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ON COMPARISON PRINCIPLES FOR WEAK SOLUTIONS OF DOUBLY NONLINEAR REACTION-DIFFUSION EQUATIONS

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Abstract. The weak comparison principle for weak solutions of scalar doubly nonlinear reaction-diffusion equations

$$\frac{\partial}{\partial t}b(u) - \operatorname{div}(a(b(u), \nabla u)) = f$$

is proved under slightly more general conditions on a than those used by Otto [11], Díaz [4] and Kurta [8].

Keywords: doubly nonlinear reaction-diffusion equations, comparison principle.

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§1. Introduction

Consider a scalar doubly nonlinear reaction-diffusion equation

$$\frac{\partial}{\partial t}b(u) - \operatorname{div}(a(b(u), \nabla u)) = f \tag{1.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$, where a, b are allowed to have degenerate or singular derivatives and f denotes an inhomogeneity or nonlinearity. This article is concerned with conditions on the nonlinear mappings a, b which allow to prove the weak comparison principle for two weak solutions $u, \tilde{u}$ of (1.1) to initial values $u_0 \leq \tilde{u}_0$ and inhomogeneities $f \leq \tilde{f}$ (or nonlinearities $f = f(b(u))$). Otto [11] has proved the validity of the weak comparison principle for a continuous non-decreasing monotone function b under certain conditions on a, Díaz [4] has proved the same principle (for strong solutions) under other conditions on a, and Kurta [8] again invokes different assumptions on a. The aim of this article is to show the weak comparison principle under a slightly more general condition on a, which unifies the conditions of [11, 4, 8] and seems to be at the heart of the method of proof.

The validity of the weak comparison principle for (1.1) has several consequences, e.g. it directly implies uniqueness of solutions, continuous dependence and the weak maximum principle, see [12]. Moreover, it allows to develop an L^1-theory for doubly nonlinear reaction-diffusion equations, and positivity of solutions can be proved in some situations by comparison with an explicit solution (for a general discussion of positivity let us refer to [5]). Therefore, it is important to find general conditions which guarantee the validity of the weak comparison principle for a large class of equations.

Outline

In section 2 conditions on the functions a, b, f needed to prove existence of weak solutions with finite energy are formulated. The weak comparison principle is established in section 3 via a signed L^1-contraction principle under conditions, which unify and slightly generalize the conditions of [11, 4, 8], as is shown in the final section.

§2. Weak solutions with finite energy

Before we discuss the weak comparison principle for scalar doubly nonlinear reaction-diffusion equations

$$\frac{\partial}{\partial t}b(u) - \operatorname{div}(a(b(u), \nabla u)) = f\,, \tag{1.1}$$

let us formulate conditions which play an important role in proving existence of weak solutions of (1.1) with finite energy.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $b : \mathbb{R} \to \mathbb{R}$ be a continuous non-decreasing monotone function. Denote by $\phi_b(u) := \int_0^u b(u)\,du$ the convex potential of b with $\phi_b(0) = 0$ and define the Legendre-Fenchel transform of ϕ_b by

$$\phi_b^*(v) := \sup_{u\in\mathbb{R}} (vu - \phi_b(u))\ .$$

If ϕ_b is not superlinear, then there may exist $v \in \mathbb{R}$ with $\phi_b^*(v) = +\infty$. However, $\phi_b^*(b(u)) = b(u)u - \phi_b(u) < +\infty$ is generally valid, and for every $\delta > 0$ there exists a constant $C_\delta < +\infty$ such that $|b(u)| \leq \delta\phi_b^*(b(u)) + C_\delta$.

