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SCHAUDER ESTIMATES FOR DISCRETE FRACTIONAL INTEGRALS

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Abstract. In this note we focus on the discrete fractional integrals as a natural continuation of our previous work about nonlocal fractional derivatives, discrete and continuous. We define the discrete fractional integrals by using the semigroup theory and we study the regularity of discrete fractional integrals on the discrete Hölder spaces, which it is known in the differential equations field as the discrete Schauder estimates.

Keywords: Discrete fractional integrals, Schauder estimates, Semigroups.

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§1. Introduction

Fractional calculus extends the definitions of derivatives and integrals to noninteger orders. It was born in 1695, with a letter from L' Hôpital to Leibniz, where he asked what would be the derivative of order $1/2$. After this moment, a lot of authors have worked in this field, and in the last century, the interest in fractional operators and fractional differential equations has grown exponentially because of the big amount of applications it has.

In this note we will focus on discrete fractional operators, in particular on the discrete fractional integrals as a continuation of our recent work [2]. We will define the discrete fractional integrals by the semigroup theory approach and we will take some advantages of the method to get some regularity results. This point of view to treat discrete fractional operators has been recently used in [5],[6] and [7], among others works. Of course, in the last years the discrete fractional integrals have also been considered in a lot of papers (see for instance [1, 3, 8] and references therein), but not from the point of view of the semigroup theory as fractional powers.

For $f : \mathbb{Z} \to \mathbb{R}$, we define *"the discrete derivative from the right"* and *"the discrete derivative from the left"* as the operators given by the formulas

$$\delta_{\text{right}} f(n) = f(n) - f(n+1) \ \ and \ \ \delta_{\text{left}} f(n) = f(n) - f(n-1).$$

As we did in [2], we shall use semigroup language as an alternative approach to discrete fractional integrals. Given the function $G_t(n) = e^{-t}\frac{t^n}{n!}$, $n \in \mathbb{N}_0$, we prove in [2, Proposition 2.2] that the operators

$$T_{t,+}f(n) = \sum_{j=0}^{\infty} G_t(j) f(n+j), \text{ and } T_{t,-}f(n) = \sum_{j=0}^{\infty} G_t(j) f(n-j), \qquad t > 0,\, n \in \mathbb{Z}.$$

are markovian semigroups on $\ell^p(\mathbb{Z})$, $1 \le p \le \infty$, whose infinitesimal generators are $-\delta_{\text{right}}$ and $-\delta_{\text{left}}$, respectively. In addition, we proved that $u(n,t) = T_{t,+}f(n)$ solves the first order

Cauchy problem

$$\begin{cases} \partial_t u(n,t) + \delta_{\text{right}} u(n,t) = 0, & n \in \mathbb{Z},\ t \geq 0, \\ u(n,0) = f(n), & n \in \mathbb{Z}, \end{cases}$$

and $v(n,t) = T_{t,-}f(n)$ satisfies the analogous Cauchy problem for δ_{left}.

We recall to the reader the following Gamma function formulas for an operator L,

$$L^{\alpha} = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} (e^{-tL} - 1) \frac{dt}{t^{1+\alpha}} \quad \text{and} \quad L^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-tL} \frac{dt}{t^{1-\alpha}},$$

where $0 < \alpha < 1$ and e^{-tL} is the associated semigroup, see [4, 9, 10, 12]. In particular, we have that the powers of order α of the discrete derivatives can be written by

$$(\delta_{\text{right}})^{\alpha} f(n) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{T_{t,+}f(n) - f(n)}{t^{1+\alpha}} dt,\ 0 < \alpha < 1,$$

and

$$(\delta_{\text{right}})^{-\alpha} f(n) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \frac{T_{t,+}f(n)}{t^{1-\alpha}} dt,\ 0 < \alpha < 1, \tag{1}$$

whenever the integrals converges, and the corresponding formula for $(\delta_{\text{left}})^{\alpha}$, $-1 < \alpha < 1$.

In order to get regularity results for the discrete fractional integrals in a more general setting, we will consider our operators on a mesh of step length $h > 0$ instead of the integers mesh, that is, our functions will be defined on $\mathbb{Z}_h = \{jh : j \in \mathbb{Z}\}$, for $h > 0$. Hence, for $u : \mathbb{Z}_h \to \mathbb{R}$, with $h > 0$, the first order difference operators on $\mathbb{Z}_h$ are given by

$$\delta_{\text{right}} u(hn) = \frac{u(hn) - u(h(n+1))}{h}, \quad \delta_{\text{left}} u(hn) = \frac{u(hn) - u(h(n-1))}{h}, \quad n \in \mathbb{Z}.$$

In [2] we also prove that $\{T_{\frac{t}{h},\pm}\}_{t\geq 0}$ are the associated semigroups on $\ell^p(\mathbb{Z}_h)$.

