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FROM THE HEAT EQUATION TO THE SOBOLEV EQUATION

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Abstract. In this paper, we consider the theorem of Lions-Tartar in $W(0, T, V, V')$ with different "pivot-spaces" H. In a first part, depending on H, we have a look at the corresponding solved problem. Then, the second energy equality set forth in a second part.

Keywords: Lions-Tartar, pivot space, second energy.

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§1. Introduction

Considering two separable Hilbert spaces V and H, V being continuously embedded in H and dense in H, the interpretation of the equation $du/dt + Au = f$ in V', with initial condition u_0 in H, is under discussion in the situation where one changes the pivot space H in the usual Gelfand-Lions framework.

In [5], J. Simon warns us against the use of the common identification of H with its dual space in the functional frame $V \hookrightarrow H \equiv H' \hookrightarrow V'$. In particular, it is mentioned that if $D(\Omega)^{\dagger}$ is not dense in V, the study is incompatible with the distributional frame for some standard PDE's. Remaining with $D(\Omega)$ dense V, we present in this paper the different type of solved problems by the theorem of Lions-Tartar when one changes the pivot's space. We systematically illustrate our remarks with the rigged Hilbert space $(H^s(\mathbb{R}^d), H^1(\mathbb{R}^d))$ when $s \in [0, 1]$. For example, the equation $du/dt - \Delta u = f$ would correspond to the heat equation $\partial u/\partial t - \Delta u = f$ if $s = 0$. Here, $\partial u/\partial t$ denotes the time derivative of u in the sense of the distribution of $D'(Q)$. It would correspond to the Sobolev equation $(I - \Delta)\partial u/\partial t - \Delta u = f$ if $s = 1$.

In a last part of the paper, we will be interested in the "second energy equality" for the solution to the lemma of Lions-Tartar. More precisely, Theorem 4 asserts that if $u_0 \in V$, $g \in L^2(0, T, H)$ and assuming that the bilinear form a is independent of time, symmetric and coercive, then the corresponding solution u to the lemma of Lions-Tartar belongs to $C([0, T], V)$ and for any $t \in [0, T]$,

$$\int_{]0,t[} \left|\frac{du}{dt}\right|^2 d\sigma + \frac{1}{2}a(u(t), u(t)) = \frac{1}{2}a(u(0), u(0)) + \int_{]0,t[} \left(g(\sigma), \frac{du}{dt}(\sigma)\right) d\sigma.$$

Outlines of the paper

One presents in Section 2 some notations, then, in Section 3, one reminds the reader of the embedding of V in V' when the Riesz-identification $H \equiv H'$ is assumed. In particular, what is

[†] The space of infinitely differentiable functions with a compact support.

the characterization of the image of $H^1(\mathbb{R}^d)$ when $H = H^s(\mathbb{R}^d)$. Then, thanks to this, we will be interested in the sense given to the space $W(0,T) = \{u \in L^2(0,T,V),\ du/dt \in L^2(0,T,V')\}$. We will look more closely to the case $V = H^1(\mathbb{R}^d)$ and $H = H^s(\mathbb{R}^d)$ when $s \in [0,1]$ and to the link with fractional operators.

Section 4 will be devoted to the lemma of Lions-Tartar and Section 5 to the second energy equality. Then, we end this paper with an annex that precise the regularization of Landes, used in the proof of the result of Section 5.

§2. Notations

Let V and H be two separable Hilbert spaces, with norm $\|\cdot\|$ for V, associated with the scalar product $((\cdot,\cdot))$, and norm $|\cdot|$ for H, associated with the scalar product $(\cdot,\cdot)$. Assume moreover that V is continuously embedded in H with a dense injection. Then, the dual space H' is continuously embedded in V' and dense. The norm in V' is denoted by $\|\cdot\|_*$.

$\Omega \subset \mathbb{R}^d$ denotes a regular open set and for any positive T, $Q =]0,T[\times\Omega$.

As usual, $D(A)$ denotes the class of C^∞-derivable functions in a given open set A, with compact support in A and its dual space $D'(A)$ denotes the space of distributions in A.

$\mathcal{S}$ denotes the Schwartz space in $\mathbb{R}^d$ and $\mathcal{S}'$ the tempered distributions.

For any $s \in [0,1]$, $H^s(\mathbb{R}^d)$ denotes the fractional Sobolev space defined, for any s, by $H^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d),\ |\xi|^s\mathcal{F}_x(u) \in L^2(\mathbb{R}^d)\}$, where $\mathcal{F}_x$ is the Fourier transform of variable $x \in \mathbb{R}^d$.