In this context, a measurable function u on Ω is said to have finite energy, if $\phi_b^*(b(u))$ is integrable over Ω. Consequently, $b(u) \in L^1(\Omega)$ holds for functions u with finite energy by the former inequality. Functions with finite energy play an important role in the theory of doubly nonlinear reaction-diffusion equations (1.1), as the energy identity

$$\frac{d}{dt}\int_\Omega \phi_b^*(b(u))\,dx = \langle\frac{\partial}{\partial t}b(u), u\rangle$$

(which generalizes the energy identity $\frac{d}{dt}\frac{1}{2}\|u\|_H^2 = \langle\frac{\partial u}{\partial t}, u\rangle$ on a Hilbert space H) is valid for distributions $\frac{\partial}{\partial t}b(u)$ acting on u. Thus, for initial values with finite energy a priori estimates of weak solutions can be obtained by testing (1.1) with u, provided that a satisfies appropriate conditions. With a parameter $1 < p < +\infty$ such conditions on $a = a(v, \xi)$ read as

(A1) $a : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, $(v, \xi) \mapsto a(v, \xi)$, is continuous,

(A2) a satisfies the growth condition

$$|a(v, \xi)| \leq C_1|\xi|^{p-1} + C_2\phi_b^*(v)^{1/p'} + C_3 \tag{2.1}$$

with constants $C_1, C_2, C_3 < +\infty$,

(A3) a satisfies the semicoercivity condition

$$a(v, \xi)\cdot\xi \geq c_1|\xi|^p - C_2|\xi| - C_3\phi_b^*(v) \tag{2.2}$$

with constants $c_1 > 0$, $C_2, C_3 < +\infty$, (A4) a is monotone in the main part, i.e.

$$(a(v,\xi) - a(v,\tilde{\xi})) \cdot (\xi - \tilde{\xi}) \geq 0\,. \tag{2.3}$$

Under these conditions on a, b, for an inhomogeneity $f \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ the following notion of a weak solution u of (1.1) under Dirichlet boundary conditions to initial values u_0 with finite energy is appropriate, see [1, 1.4]. Note that if $\partial\Omega$ is C^1, then $u(t) \in W_0^{1,p}$ implies $u(t) = 0$ on $\partial\Omega$ in the sense of traces.

Definition 1. A function $u \in L^p(0,T;W_0^{1,p}(\Omega))$ is called a weak solution of (1.1) to the initial value u_0 with finite energy $\phi_b^*(b(u_0)) \in L^1(\Omega)$, if $\phi_b^*(b(u)) \in L^\infty(0,T;L^1(\Omega))$, if $b(u) \in L^\infty(0,T;L^1(\Omega))$ has a weak derivative $\frac{\partial}{\partial t}b(u) \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and the initial value $b(u_0) \in L^1(\Omega)$, and if (1.1) holds as an equation in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Moreover, if $f = f(b(u))$ is a nonlinearity, let us assume that

(F1) $f : \mathbb{R} \to \mathbb{R}$, $v \mapsto f(v)$, is continuous,

(F2) f satisfies the growth condition

$$|f(v)| \leq C_1 \phi_b^*(v)^{1/p'} + C_2 \tag{2.4}$$

with constants $C_1, C_2 < +\infty$.

Example 1. If $b(u) = |u|^{m-2}u$ for $1 < m < \infty$, then $\phi_b(u) = \frac{1}{m}|u|^m$, $\phi_b^*(v) = \frac{1}{m'}|v|^{m'}$ and $\phi_b^*(b(u)) = \frac{1}{m'}|u|^m$. Thus, a function u has finite energy iff $u \in L^m(\Omega)$. As the superposition operator associated with b maps $L^m(\Omega)$ continuously into $L^{m'}(\Omega)$, here even $b(u) \in L^\infty(0,T;L^{m'}(\Omega))$ holds for a weak solution u. Particularly, for inhomogeneities f in the space $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(0,T;L^{m'}(\Omega))$ under the condition $m < p^*$ existence of weak solutions can be shown, where in Definition 1 more generally existence of $\frac{\partial}{\partial t}b(u)$ and validity of (1.1) in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(0,T;L^{m'}(\Omega))$ is allowed, see [9]. Moreover, with the parameter $p^{\circledast} := p(1 + \frac{m}{n})$ [1] the growth condition (F2) for a nonlinearity f can be replaced by the more general condition $|f(v)| \leq C_1\phi_b^*(v)^{1/q'} + C_2$ with a parameter $1 < q < p^{\circledast}$. Then, at least short-time existence of weak solutions satisfying (1.1) as an equation in $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^{q'}(0,T;L^{q'}(\Omega))$ can be shown, and long-time existence holds for $q < \max(m,p)$.