The main results of this note are the discrete Schauder estimates for the discrete fractional integrals. Schauder estimates are very useful in the field of differential equations because they concern the regularity of solutions to partial differential equations. Recently, Schauder estimates have been used to get the regularity of fractional operadors in the adapted Hölder spaces, see for instance [11].

In our case, we need some special discrete Hölder spaces, called $C_h^{k,\beta}$. These spaces were introduced in [7]. To see the definition of these spaces, see Section 3.

Theorem 1 (Discrete Schauder estimates). *Let* $0 < \beta, \alpha < 1$*, and* $u \in \ell_{-\alpha,h}$*, see* (4).

(i) Let $u \in C_h^{0,\beta}$ *and* $\alpha + \beta < 1$*. Then* $(\delta_{\text{right}})^{-\alpha} u \in C_h^{0,\beta+\alpha}$ *and*

$$\|(\delta_{\text{right}})^{-\alpha} u\|_{C_h^{0,\beta+\alpha}} \leq C \|u\|_{C_h^{0,\beta}}.$$

(ii) Let $u \in C_h^{0,\beta}$ *and* $\alpha + \beta > 1$*. Then* $(\delta_{\text{right}})^{-\alpha} u \in C_h^{1,\beta+\alpha-1}$ *and*

$$\|(\delta_{\text{right}})^{-\alpha} u\|_{C_h^{1,\beta+\alpha-1}} \leq C \|u\|_{C_h^{0,\beta}}.$$

(iii) Let $u \in C_h^{k,\beta}$ *and assume that* $k + \beta + \alpha$ *is not an integer. Then* $(\delta_{\text{right}})^{-\alpha} u \in C_h^{l,s}$ *where* l *is the integer part of* $k + \beta + \alpha$ *and* $s = k + \beta + \alpha - l$*.*

(iv) Let $u \in \ell^\infty$. Then $(\delta_{\text{right}})^{-\alpha} u \in C_h^{0,\alpha}$ and

$$\|(\delta_{\text{right}})^{-\alpha} u\|_{C_h^{0,\alpha}} \le C\|u\|_\infty.$$

The positive constants C are independent of h and u.

§2. The approach via semigroup theory.

Let $\alpha \in \mathbb{R}$. Along this paper we denote

$$\Lambda^{-\alpha}(m) = \frac{\alpha(\alpha+1)\cdots(\alpha+m-1)}{m!}, \quad m \in \mathbb{N},$$

and $\Lambda^{-\alpha}(0) = 1$. Note that if $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$ we have that $\Lambda^{-\alpha}(m) = \binom{m+\alpha-1}{m}$ for $m \in \mathbb{N}_0$. Here we highlight some properties of this kernel. Also, if $0 < \alpha < 1$, then $\Lambda^{-\alpha}$ is decreasing as a function of n, while if $-1 < \alpha < 0$, we have $\sum_{n=0}^\infty \Lambda^{-\alpha}(n) = 0$, so $\sum_{n=1}^\infty \Lambda^{-\alpha}(n) = -1$.

Also, the kernel $(\Lambda^{-\alpha}(n))_{n\in\mathbb{N}_0}$ could be defined by the generating function, that is,

$$\sum_{n=0}^\infty \Lambda^{-\alpha}(n) z^n = \frac{1}{(1-z)^\alpha}, \quad |z| < 1,$$

and therefore we have

$$\Lambda^{-(\alpha+\beta)}(n) = \sum_{j=0}^n \Lambda^{-\alpha}(n-j)\Lambda^{-\beta}(j), \quad \alpha,\beta \in \mathbb{R},\ n \in \mathbb{N}_0. \tag{2}$$

In the following, we will use the asymptotic behaviour of the sequences $\Lambda^{-\alpha}$. It is known that for every $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$,

$$\Lambda^{-\alpha}(n) = \frac{1}{n^{1-\alpha}\Gamma(\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \in \mathbb{N}, \tag{3}$$

see [13, Vol.I, p.77, (1.18)]. In the case $\alpha \in \{0, -1, -2, \dots\}$, $\Lambda^{-\alpha}(n) = 0$ for $n > -\alpha$. To see more properties of $\{\Lambda^{-\alpha}(n)\}_{n\in\mathbb{N}_0}$ in a general setting, see [13].