Given $s \in]-d/2,1]$ and $f \in \mathcal{S}$, we recall the fractional operators $(-\Delta)^s f$ as $(-\Delta)^s f = \mathcal{F}_x^{-1}[|\xi|^{2s}\mathcal{F}_x(f)]$ and $(I-\Delta)^s f$ as $(I-\Delta)^s f = \mathcal{F}_x^{-1}[(1+|\xi|^2)^s\mathcal{F}_x(f)]$.

Then, one denotes by $W_{(H,V)}(0,T) = \{u \in L^2(0,T,V),\ du/dt \in L^2(0,T;V')\}$.

§3. The space $W_{(H,V)}(0,T)$

3.1. How to embed V in V'?

In this section, we lay stress on the question: how to embed V in V'? Since V is not *a priori* a space with a finite dimension, there exist many possibilities to identify V with its image in V' when one says that $V \hookrightarrow V'$?

Classically, the rigged Hilbert space (H,V) is considered (or Gelfand-Lions triple):

1. Either $H = V$. Then, thanks to the theorem of Riesz, V is identified with its dual V'. Indeed,
$$J : V \to V',\ u \mapsto Ju \quad \text{such that} \quad Ju : v \in V \mapsto ((u,v))$$
is an isometric mapping.
2. Or, $H \subsetneq V$. Then, H is identified with its dual H' (Riesz's theorem) and V is embedded in V' by "passing through $H \equiv H'$". H is called the pivot-space, or intermediate space. Then,
$$J_H : V \to V',\ u \mapsto J_H u \quad \text{such that} \quad J_H u : v \in V \mapsto (u,v)$$
is an injective mapping.

Remark 1.

1. Note that if $H = V$, then $J_H = J$.
2. If H and $\widetilde{H}$ are two pivot-spaces with $H \subsetneq \widetilde{H}$ and V is densely embedded in H and $\widetilde{H}$, then we get that $J_{\widetilde{H}}(V) \subsetneq J_H(V)$.
3. If $V = H^1(\mathbb{R}^d)$ and $H = L^2(\mathbb{R}^d)$ then, for any $u \in V$, we get that $J^{-1} \circ J_H u = w$ where w is the unique solution in $H^1(\mathbb{R}^d)$ of the problem: $w - \Delta w = (u, \cdot)_{L^2(\mathbb{R}^d)}$.

3.1.1. Fractional Sobolev spaces

Let us recall some basics about $H^s(\mathbb{R}^d)$ from J.-L. Lions *et al.* [2] and L. Tartar [7]. Remind that $\mathcal{S}$ denotes the Schwartz space and $\mathcal{S}'$ the tempered distributions.

Definition 1. Let us denote by $\mathcal{F}_x$ the Fourier transform of variable $x \in \mathbb{R}^d$. Then, for a real number $s \geq 0$, $H^s(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d),\ |\xi|^s \mathcal{F}_x(u) \in L^2(\mathbb{R}^d)\}$, and, for a real number s, $H^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d),\ (1 + |\xi|^2)^{s/2} \mathcal{F}_x(u) \in L^2(\mathbb{R}^d)\}$.

Then,

Lemma 1.

1. *When $s \in \mathbb{N}$, $H^s(\mathbb{R}^d)$ denotes the classical Sobolev space (with $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$).*
2. *$D(\mathbb{R}^d)$ is dense in the Hilbert space $H^s(\mathbb{R}^d)$ for the norm $u \mapsto \left\|[1 + |\xi|^2]^{s/2} \mathcal{F}_x(u)\right\|_{L^2(\mathbb{R}^d)}$.*
3. *If $s \in]0, 1[$, then $u \in H^s(\mathbb{R}^d)$ if and only if $u \in L^2(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}}\, dx\, dy < \infty.$$

For an open set Ω, one could define $H^s(\Omega)$ for $0 < s < 1$ in (at least) three different ways:

1. $u \in L^2(\Omega)$ and $\displaystyle\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}}\, dx\, dy < \infty$.
2. u is the restriction to Ω of an element U in $H^s(\mathbb{R}^d)$.
3. One may define $H^s(\Omega)$ by interpolation $H^s(\Omega) = [H^1(\Omega), L^2(\Omega)]_{1-s,2}$.

For a bounded open set with a Lipschitz boundary, the three definitions give the same space with equivalent norms.

