}{\nu - \mu} > 0\ if\ \mu < \nu\,;\ (\tilde{u}_1)^1 \leq \frac{K}{\nu - \mu} < 0\ if\ \mu > \nu$$

where K is a positive constant depending only on F and A.
Now, it follows from $U = P\tilde{U}$, that

$$u_1 = b(\tilde{u}_1 + \tilde{u}_2),\ u_2 = (\xi_1 - a)\tilde{u}_1 + (\xi_2 - a)\tilde{u}_2.$$

As $\nu - \mu \to 0$, since $\tilde{u}_2/\phi$ stays bounded, u_1 behaves as $b(\tilde{u}_1)^1\phi > 0$; u_2 behaves as $(\xi_1 - a)(\tilde{u}_1)^1\phi$.
Therefore 3 cases appear according to matrix A:

If A is cooperative $(b > 0, c > 0)$, then $\xi_2 < a < \xi_1$ so that $(\xi_1 - a) > 0$ and $u_2 > 0$.

If A is non-cooperative with $b > 0, c < 0$, $d > a$, then $a < \xi_2 < \xi_1 \ \Rightarrow\ (\xi_1 - a) > 0,\ u_2 > 0$.

If A is non-cooperative with $b > 0, c < 0, a > d$, then $\xi_2 < \xi_1 < a \ \Rightarrow\ (\xi_1 - a) < 0,\ u_2 < 0$.

Remark 3. Indeed, we always assume that $b > 0$, hence $u_1 > 0$ for $\nu - \mu > 0$ small enough.

3.1.2. Behavior of the solution for $\mu \to \nu' := \Lambda - \xi_2$.

Obviously, $\nu' := \Lambda - \xi_2$ is also an eigenvalue of the system with associated eigenvector $X_2\phi$. Moreover we assume that ν' is the second eigenvalue of the system: $\nu < \nu' < \lambda_2 - \xi_1$.

Theorem 5. *Assume (H_q'), $b > 0$, $D > 0$ and $\nu' < \lambda_2 - \xi_1$. Assume also that f_1 and f_2 are in $\mathcal{X}$ and*

$$(\xi_1 - a)f_1^1 - bf_2^1 > 0. \tag{23}$$

Then, for $0 < \nu' - \mu$ small enough, there exists a positive constant γ' depending only on F and Matrix A such that
For cooperative systems, ($c > 0$), then

$$u_1, -u_2 \geq \frac{\gamma'}{\nu' - \mu}\phi > 0,$$

For non-cooperative systems ($c < 0$), then

$$d > a \Rightarrow u_1, u_2 \geq \frac{\gamma'}{\nu' - \mu}\phi > 0. \tag{24}$$

$$a > d \Rightarrow u_1, -u_2 \geq \frac{\gamma'}{\nu - \mu}\phi > 0. \tag{25}$$

If $0 < \mu - \nu'$ small enough, the sign are reversed.

Proof The proof is exactly the same as for Theorem 4 except that we derive from (21) that $(\tilde{u}_1)^1$ stays bounded and $(\tilde{u}_2)^1 = \frac{(\tilde{f}_2)^1}{\nu' - \mu} \to \infty$ as $\nu' - \mu \to 0$. This holds also since for $0 < \mu - \nu'$ small enough, $\mu + \xi_2 < \mu + \xi_1 < \lambda_2$. Now u_1 behaves as $b(\tilde{u}_2)$ and u_2 as $(\xi_2 - a)(\tilde{u}_2)$, and the result follows.

3.2. Case $\xi_1 = \xi_2$

Consider now the case where the coefficients of the matrix A satisfy $b > 0$ and

$$D := (a - d)^2 + 4bc = 0.$$

Of course this implies $bc < 0$ and since $b > 0$, then $c < 0$: only for non-cooperative systems a double root can appear.
Now $\xi_1 = \xi_2 = \xi = \frac{a+d}{2}$ and $\nu = \Lambda - \xi$. The proof of Theorem 4 is no more valid since *e.g.* in (19) there is a factor of the form $\frac{1}{\xi_1 - \xi_2}$. Moreover Matrix J is triangular and the system in $\tilde{U}$ is no more decoupled. We prove here

Theorem 6. *Assume (H_q') and $b > 0, c < 0$ with $(a - d)^2 + 4bc = 0$; assume also that f_1, f_2 are in $\mathcal{X}$ and :*

$$\frac{(a - d)}{2}f_1^1 + bf_2^1 > 0. \tag{26}$$

If $\nu - \delta < \mu < \nu + \delta$, δ small enough, there exists a positive constant γ such that

if $a > d$ $\quad u_1 \geq \frac{\gamma}{|\nu - \mu|}\phi, \quad u_2 \leq -\frac{\gamma}{|\nu - \mu|}\phi.$

$$\text{if } d > a \qquad u_1 \geq \frac{\gamma}{|\nu - \mu|}\phi, \;\; u_2 \geq \frac{\gamma}{|\nu - \mu|}\phi.$$