As it was done in [7], we also need to consider our functions in a particular space in order to assure the convergence of our operators. For $0 < \alpha < 1$, we define the space $\ell_{-\alpha,h}$ as follows:

$$\ell_{-\alpha,h} = \left\{ u : \mathbb{Z}_h \to \mathbb{R} : \text{ for every } n \in \mathbb{Z}, \sum_{m=0}^\infty \frac{|u(m \pm n)h)|}{(1+m)^{1-\alpha}} < \infty \right\}. \tag{4}$$

Hence, by using (1), for $0 < \alpha < 1$, and $f \in \ell_{-\alpha,1}$, we have

$$\begin{aligned}(\delta_{\text{right}})^{-\alpha} f(n) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-t}\sum_{j=0}^\infty \frac{t^j}{j!} f(n+j)}{t^{1-\alpha}} dt = \sum_{j=0}^\infty f(n+j) \int_0^\infty \frac{e^{-t} t^{j+\alpha}}{j!\Gamma(\alpha)} \frac{dt}{t} \\ &= \sum_{j=0}^\infty f(n+j) \frac{\Gamma(\alpha+j)}{\Gamma(\alpha) j!} = \sum_{j=0}^\infty \Lambda^{-\alpha}(j) f(n+j) = \sum_{m=n}^\infty \Lambda^{-\alpha}(m-n) f(m),\end{aligned}$$

where the interchange of the sum and the integral is justified because of the integral converges absolutely. By a similar way we also get

$$(\delta_{\text{left}})^{-\alpha}f(n)=\sum_{j=0}^{\infty}\Lambda^{-\alpha}(j)f(n-j)=\sum_{m=-\infty}^{n}\Lambda^{-\alpha}(n-m)f(m).$$

Observe that, as we did in [2], by proceeding similarly we get

$$(\delta_{\text{right}})^{\alpha}f(n)=\sum_{m=n}^{\infty}\Lambda^{\alpha}(m-n)f(m),\quad (\delta_{\text{left}})^{\alpha}f(n)=\sum_{m=-\infty}^{n}\Lambda^{\alpha}(n-m)f(m),\quad n\in\mathbb{N}_0.$$

Now we will consider our functions on $\mathbb{Z}_h=h\mathbb{Z}$, for $h>0$. Let $u:\mathbb{Z}_h\to\mathbb{R}$. Then, for $0<\alpha<1$, we can write

$$(\delta_{\text{right}})^{-\alpha}u(nh)=h^{\alpha}\sum_{m=n}^{\infty}\Lambda^{-\alpha}(m-n)u(mh),\quad (\delta_{\text{left}})^{-\alpha}u(nh)=h^{\alpha}\sum_{m=-\infty}^{n}\Lambda^{-\alpha}(n-m)u(mh),\tag{5}$$

$$(\delta_{\text{right}})^{\alpha}u(nh)=\frac{1}{h^{\alpha}}\sum_{m=n}^{\infty}\Lambda^{\alpha}(m-n)u(mh)\ \text{ and }\ (\delta_{\text{left}})^{\alpha}u(nh)=\frac{1}{h^{\alpha}}\sum_{j=-\infty}^{n}\Lambda^{\alpha}(n-m)u(mh),\tag{6}$$

whenever the series converge.

In general, for any $\alpha>0$, it is defined

$$(\delta_{\text{right}})^{\alpha}u=(\delta_{\text{right}})^{m}(\delta_{\text{right}})^{\alpha-m}u,\quad (\delta_{\text{right}})^{-\alpha}u=(\delta_{\text{right}})^{-m}(\delta_{\text{right}})^{-(\alpha-m)}u,$$

where $m=[\alpha]$. In addition, in our case, by (2) we have that formulas (5) and (6) are valid for every $\alpha>0$. Also, by (2) we have

$$(\delta_{\text{right}})^{-\alpha}(\delta_{\text{right}})^{\alpha}u(nh)=u(nh),\quad n\in\mathbb{Z},u\in\ell^p(\mathbb{Z}_h).$$

Furthermore, for $\alpha,\beta\in\mathbb{R}$, we have

$$(\delta_{\text{right}})^{\alpha}(\delta_{\text{right}})^{\beta}u(nh)=(\delta_{\text{right}})^{\alpha+\beta}u(nh),\quad n\in\mathbb{Z},$$

for u such that the series involved in the identity converge.