Remark 1. Note that there exist various extensions of the mentioned conditions, under which existence of weak solutions of (1.1) with finite energy can be studied. For example, convection terms not restricted by a growth condition could be allowed, or b could be a maximal monotone graph such that b^{-1} is continuous, see [3]. However, as the focus of this article lies on the weak comparison principle, such extensions are not discussed here.

§3. The weak comparison principle

Like [11] we can prove the validity of the weak comparison principle only under a more restrictive monotonicity assumptions than (A4), but we are able to avoid the assumption

[1] The parameter $p^{\circledast}$ is the optimal parameter such that $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^m(\Omega))$ is continuously embedded into $L^{p^{\circledast}}(0,T,L^{p^{\circledast}}(\Omega))$, and $p^{\circledast}$ coincides for $m = 2$ with the parameter defined in [13, (8.116)].

of uniform monotonicity (4.1). For the proof of theorem 2, let us introduce the following notation, see [2, Definition 2.1]

Definition 2. For $\eta \in C^1(\mathbb{R})$ the primitive b_η of η' w.r.t. b is defined up to a constant by

$$b_\eta(u) := \int^u \eta'(\tilde{u})\, db(\tilde{u}) \quad \left(= \int^{b(u)} \eta'(b^{-1}(\tilde{v}))\, d\tilde{v} \quad \text{if } b^{-1} \text{ is continuous}\right).$$

In [11] the function η is called an entropy and b_η is called the corresponding entropy flux. Note that $b_{\mathrm{Id}} = b$, but instead of the identity usually functions η with a fast decreasing derivative are used to cut off b appropriately. A basic tool is the following integration by parts formula for weakly differentiable functions, which originates from the work of Alt-Luckhaus [1] and Mignot-Bamberger (see [6, p.31]). For differentiable functions $t \mapsto b_{\eta(u(t)-\tilde{u})}$ and $t \mapsto b(u(t))$ this integration by parts formula simply follows from $\frac{\partial}{\partial t} b_{\eta(\cdot-\tilde{u})}(u) = \eta'(u(t)-\tilde{u})\frac{\partial b(u)}{\partial t}$.

Lemma 1. *Let $b : \mathbb{R} \to \mathbb{R}$ be continuous non-decreasing monotone. Assume that $u \in L^p(0,T;W_0^{1,p}(\Omega))$ and the measurable function u_0 are such that $b(u) \in L^\infty(0,T;L^1(\Omega))$ has a weak derivative $\frac{\partial b(u)}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(0,T;L^1(\Omega))$ and an initial value $b(u_0) \in L^1(\Omega)$. Then,*

$$\int_0^T \langle \frac{\partial b(u)}{\partial t}(t), \eta'(u(t)-\tilde{u})\phi(t)\rangle\, dt = -\int_0^T \int_\Omega (b_{\eta(\cdot-\tilde{u})}(u(t)) - b_{\eta(\cdot-\tilde{u})}(u_0))\frac{\partial \phi}{\partial t}\, dx\, dt \tag{3.1}$$

is valid for every $\eta \in C^{1,1}(\mathbb{R})$, every fixed $\tilde{u} \in W^{1,p}(\Omega)$ and every $\phi \in W^{1,\infty}((0,T)\times\Omega)$ with $\phi(T) = 0$ such that $\eta'(u-\tilde{u})\phi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))$.

Remark 2. In [11, Lemma 1] a variant of Lemma 1 for sub- and super-solutions is formulated, and in [3, Lemma 2.1] a variant for monotone graphs b is shown.

In the following main theorem of this article a signed L^1-contraction principle is shown for weak solutions of (1.1) under conditions, which unify and slightly generalize the conditions of [11, 4, 8]. The proof of this contraction principle is very close to the proof in [11], and the weak comparison principle more or less directly follows.