Remark 4. We notice that u_1 is always positive whatever the sign of $d-a$ or of $\nu-\mu$. Also u_2 keeps the same sign for μ going over ν. Things work as having 2 eigenvalues ξ_1 and ξ_2 with $\xi_1 - \xi_2 \to 0$. If (15) near ξ_1 and (23) near ξ_2 are valid together (that is if $(f_1)^1 > 0$ and

$$(\xi_1 - a)(f_1)^1 \geq b(f_2)^1 \geq (\xi_2 - a)(f_1)^1 \text{ if } d > a;$$

$$(a - \xi_2)(f_1)^1 > -b(f_2)^1 \geq (a - \xi_1))(f_1)^1 \text{ if } a > d,)$$

we apply the theorems above and derive that the functions u_1 and u_2 change sign twice (as μ goes over ν and ν') and finally they keep the same sign. Finally for $\xi_1 = \xi_2$, the 2 conditions reduce to $(f_1)^1 > 0$ and (26).

Proof The eigenvector associated to eigenvalue ξ is

$$X = \begin{pmatrix} b \\ \frac{d-a}{2} \end{pmatrix}.$$

The vector $X\phi$ is thus an eigenvector for $L - A$,

$$L(X\phi) - AX\phi = (\Lambda - \xi)X\phi = \nu X\phi.$$

We will need to use two different decompositions of the matrix A. For the decomposition 1 we choose

$$P_1 = \begin{pmatrix} b & \frac{2b}{a-d} \\ \frac{d-a}{2} & 0 \end{pmatrix}, \quad P_1^{-1} = \frac{1}{b}\begin{pmatrix} 0 & -\frac{2b}{a-d} \\ \frac{a-d}{2} & b \end{pmatrix}.$$

So the associated triangular matrix J_1 is

$$J_1 = P_1^{-1} A P_1 = \begin{pmatrix} \xi & 1 \\ 0 & \xi \end{pmatrix}.$$

As above, setting $\tilde{U} = P_1^{-1}U$ and $\tilde{F} = P_1^{-1}F$, we derive from System (S)

$$L\tilde{U} = J_1\tilde{U} + \mu\tilde{U} + \tilde{F}.$$

We do not have anymore a decoupled system but

$$\begin{cases} L\tilde{u}_1 = (\xi + \mu)\tilde{u}_1 + \tilde{u}_2 + \tilde{f}_1, \\ L\tilde{u}_2 = \qquad\qquad\; + (\xi + \mu)\tilde{u}_2 + \tilde{f}_2; \end{cases} \tag{27}$$

here $\tilde{f}_1 = \frac{-2}{a-d} f_2$ and $\tilde{f}_2 = \frac{(a-d)}{2b} f_1 + f_2$ are in $\mathcal{X}$ and $(\tilde{f}_2)^1 > 0$ by (26).

• If $\xi + \mu < \Lambda$ (that is $\mu < \nu$), by Theorem 3 applied to the second equation, there exists a constant $K > 0$, such that $\tilde{u}_2 > \frac{K}{\nu-\mu}\phi$. Hence, for $\nu - \mu$ small enough for any $\tilde{f}_1 \in \mathcal{X}$, $\tilde{u}_2 + \tilde{f}_1$ is strictly positive and is in $\mathcal{X}$; then again Theorem 3 applied to the first equation implies that there exists a constant $K' > 0$, such that $\tilde{u}_1 > \frac{K'}{\nu-\mu}\phi$.

For $a > d$, we can conclude that there exists a constant $\gamma > 0$,

$$U = P_1\tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 + \frac{2b}{a-d}\tilde{u}_2 > \frac{\gamma}{\nu-\mu}\phi, \\ u_2 = \frac{d-a}{2}\tilde{u}_1 < -\frac{\gamma}{\nu-\mu}\phi. \end{cases}$$

• If $\mu > \nu$ we have reversed sign for $\tilde{u}_2$. Hence, for $\mu - \nu$ small enough for any $\tilde{f}_1 \in \mathcal{X}$, $\tilde{u}_2 + \tilde{f}_1$ is stricly negative and is in $\mathcal{X}$; then again Theorem 3 for the first equation implies that there exists a constant $K' > 0$, such that $\tilde{u}_1 > \frac{K'}{\mu-\nu}\phi$.
For $d > a$, we can conclude that there exists a constant $\gamma > 0$,

$$U = P_1\tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 + \frac{2b}{a-d}\tilde{u}_2 > \frac{\gamma}{\mu-\nu}\phi, \\ u_2 = \frac{d-a}{2}\tilde{u}_1 > \frac{\gamma}{\mu-\nu}\phi. \end{cases}$$