3.1.2. Fractional Laplace operator

Let us now remind some basics on the fractional operators (cf. L. E. Silvestre [3]):

Definition 2. Given $s \in]-d/2, 1]$, and $f \in \mathcal{S}$, we define:

1. $(-\Delta)^s f$ as $(-\Delta)^s f = \mathcal{F}_x^{-1}[|\xi|^{2s} \mathcal{F}_x(f)]$.
2. $(I - \Delta)^s f$ as $(I - \Delta)^s f = \mathcal{F}_x^{-1}[(1 + |\xi|^2)^s \mathcal{F}_x(f)]$.

Clearly, if $s = 1$, then $(-\Delta)^s = -\Delta$; if $s = 0$, then $(-\Delta)^s = Id$; and $(-\Delta)^{s_1} \circ (-\Delta)^{s_2} = (-\Delta)^{s_1+s_2}$, respectively with $I - \Delta$ instead of $-\Delta$.

When $f \in \mathcal{S}$, we can also compute the same operator by using the singular integral

$$(-\Delta)^s f(x) = c_{n,s} PV \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x - y|^{d+2s}}\, dy.$$

Let us remark also that for any $f, g \in \mathcal{S}$,

$$\begin{aligned}
\int_{\mathbb{R}^d} [(Id - \Delta)^s] f\, g\, dx &= \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \mathcal{F}_x(f) \mathcal{F}_x(g)\, d\xi \\
&= \int_{\mathbb{R}^d} (1 + |\xi|^2)^{s/2} \mathcal{F}_x(f) (1 + |\xi|^2)^{s/2} \mathcal{F}_x(g)\, d\xi \\
&= \int_{\mathbb{R}^d} (I - \Delta)^{s/2} f (I - \Delta)^{s/2} g\, dx,
\end{aligned}$$

which is the scalar product of $H^s(\mathbb{R}^d)$.

3.1.3. Intermediate spaces

If one assumes that $V \hookrightarrow H \equiv H' \hookrightarrow V'$, then (J.-L. Lions *et al.* [2]) there exists an unbounded operator A on V' such that $[D(A), H]_{1/2} = V$, $[V, V']_{1/2} = H$ and $D(A^{1/2}) = V$.

Classically, when $V = H^1(\mathbb{R}^d)$, we consider that $H = L^2(\mathbb{R}^d)$. Therefore, the image of the dual of $H^1(\mathbb{R}^d)$ by the identification $L^2(\mathbb{R}^d)' \equiv L^2(\mathbb{R}^d)$ is $H^{-1}(\mathbb{R}^d)$, the space of "derivatives of order less than one of elements of L^2", and $[H^1(\mathbb{R}^d), H^{-1}(\mathbb{R}^d)]_{1/2} = H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. Moreover, since we have

$$D(\mathbb{R}^d) \hookrightarrow H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \equiv L^2(\mathbb{R}^d)' \hookrightarrow H^{-1}(\mathbb{R}^d) \hookrightarrow D'(\mathbb{R}^d),$$

any element u of $H^1(\mathbb{R}^d)$ is a distribution *via* the identification $L^2(\mathbb{R}^d) \equiv L^2(\mathbb{R}^d)'$, *i.e.*, u is identifiable with the distribution: $\varphi \in D(\mathbb{R}^d) \mapsto \int_{\mathbb{R}^d} u\varphi\, dx$.

Consider now that the pivot-space is $H^s(\mathbb{R}^d)$, for a given $s \in [0, 1]$. Then,

$$D(\mathbb{R}^d) \hookrightarrow H^1(\mathbb{R}^d) \hookrightarrow H^s(\mathbb{R}^d) \equiv H^s(\mathbb{R}^d)' \hookrightarrow H^1(\mathbb{R}^d)' \hookrightarrow D'(\mathbb{R}^d),$$

and an element u of $H^1(\mathbb{R}^d)$ is a distribution *via* the identification $H^s(\mathbb{R}^d) \equiv H^s(\mathbb{R}^d)'$, *i.e.*, u is identifiable with the distribution: $\varphi \in D(\mathbb{R}^d) \mapsto (u, \varphi)_{H^s}$.