Theorem 2. *Assume that b is continuous non-decreasing monotone, and suppose that a satisfies beneath (A1)-(A3) instead of (A4) the stronger condition*

$$(a(v,\xi) - a(\tilde{v},\tilde{\xi}))\cdot(\xi-\tilde{\xi}) \ge -C(1+|\xi|^p+|\tilde{\xi}|^p+\phi_b^*(v)+\phi_b^*(\tilde{v}))|u-\tilde{u}| \tag{3.2}$$

with $v = b(u)$, $\tilde{v} = b(\tilde{u})$. Then, if $u, \tilde{u}$ are weak solutions of (1.1) to initial values $u_0, \tilde{u}_0$ with finite energy and inhomogeneities $f, \tilde{f}$ with $f, \tilde{f} \in L^1(0,T;L^1(\Omega))$, the inequality

$$\int_\Omega (b(u(t)) - b(\tilde{u}(t)))^+\, dx \le \int_\Omega (b(u_0) - b(\tilde{u}_0))^+\, dx + \int_0^t \int_\Omega (f-\tilde{f})^+\, dx\, ds \tag{3.3}$$

is valid for a.e. $t \in (0,T)$. If instead of an inhomogeneity the right hand side of (1.1) is a nonlinearity $f = f(b(u))$, which does not only satisfy (F1)-(F2) but also a one-sided Lipschitz condition $(f(v) - f(\tilde{v}))(v-\tilde{v}) \le L|v-\tilde{v}|^2$, then for a.e. $t \in (0,T)$

$$\int_\Omega (b(u(t)) - b(\tilde{u}(t)))^+\, dx \le e^{tL}\int_\Omega (b(u_0) - b(\tilde{u}_0))^+\, dx\,. \tag{3.4}$$

Proof. We follow the proof of [11] and just mention the necessary modifications. Let $\eta' = \eta_k'$ be a smooth non-decreasing monotone approximation of $u \mapsto \operatorname{sign}^+(u) := \max(\operatorname{sign}(u), 0)$. Double the time variable (see [7]) and apply Lemma 1, then as in [11, (39)] we end up with

$$\begin{aligned}
&-\int_0^T\int_0^T\int_\Omega ((b_{\eta(\cdot-\tilde u(s))}(u(t)) - b_{\eta(\cdot-\tilde u(s))}(u_0))\frac{\partial\psi}{\partial t}\,dx\,ds\,dt\\
&-\int_0^T\int_0^T\int_\Omega (b_{\eta(u(t)-\cdot)}(\tilde u(s)) - b_{\eta(u(t)-\cdot)}(\tilde u_0))\frac{\partial\psi}{\partial s})\,dx\,ds\,dt\\
&+\int_0^T\int_0^T\int_\Omega (a(\nabla u) - a(\nabla\tilde u))\cdot\nabla(\eta'(u(t)-\tilde u(s))\psi(t,s))\,dx\,ds\,dt\\
=&\int_0^T\int_0^T\int_\Omega (f-\tilde f)\eta'(u(t)-\tilde u(s))\psi(t,s)\,dx\,ds\,dt
\end{aligned} \tag{3.5}$$

for every sufficiently smooth $\psi = \psi(t, s, x)$ with $\psi(T) = 0$. Hereby,

$$\begin{aligned}
&\int_0^T\int_0^T\int_\Omega (a(\nabla u) - a(\nabla\tilde u))\cdot\nabla(\eta'(u(t)-\tilde u(s)\psi(t,s)))\,dx\,ds\,dt\\
=&\int_0^T\int_0^T\int_\Omega \eta'(u(t)-\tilde u(s)))(a(\nabla u) - a(\nabla\tilde u))\cdot\nabla\psi(t,s)\,dx\,ds\,dt\\
&+\int_0^T\int_0^T\int_\Omega \psi(t,s)\eta''(u(t)-\tilde u(s)))(a(\nabla u) - a(\nabla\tilde u))\cdot\nabla(u(t)-\tilde u(s))\,dx\,ds\,dt\,,
\end{aligned}$$