§3. Regularity results of Discrete Fractional Integrals

Following the notation in [2] and [7], for $l,s\in\mathbb{N}_0$, we denote $\delta^{l,s}_{\text{right,left}}:=(\delta_{\text{right}})^{l}(\delta_{\text{left}})^{s}$.

Definition 1. ([7, Definition 2.1]). Let $0<\beta\le 1$ and $k\in\mathbb{N}_0$. A function $u:\mathbb{Z}_h\to\mathbb{R}$ belongs to the discrete Hölder space $C_h^{k,\beta}$ if

$$[\delta^{l,s}_{\text{right,left}}u]_{C_h^{0,\beta}}=\sup_{m\neq j}\frac{|\delta^{l,s}_{\text{right,left}}u(jh)-\delta^{l,s}_{\text{right,left}}u(hm)|}{h^{\beta}|j-m|^{\beta}}<\infty$$

for each pair $l,s\in\mathbb{N}_0$ such that $l+s=k$. The norm in the spaces $C_h^{k,\beta}$ is given by

$$\|u\|_{C_h^{k,\beta}}=\max_{l+s\le k}\sup_{m\in\mathbb{Z}}|\delta^{l,s}_{\text{right,left}}u(mh)|+\max_{l+s=k}[\delta^{l,s}_{\text{right,left}}u]_{C_h^{0,\beta}}.$$

For simplicity, we only write the following theorem for $(\delta_{\text{right}})^{-\alpha}$ since it is analogous for $(\delta_{\text{left}})^{-\alpha}$.

To prove this theorem we need a lemma about the kernel $\Lambda^{-\alpha}$.

Lemma 2. *1. For every $j \in \mathbb{N}_0$, and $\alpha \in \mathbb{R}$, $\Lambda^{-\alpha}(j+1) - \Lambda^{-\alpha}(j) = \Lambda^{-(\alpha-1)}(j+1)$.*

2. For every $n, l \in \mathbb{Z}$, with $n > l$, and $0 < \alpha < 1$,

$$\sum_{m=n}^{\infty}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l)) - \sum_{m=l}^{n-1}\Lambda^{-\alpha}(m-l) = 0.$$

Proof. At first we prove (1). Observe that $\Lambda^{-\alpha}(1) - \Lambda^{-\alpha}(0) = \alpha - 1 = \Lambda^{-(\alpha-1)}(1)$. Let $j \in \mathbb{N}$. We have that

$$\begin{aligned}\Lambda^{-\alpha}(j+1) - \Lambda^{-\alpha}(j)) &= \frac{\alpha(\alpha+1)\dots(\alpha+j-1)}{j!}\left(\frac{\alpha+j}{j+1} - 1\right) = \frac{(\alpha+j-1)!}{(\alpha-1)!j!}\left(\frac{\alpha-1}{j+1}\right) \\ &= \frac{\Gamma(j+\alpha)}{(j+1)!\Gamma(\alpha-1)} = \Lambda^{-(\alpha-1)}(j+1).\end{aligned}$$

Now we prove (2). Let $n, l \in \mathbb{Z}$, with $n > l$, and $0 < \alpha < 1$. By using the identity in (1) we obtain

$$\begin{aligned}\sum_{m=n}^{\infty}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l)) &= \sum_{m=n}^{\infty}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-(n-1)) + \Lambda^{-\alpha}(m-(n-1)) + \dots \\ &\quad + \Lambda^{-\alpha}(m-l-1) - \Lambda^{-\alpha}(m-l)) \\ &= \sum_{m=n}^{\infty}(-\Lambda^{-(\alpha-1)}(m-(n-1)) - \Lambda^{-(\alpha-1)}(m-(n-2)) - \dots - \Lambda^{-(\alpha-1)}(m-l))\end{aligned}$$