Now, the question is: since in this case $[H^1(\mathbb{R}^d), H^1(\mathbb{R}^d)']_{1/2} = H^s(\mathbb{R}^d)$, what is the image in the dual of $H^1(\mathbb{R}^d)$ by the identification $H^s(\mathbb{R}^d)' \equiv H^s(\mathbb{R}^d)$? More precisely, since we have to obtain $[H^1(\mathbb{R}^d), H^1(\mathbb{R}^d)']_{1/2} = H^s(\mathbb{R}^d)$, why can we identify $H^1(\mathbb{R}^d)'$ with $H^{2s-1}(\mathbb{R}^d)$? Indeed, let us denote by

$$\Phi : H^{2s-1}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)';\ w \mapsto \Phi_w$$

where

$$\Phi_w : H^1(\mathbb{R}^d) \to \mathbb{R};\ u \mapsto \int_{\mathbb{R}^d} (1 + |\xi|^2)^s \mathcal{F}_x w\, \mathcal{F}_x u\, d\xi.$$

Clearly, Φ exists and is an injection.

Consider $T \in H^1(\mathbb{R}^d)'$. Since $H^s(\mathbb{R}^d)'$ is dense in $H^1(\mathbb{R}^d)'$, T is the limit of a sequence $(T_n) \subset H^s(\mathbb{R}^d)'$ in $H^1(\mathbb{R}^d)'$. Since $H^s(\mathbb{R}^d)$ is the pivot space, there exists $w_n \in H^s(\mathbb{R}^d)$ such that

$$\forall v \in H^1(\mathbb{R}^d),\ \langle T_n, v\rangle = (w_n, v)_{H^s} = \int_{\mathbb{R}^d} (1+|\xi|^2)^s \mathcal{F}_x w_n \mathcal{F}_x v\, d\xi.$$

Since $H^1(\mathbb{R}^d) = \left\{u \in \mathcal{S}',\ \int_{\mathbb{R}^d}(1+|\xi|^2)|\mathcal{F}_x v|^2\, d\xi < +\infty\right\}$ and $\|v\|^2_{H^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(1+|\xi|^2)|\mathcal{F}_x|^2 v\, d\xi$,

$$\|T_n\|_{H^1(\mathbb{R}^d)'} = \sup_{v \in H^1(\mathbb{R}^d)\setminus\{0\}} \frac{\int_{\mathbb{R}^d}(1+|\xi|^2)^{s-1/2}\mathcal{F}_x w_n (1+|\xi|^2)^{1/2}\mathcal{F}_x v\, d\xi}{\|v\|_{H^1(\mathbb{R}^d)}} = \|w_n\|_{H^{2s-1}(\mathbb{R}^d)}.$$

Then, the result holds by passing to the limit and Φ is an isometry.

3.2. Time derivation

Consider a positive real number T and assume that $u \in L^2(0,T,V)$. Then, u is said to belong to $W_{(H,V)}(0,T)$ if $u \in L^2(0,T,V)$, $du/dt \in L^2(0,T;V')$ and $V \hookrightarrow H \equiv H' \hookrightarrow V'$. Then, in this section, we wish to discuss about the sense given to $du/dt \in L^2(0,T;V')$ (cf. J. Simon [4]).

1. On the one hand, one can consider u as an element of $D'(0,T;V)$, the V-valued distributions. Thus, du/dt, the time derivative of u in the sense of $D'(0,T;V)$, exists and

$$\forall \varphi \in D(0,T),\ \frac{du}{dt}(\varphi) = -\int_0^T u(t)\varphi'(t)\, dt \text{ in } V.$$

Then, by using $J_H : V \hookrightarrow H \equiv H' \hookrightarrow V'$, we have that

$$\forall \varphi \in D(0,T),\ \forall v \in V,\ \left\langle J_H\left[\frac{du}{dt}(\varphi)\right], v\right\rangle = -\left(\int_0^T u(t)\varphi'(t)\, dt, v\right).$$

Since $u\varphi' \in L^1(0,T,V)$ and $T : V \to \mathbb{R}, u \mapsto (u,v)$ is a linear and continuous mapping, we get that $\left(\int_0^T u(t)\varphi'(t)\, dt, v\right) = \int_0^T \varphi'(t)(u(t),v)\, dt$. Note that this result is an obvious fact for simple functions u, then for any u by passing to the limit. Therefore, for all $\varphi \in D(0,T)$ and $v \in V$,

$$\left\langle J_H\left[\frac{du}{dt}(\varphi)\right], v\right\rangle = -\int_0^T \varphi'(t)(u(t),v)\, dt = \left\langle \frac{d}{dt}(u(t),v), \varphi\right\rangle_{D'(0,T),D(0,T)}.$$