where $\eta'' = \eta_k''$ is non-negative but blows up as $k \to \infty$. However, by (3.2)

$$\begin{aligned}
&(a(b(u),\nabla u) - a(b(\tilde u),\nabla\tilde u))\cdot\nabla(u-\tilde u)\\
\geq& -C(1+|\nabla u|^p + |\nabla\tilde u|^p + \phi_b^*(b(u)) + \phi_b^*(b(\tilde u)))|u-\tilde u|\,,
\end{aligned}$$

thus in the limit $k \to \infty$ for non-negative ψ we obtain due to $\eta''(u(t) - \tilde u(s)))|u(t) - \tilde u(s)| \to 0$ and dominated convergence (see also Remark 3)

$$\begin{aligned}
&-\int_0^T\int_0^T\int_\Omega (((b(u(t)) - b(\tilde u(s)))^+ - (b(u_0) - b(\tilde u(s)))^+)\frac{\partial\psi}{\partial t}\,dx\,ds\,dt\\
&-\int_0^T\int_0^T\int_\Omega ((b(u(t)) - b(\tilde u(s)))^+ - (b(u(t)) - b(\tilde u_0))^+)\frac{\partial\psi}{\partial s})\,dx\,ds\,dt\\
&+\int_0^T\int_0^T\int_\Omega \operatorname{sign}^+(u(t)-\tilde u(s))(a(\nabla u(t)) - a(\nabla\tilde u(s)))\cdot\nabla\psi\,dx\,ds\,dt\\
\leq&\int_0^T\int_0^T\int_\Omega \operatorname{sign}^+(u(t)-\tilde u(s))(f-\tilde f)\psi\,dx\,ds\,dt\,.
\end{aligned} \tag{3.6}$$

Now substitute $\psi(t, s, x) := \frac{1}{\epsilon}\tilde\phi(\frac{t-s}{\epsilon})\phi(\frac{t+s}{2}, x)$ with a non-negative function $\tilde\phi \in C_c^\infty(\mathbb{R})$ of unit mass and a sufficiently smooth non-negative function ϕ with $\phi(T, \cdot) = 0$. Then $\frac{\partial\psi}{\partial t} + \frac{\partial\psi}{\partial s} = \frac{1}{\epsilon}\tilde\phi(\frac{t-s}{\epsilon})\frac{\partial\phi}{\partial t}(\frac{t+s}{2}, x)$, i.e. those parts of the time derivative cancel which become singular as

$\epsilon \to 0$. Therefore, in the limit $\epsilon \to 0$ we obtain

$$\begin{aligned}
&- \int_\Omega (b(u_0) - b(\tilde{u}_0))^+ \phi(0,x)\,dx \\
&- \int_0^T \int_\Omega (b(u(t)) - b(\tilde{u}(t)))^+ \frac{\partial \phi}{\partial t}\,dx\,dt \\
&+ \int_0^T \int_\Omega \operatorname{sign}^+(u(t) - \tilde{u}(t))(a(\nabla u(t)) - a(\nabla \tilde{u}(t))) \cdot \nabla \phi\,dx\,dt \\
\leq& \int_0^T \int_\Omega \operatorname{sign}^+(u(t) - \tilde{u}(t))(f - \tilde{f})\phi\,dx\,dt\,,
\end{aligned} \tag{3.7}$$

provided that the integral in (3.6) involving a and the space derivative $\nabla\psi$ really converges to the corresponding integral in (3.7). In [11, below (43)] this convergence is established by showing that certain integrals containing the terms T_i, $i = 1, \dots, 5$, vanish in the limit $\epsilon \to 0$, but for these convergences already the conditions (A1)-(A3) and (3.2) are sufficient [2]. Finally, in the case of inhomogeneities $f, \tilde{f}$ inequality (3.7) implies for non-negative $\phi = \phi(t)$ independent of $x \in \Omega$ with $\phi(T) = 0$

$$\begin{aligned}
&- \int_0^T \int_\Omega \left((b(u(t)) - b(\tilde{u}(t)))^+ - (b(u_0) - b(\tilde{u}_0))^+\right) \frac{\partial \phi}{\partial t}\,dx\,dt \\
\leq& \int_0^T \int_\Omega (f - \tilde{f})^+ \phi\,dx\,dt\,.
\end{aligned}$$