For the remaining cases, we need to use an other decomposition of matrix A. For the decomposition 2 we choose

$$P_2 = \begin{pmatrix} b & 0 \\ \frac{d-a}{2} & 1 \end{pmatrix}, \quad P_2^{-1} = \frac{1}{b}\begin{pmatrix} 1 & 0 \\ \frac{a-d}{2} & b \end{pmatrix}.$$

So the associated triangular matrix J_2 is

$$J_2 = P_2^{-1}AP_2 = \begin{pmatrix} \xi & 1 \\ 0 & \xi \end{pmatrix}.$$

As above, setting $\tilde{U} = P_2^{-1}U$ and $\tilde{F} = P_2^{-1}F$, we derive from System (S) the same system with the same function $\tilde{f}_2 = \dfrac{(a-d)}{2b}f_1 + f_2$:

$$\begin{cases} L\tilde{u}_1 = (\xi+\mu)\tilde{u}_1 + \tilde{u}_2 + \tilde{f}_1, \\ L\tilde{u}_2 = \quad + (\xi+\mu)\tilde{u}_2 + \tilde{f}_2. \end{cases} \tag{28}$$

• If $\xi + \mu < \Lambda$ (that is $\mu < \nu$), since $(\tilde{f}_2)^1 = \frac{(a-d)}{2b}f_1^1 + f_2^1 > 0$, we get (exactly as for decomposition 1) that there exists a constant $K > 0$, such that $\tilde{u}_2 > \frac{K}{\nu-\mu}\phi$ and there exists a constant $K' > 0$, such that $\tilde{u}_1 > \frac{K'}{\nu-\mu}\phi$.
For $d > a$, we can conclude that there exists a constant $\gamma > 0$,

$$U = P_2\tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 > \frac{\gamma}{\nu-\mu}\phi, \\ u_2 = \frac{d-a}{2}\tilde{u}_1 + \tilde{u}_2 > \frac{\gamma}{\nu-\mu}\phi. \end{cases}$$

• If $\mu > \nu$ we have reversed sign for $\tilde{u}_2$. Hence, there exists a constant $K' > 0$, such that $\tilde{u}_1 > \frac{K'}{\nu-\mu}\phi$.
For $a > d$, we can conclude that there exists a constant $\gamma > 0$,

$$U = P_2\tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 > \frac{\gamma}{\mu-\nu}\phi, \\ u_2 = \frac{d-a}{2}\tilde{u}_1 + \tilde{u}_2 < -\frac{\gamma}{\mu-\nu}\phi. \end{cases}$$

References

[1] Alziary, B., and Fleckinger, J. Sign of the solution to a non-cooperative system. *Ro-MaKo 71* (2016), 3–13.

[2] Alziary, B., Fleckinger, J., and Takac, P. Maximum and anti-maximum principles for some systems involving Schrödinger operator. *Operator Theory: Advances and applications 110* (1999), 13–21.

[3] Alziary, B., Fleckinger, J., and Takac, P. An extension of maximum and anti-maximum principles to a Schrödinger equation in $\mathbb{R}^n$. *Positivity 5*, 4 (2001), 359–382.

[4] Alziary, B., Fleckinger, J., and Takac, P. Groundstate positivity, negativity, and compactness for Schrödinger operator in $\mathbb{R}^n$. *Jal Funct. Anal. 245* (2007), 213–248.

[5] Alziary, B., and Takac, P. Compactness for a schrödinger operator in the groundstate space over $\mathbb{R}^n$. *Electr. J Diff. Eq., Conf. 16* (2007), 35–58.

[6] Alziary, B., and Takac, P. Intrinsic ultracontractivity of a Schrödinger semigroup in $\mathbb{R}^n$. *J. Funct. Anal. 256*, 12 (2009), 4095–4127.

[7] Besbas, N. Principe d'antimaximum pour des équations et des systèmes de type Schrödinger dans $\mathbb{R}^n$.

[8] Clément, P., and Peletier, L. An anti-maximum principle for second order elliptic operators. *J. Diff. Equ. 34* (1979), 218–229.

[9] Edmunds, D.-E., and Evans, W.-D. *Spectral Theory and Differential Operators*. Classics in Applied Mathematics. Oxford Science Publ. Clarendon Press, 1987.

[10] Lécureux, M.-H. Comparison with groundstate for solutions of non cooperative systems for Schrödinger operators in $\mathbb{R}^n$. *RoMaKo 65* (2010), 51–69.

[11] Reed, M., and Simon, B. *Methods of modern mathematical physics IV. Analysis of operators.* Acad.Press, New York, 1978.

Bénédicte Alziary and Jacqueline Fleckinger
Université de Toulouse
Institut de Mathématiques -CeReMath UT1
CNRS UMR 5219
alziary@ut-capitole.fr and Jfleckinger@gmail.com