Again, as $\sum_{m=k}^{\infty}\Lambda^{-(\alpha-1)}(m-k) = 0$, we have that

$$\begin{aligned}-\sum_{m=n}^{\infty}\Lambda^{-(\alpha-1)}(m-(n-1)) &= \Lambda^{-(\alpha-1)}(0) = \Lambda^{-\alpha}(0) \\ -\sum_{m=n}^{\infty}\Lambda^{-(\alpha-1)}(m-(n-2)) &= \Lambda^{-(\alpha-1)}(0) + \Lambda^{-(\alpha-1)}(1) \\ &\vdots \\ -\sum_{m=n}^{\infty}\Lambda^{-(\alpha-1)}(m-l) &= \Lambda^{-(\alpha-1)}(0) + \Lambda^{-(\alpha-1)}(1) + \dots + \Lambda^{-(\alpha-1)}(n-l-1).\end{aligned}$$

Thus, using again identity on (1) we get

$$\begin{aligned}
&\sum_{m=n}^{\infty}(\Lambda^{-\alpha}(m-n)-\Lambda^{-\alpha}(m-l))\\
&=(n-l)\Lambda^{-\alpha}(0)+(n-l-1)\Lambda^{-(\alpha-1)}(1)+\cdots+\Lambda^{-(\alpha-1)}(n-l-1)\\
&=\sum_{m=l}^{n-1}\Lambda^{-\alpha}(m-l),
\end{aligned}$$

and the result follows. □

Now we can prove Theorem 1.

Proof of Theorem 1.

Let $n, l \in \mathbb{Z}$, we assume $n > l$ without loss of generality. First let $u \in C_h^{0,\beta}$ and $\alpha+\beta<1$. By using Lemma 2 (2) we can write

$$\begin{aligned}
h^{-\alpha}[(\delta_{\text{right}})^{-\alpha}u(nh)-(\delta_{\text{right}})^{-\alpha}u(lh)] &= \sum_{m=n}^{\infty}\Lambda^{-\alpha}(m-n)u(mh)-\sum_{m=l}^{\infty}\Lambda^{-\alpha}(m-l)u(mh)\\
&=\sum_{m=n}^{\infty}(\Lambda^{-\alpha}(m-n)-\Lambda^{-\alpha}(m-l))(u(mh)-u(lh))-\sum_{m=l}^{n-1}\Lambda^{-\alpha}(m-l)(u(mh)-u(lh))\\
&=I+II.
\end{aligned}$$

On the one hand, by using estimate (3) and the hypothesis on u, we get

$$|II|\le C[u]_{C_h^{0,\beta}}h^{\beta}\sum_{m=l+1}^{n-1}\frac{|m-l|^{\beta}}{|m-l|^{1-\alpha}}=C[u]_{C_h^{0,\beta}}h^{\beta}\sum_{k=1}^{n-1-l}\frac{1}{k^{1-\alpha-\beta}}\le C[u]_{C_h^{0,\beta}}h^{\beta}(n-l)^{\alpha+\beta}.$$

Before doing the estimation for I, observe that, as $n > l$, by (3) we have that $|\Lambda^{-\alpha}(m-n)| \le \frac{C}{(m-n)^{1-\alpha}}$ and $|\Lambda^{-\alpha}(m-l)| \le \frac{C}{(m-n)^{1-\alpha}}$ for $m \ge n+1$. Also, by using Lemma 2 (1) and (3) we get that

$$\begin{aligned}
&|\Lambda^{-\alpha}(m-n)-\Lambda^{-\alpha}(m-l)|\\
&=|-\Lambda^{-(\alpha-1)}(m-(n-1))-\Lambda^{-(\alpha-1)}(m-(n-2))-\cdots-\Lambda^{-(\alpha-1)}(m-l)|\\
&\le\frac{C|n-l|}{(m-(n-1))^{2-\alpha}}\le\frac{C|n-l|}{(m-n)^{2-\alpha}}\quad m\ge n+1. \qquad (7)
\end{aligned}$$

Hence, by using the comments above, the hypothesis on u and (3), we obtain that

$$\begin{aligned}
|I|&\le C[u]_{C_h^{0,\beta}}h^{\beta}\left(|n-l|^{\beta}+\frac{|n-l|^{\beta}}{(n-l)^{1-\alpha}}+\sum_{m=n+1}^{2n-l}\frac{|m-l|^{\beta}}{(m-n)^{1-\alpha}}+\sum_{m=2n-l+1}^{\infty}\frac{(n-l)|m-l|^{\beta}}{(m-n)^{2-\alpha}}\right)\\
&\le C[u]_{C_h^{0,\beta}}h^{\beta}\left((|n-l|^{\alpha+\beta}+\sum_{k=1}^{n-l}\frac{k^{\beta}+(n-l)^{\beta}}{k^{1-\alpha}}+\sum_{k=n-l+1}^{\infty}\frac{(n-l)(k^{\beta}+(n-l)^{\beta})}{k^{2-\alpha}}\right)\\
&\le C[u]_{C_h^{0,\beta}}h^{\beta}(n-l)^{\alpha+\beta}.
\end{aligned}$$