This expression is equivalent to $\frac{d}{dt}\|b(u) - b(\tilde{u})\|_1 \leq \|f - \tilde{f}\|_1$ (in the sense of scalar distributions) and establishes the signed L^1-contraction principle. In the case of a nonlinearity $f = f(b(u))$, $\tilde{f} = f(b(\tilde{u}))$, and due to $\operatorname{sign}(u - \tilde{u})(f(b(u)) - f(b(\tilde{u}))) = \operatorname{sign}(b(u) - b(\tilde{u}))(f(b(u)) - f(b(\tilde{u})))$ inequality (3.7) and the one-sided Lipschitz condition imply

$$\begin{aligned}
&- \int_0^T \int_\Omega \left((b(u(t)) - b(\tilde{u}(t)))^+ - (b(u_0) - b(\tilde{u}_0))\right) \frac{\partial \phi}{\partial t}\,dx\,dt \\
\leq& L \int_0^T \int_\Omega (b(u(t)) - b(\tilde{u}(t)))^+ \phi\,dx\,dt\,,
\end{aligned}$$

i.e. $\frac{d}{dt}\|b(u) - b(\tilde{u})\|_1 \leq L\|b(u) - b(\tilde{u})\|_1$. Thus, Gronwall's lemma allows to obtain the contraction principle. □

Corollary 3. *If $u, \tilde{u}$ are weak solutions of* (1.1) *to initial values $u_0 \leq \tilde{u}_0$ and inhomogeneities $f \leq \tilde{f}$ resp. a nonlinearity $f = f(b(u))$, then $u \leq \tilde{u}$.*

Proof. The right hand sides of (3.3) resp. (3.4) vanish, thus $b(u) \leq b(\tilde{u})$ and $u \leq \tilde{u}$ a.e. . □

[2] Note that due to continuity of a, b and the growth condition (2.1) the induced superposition operator maps the space $\{u \in W_0^{1,p}(\Omega) \mid \phi_b^*(b(u_0)) \in L^1(\Omega)\}$ continuously into $L^{p'}(\Omega)$, and particularly for $s \to t$ in $L^{p'}(\Omega)$ it holds that

$$\operatorname{sign}^+(u(t) - \tilde{u}(s))(a(b(u(t)), \nabla u(t)) - a(b(\tilde{u}(s)), \nabla \tilde{u}(s))) \to \operatorname{sign}^+(u(t) - \tilde{u}(t))(a(b(u(t)), \nabla u(t)) - a(b(\tilde{u}(t)), \nabla \tilde{u}(t)))\,.$$

Remark 3. In the proof it is important that $\eta_k''(z)|z|$ is uniformly bounded for the smooth approximation η_k of sign^+ and $\eta_k''(z)|z| \to 0$ holds pointwisely as $k \to \infty$, as else dominated convergence can not be applied to conclude that

$$-C \int_0^T \int_0^T \int_\Omega \psi(t,s)\eta_k''(u(t)-\tilde{u}(s)))|u(t)-\tilde{u}(s)|(1+|\nabla u|^p+|\nabla\tilde{u}|^p+\phi_b^*(b(u))+\phi_b^*(b(\tilde{u}))) \, dx \, ds \, dt$$

converges to zero. In this sense the Hölder condition (3.2) is optimal for the method of proof, because there does not exist a smooth approximation η_k of sign^+ with $\eta_k''(z)|z|^\alpha \to 0$ for $\alpha < 1$ as $k \to \infty$.

Remark 4. Note that (3.2) implies (A4), i.e. monotonicity in the main part (2.3), because in the case $u = \tilde{u}$ the right hand side of (3.2) vanishes. Thus, monotonicity (2.3) of a in the main part is a weaker requirement than (3.2). Further, note that the terms inside the bracket on the right hand side of (3.2) can be replaced by other terms, as long as a priori estimates can be obtained for these terms. Particularly, let us explicitly point out that the inhomogeneities $f, \tilde{f}$ resp. the nonlinearity $f = f(b(u))$ do not only have to lie in $L^1(0,T;L^1(\Omega))$ resp. have to satisfy a one-sided Lipschitz condition, but additionally the a priori estimates of ∇u in $L^p(0,T;L^p(\Omega))$ and of $\phi_b^*(b(u))$ in $L^\infty(0,T;L^1(\Omega))$ have to be available, which generally is true for inhomogeneities in $L^\infty(0,T;L^\infty(\Omega))$ or in the setting of Example 1 for inhomogeneities in $L^1(0,T;L^{m'}(\Omega))$ resp. for nonlinearities satisfying (F1)-(F2). Finally, observe that (3.2) is very similar to [13, (8.101)] in the case $b(u) = u$ resp. to [10, (7)] in the doubly nonlinear case, only that here the right hand side depends on $u, \tilde{u}$ not via an L^2-term $|u-\tilde{u}|^2$ resp. an L^m-term $(b(u)-b(\tilde{u}))(u-\tilde{u})$, but via the L^1-term $|u-\tilde{u}|$.

§4. The comparison principles of Otto, Díaz and Kurta

Let us show that (3.2) is implied by the conditions of [11], [4] and [8].

1. Otto [11] requires that a is uniformly p-monotone in the main part, i.e.

$$(a(v,\xi) - a(v,\tilde{\xi})) \cdot (\xi - \tilde{\xi}) \geq c|\xi - \tilde{\xi}|^p \tag{4.1}$$

holds for every $x \in \Omega$, $u, v \in \mathbb{R}$ and $\xi, \tilde{\xi} \in \mathbb{R}^n$ with a constant $c > 0$, and Hölder continuous w.r.t. the exponent $1/p'$ in u, i.e.

$$|a(b(u),\xi) - a(b(\tilde{u}),\xi)|^{p'} \leq C(1 + |\xi|^p + \hat{\phi}_b(u) + \hat{\phi}_b(\tilde{u}))|u - \tilde{u}| \tag{4.2}$$

for every $x \in \Omega$, $u, \tilde{u} \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$. These two conditions imply the validity of (3.2) due to

$$\begin{aligned}
&(a(b(u),\xi) - a(b(\tilde{u}),\tilde{\xi})) \cdot (\xi - \tilde{\xi}) \\
&= (a(b(\tilde{u}),\xi) - a(b(\tilde{u}),\tilde{\xi})) \cdot (\xi - \tilde{\xi}) + (a(b(u),\xi) - a(b(\tilde{u}),\xi)) \cdot (\xi - \tilde{\xi}) \\
&\geq c|\xi - \tilde{\xi}|^p - C(1 + |\xi|^p + \hat{\phi}_b(u) + \hat{\phi}_b(\tilde{u}))^{1/p'}|u - \tilde{u}|^{1/p'}|\xi - \tilde{\xi}| \\
&\geq -C(1 + |\xi|^p + \hat{\phi}_b(u) + \hat{\phi}_b(\tilde{u}))|u - \tilde{u}|\,,
\end{aligned}$$

where in the last step Young's inequality has been applied so that $c|\xi - \tilde{\xi}|^p$ cancels.