Now suppose that $u \in C_h^{0,\beta}$ with $\alpha + \beta > 1$. By the definition of the space $C_h^{1,\alpha+\beta-1}$, we have to prove that $\delta_{\text{right}}((\delta_{\text{right}})^{-\alpha}u) \in C_h^{0,\alpha+\beta-1}$. By using $\delta_{\text{right}}((\delta_{\text{right}})^{-\alpha}u) = (\delta_{\text{right}})^{1-\alpha}u$ and [2, Theorem 3.2], we conclude that $\delta_{\text{right}}((\delta_{\text{right}})^{-\alpha}u) \in C_h^{0,\alpha+\beta-1}$, so the result follows.

We prove statement (iii) for $k = 1$. The other cases follow by iteration.

Let $u \in C_h^{1,\beta}$ and $\alpha + \beta < 1$. By hypothesis, $\delta_{\text{right}}u$ belongs to $C_h^{0,\beta}$. We want to prove that $\delta_{\text{right}}^{-\alpha}u \in C_h^{1,\alpha+\beta}$, that is, $\delta_{\text{right}}(\delta_{\text{right}}^{-\alpha})u = \delta_{\text{right}}^{-\alpha}(\delta_{\text{right}}u) \in C_h^{0,\alpha+\beta}$, and this is consequence of (i).

Now suppose that $u \in C_h^{1,\beta}$ and $\alpha + \beta > 1$. By hypothesis, $\delta_{\text{right}}u \in C_h^{0,\beta}$. We want to prove that $\delta_{\text{right}}^{-\alpha}u \in C_h^{2,\alpha+\beta-1}$, that is, $(\delta_{\text{right}})^2(\delta_{\text{right}}^{-\alpha}u) = \delta_{\text{right}}(\delta_{\text{right}}^{1-\alpha})u \in C_h^{0,\alpha+\beta-1}$. By using (ii), we have that $\delta_{\text{right}}^{-\alpha}(\delta_{\text{right}}u) = \delta_{\text{right}}^{1-\alpha}u \in C_h^{1,\alpha+\beta-1}$, and by the definition of the space $C_h^{1,\alpha+\beta-1}$, we conclude that $\delta_{\text{right}}(\delta_{\text{right}}^{1-\alpha})u \in C_h^{0,\alpha+\beta-1}$.

Finally, assume that $u \in \ell^\infty$. Again, we can write

$$\begin{aligned} h^{-\alpha}[(\delta_{\text{right}})^{-\alpha}u(nh) - (\delta_{\text{right}})^{-\alpha}u(lh)] &= \sum_{m=n}^{\infty}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l))u(mh) \\ &\quad - \sum_{m=l}^{n-1}\Lambda^{-\alpha}(m-l)u(mh). \end{aligned}$$

By using (7), we have

$$\left|\sum_{m=2n-l+1}^{\infty}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l))u(mh)\right| \leq \|u\|_\infty \sum_{m=2n-l+1}^{\infty}\frac{n-l}{(m-n)^{2-\alpha}} \leq C\|u\|_\infty(n-l)^\alpha$$

and by using (3), we get that

$$\begin{aligned} &\left|\sum_{m=n}^{2n-l}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l))u(mh)\right| \\ &\leq C\|u\|_\infty\left(1 + \frac{1}{(n-l)^{1-\alpha}} + \sum_{m=n+1}^{2n-l}(|\Lambda^{-\alpha}(m-n)| + |\Lambda^{-\alpha}(m-l)|)\right) \\ &\leq C\|u\|_\infty\left(1 + \frac{1}{(n-l)^{1-\alpha}} + \sum_{m=n+1}^{2n-l}\frac{1}{(m-n)^{1-\alpha}}\right) \leq C\|u\|_\infty(n-l)^\alpha \end{aligned}$$

and

$$\left|\sum_{m=l}^{n-1}\Lambda^{-\alpha}(m-l)u(mh)\right| \leq C\|u\|_\infty\left(1 + \sum_{m=l+1}^{n-1}\frac{1}{(m-l)^{1-\alpha}}\right) \leq C\|u\|_\infty(n-l)^\alpha.$$

□

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