2. Díaz [4] considers a of the form $a(v,\xi) := \phi(\xi + K(v)e)$ with a fixed vector $e \in \mathbb{R}^n$, $\phi(\xi) := |\xi|^{p-2}\xi$ and a function K such that $K \circ b$ is Hölder continuous w.r.t. the exponent $\frac{1}{p}$ (resp. $\frac{1}{p'}$) in the case $1 < p < 2$ (resp. $p \geq 2$). Under these assumptions (3.2) is valid, because in the case $1 < p < 2$

$$\begin{aligned}
&(a(b(u),\xi) - a(b(\tilde{u}),\tilde{\xi})) \cdot (\xi - \tilde{\xi}) \\
&= (\phi(\xi + K(b(u))e) - \phi(\tilde{\xi} + K(b(\tilde{u}))e)) \cdot ((\xi + K(b(u))e) - (\tilde{\xi} + K(b(\tilde{u}))e)) \\
&\quad - (\phi(\xi + K(b(u))e) - \phi(\tilde{\xi} + K(b(\tilde{u}))e)) \cdot (K(b(u))e - K(b(\tilde{u}))e) \\
&\geq c|\phi(\xi + K(b(u))e) - \phi(\tilde{\xi} + K(b(\tilde{u}))e)|^{p'} \\
&\quad - |\phi(\xi + K(b(u))e) - \phi(\tilde{\xi} + K(b(\tilde{u}))e)| \cdot |K(b(u)) - K(b(\tilde{u}))| \cdot |e| \\
&\geq -C|u - \tilde{u}|\,,
\end{aligned}$$

where $(\phi(\zeta) - \phi(\tilde{\zeta})) \cdot (\zeta - \tilde{\zeta}) \geq c|\phi(\zeta) - \phi(\tilde{\zeta})|^{p'}$ has been applied, and in the last step Young's inequality and Hölder continuity $|K(b(u)) - K(b(\tilde{u}))|^p \leq C|u - \tilde{u}|$ have been used so that $c|\phi(\xi + K(b(u))e) - \phi(\tilde{\xi} + K(b(\tilde{u}))e)|^{p'}$ cancels. In the case $p \geq 2$, use instead $(\phi(\zeta) - \phi(\tilde{\zeta})) \cdot (\zeta - \tilde{\zeta}) \geq c|\zeta - \tilde{\zeta}|^p$, $|\phi(\zeta) - \phi(\tilde{\zeta})| \leq C|\zeta - \tilde{\zeta}|(|\zeta|^p + |\tilde{\zeta}|^p)^{\frac{p-2}{p}}$ and $|K(b(u)) - K(b(\tilde{u}))|^{p'} \leq C|u - \tilde{u}|$ to obtain

$$\begin{aligned}
&(a(b(u),\xi) - a(b(\tilde{u}),\tilde{\xi})) \cdot (\xi - \tilde{\xi}) \\
&\geq -C(|\xi + K(b(u))e|^p + |\tilde{\xi} + K(b(\tilde{u}))e|^p)^{\frac{p-2}{p-1}}|u - \tilde{u}| \\
&\geq -C(1 + |\xi|^p + |\tilde{\xi}|^p + |u|^{p-1} + |\tilde{u}|^{p-1})|u - \tilde{u}|\,,
\end{aligned}$$

which is equally well as condition (3.2), because the a priori bound of ∇u in $L^p(\Omega)$ implies for bounded Ω under Dirichlet boundary conditions an a priori bound of u in $L^{p-1}(\Omega)$.

3. Kurta [8] studies the case where a satisfies

$$(a(v,\xi) - a(\tilde{v},\tilde{\xi})) \cdot (\xi - \tilde{\xi}) \geq c|a(v,\xi) - a(\tilde{v},\tilde{\xi})|^{\alpha'} \tag{4.3}$$

with $\alpha = p$ in the case $1 < p < 2$. We already used this condition during our former discussion of [4], thus also in this case (3.2) is valid. Kurta notes that (4.3) is not only satisfied by the p-Laplacian in the case $1 < p < 2$, but also by the modified p-Laplacian $\operatorname{div}(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i})$. Further, he emphasizes that under the condition (4.3) with $\alpha = p$, $1 < p < 2$, the weak comparison principle is valid for solutions on $\Omega = \mathbb{R}^n$ which merely satisfy local a priori estimates $\nabla u \in L^p(0,T;W^{1,p}_{loc}(\mathbb{R}^n))$, in contrast to the case $p = 2$ where there exists a counter example, see [8, Remark 2].

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