2. On the other hand, one can consider that $J_H(u)$ is then an element of $L^2(0,T,V')$, thus an element of $D'(0,T;V')$, the V'-valued distributions. Therefore, dJ_Hu/dt, the time derivative of J_Hu in the sense of $D'(0,T;V')$, exists and

$$\forall \varphi \in D(0,T),\ \frac{dJ_Hu}{dt}(\varphi) = -\int_0^T J_Hu(t)\varphi'(t)\, dt \text{ in } V',$$

i.e.

$$\forall \varphi \in D(0,T),\ \forall v \in V,\ \left\langle \frac{dJ_Hu}{dt}(\varphi), v\right\rangle = -\left\langle \int_0^T J_Hu(t)\varphi'(t)\, dt, v\right\rangle.$$

Since $J_H u\varphi' \in L^1(0,T,V')$ and $T : V' \to \mathbb{R}$, $f \mapsto \langle f, v\rangle$ is a linear and continuous mapping, we get also that $\left\langle \int_0^T J_H u(t)\varphi'(t)\,dt, v\right\rangle = \int_0^T \varphi'(t)\,\langle u(t), v\rangle\,dt$. Therefore, for all $\varphi \in D(0,T)$ and $v \in V$,

$$\left\langle \frac{dJ_H u}{dt}(\varphi), v\right\rangle = -\int_0^T \varphi'(t)\,\langle J_H u(t), v\rangle\,dt$$
$$= \left\langle \frac{d}{dt}\,\langle J_H u(t), v\rangle\,, \varphi\right\rangle_{D'(0,T),D(0,T)} = \left\langle \frac{d}{dt}(u(t), v), \varphi\right\rangle_{D'(0,T),D(0,T)}.$$

Thus, $J_H \circ \dfrac{du}{dt} = \dfrac{dJ_H u}{dt}$.

Assume, for example, that $V = H^1(\mathbb{R}^d)$ and $H = H^s(\mathbb{R}^d)$ with $s \in [0,1]$. Then, for any $v \in D(\mathbb{R}^d)$ and any $\varphi \in D(0,T)$,

$$\left\langle \frac{dJ_H u}{dt}(\varphi), v\right\rangle = -\int_0^T \varphi'(t)(u(t), v)_{H^s(\mathbb{R}^d)}\,dt = \left\langle \frac{d}{dt}(u(t), v)_{H^s(\mathbb{R}^d)}, \varphi\right\rangle_{D'(0,T),D(0,T)}.$$

1. Assume that $s = 0$. Then,

$$\left\langle \frac{dJ_{L^2(\mathbb{R}^d)} u}{dt}(\varphi), v\right\rangle = -\int_0^T \varphi'(t)\int_{\mathbb{R}^d} uv\,dx\,dt = \left\langle \frac{\partial u}{\partial t}, \varphi \otimes v\right\rangle_{D'(Q),D(Q)},$$

where $\partial u/\partial t$ denotes the time derivative of u in the sense of the distribution of $D'(Q)$ where $Q =]0,T[\,\times \mathbb{R}^d$ with the classical identification $L^2 \equiv (L^2)'$.

2. Assume that $s = 1$. Then, up eventually to a constant due to the Fourier transform,

$$\left\langle \frac{dJ_{H^1(\mathbb{R}^d)} u}{dt}(\varphi), v\right\rangle = -\int_0^T \varphi'(t)\int_{\mathbb{R}^d} (uv + \nabla u \nabla v)\,dx\,dt = \left\langle \frac{\partial u}{\partial t} - \Delta\frac{\partial u}{\partial t}, \varphi \otimes v\right\rangle_{D'(Q),D(Q)},$$

where the derivations are in the sense of the distribution of $D'(Q)$ with the classical identification $L^2 \equiv (L^2)'$.

3. Assume that $s \in]0,1[$. Then,

$$\left\langle \frac{dJ_{H^s(\mathbb{R}^d)} u}{dt}(\varphi), v\right\rangle = -\int_0^T \varphi'(t)\,\langle (I-\Delta)^s u, v\rangle_{(H^s)',H^s}\,dt$$

and

$$\frac{dJ_{H^s(\mathbb{R}^d)} u}{dt} = (I-\Delta)^s \frac{du}{dt},$$

where du/dt is understood in the sense of $D'(0,T;H^s(\mathbb{R}^d))$.

§4. Lemma of Lions-Tartar

Lemma 2 (J.-L. Lions [1], J. Simon [4, 5] and L. Tartar [6]). *Let $a \in L^\infty(0,T,\mathcal{L}(V,V'))$ such that*

$$\exists\alpha > 0,\ \beta \in \mathbb{R},\ \textit{for which},\ \forall u \in V,\ a(u,u) \geq \alpha\|u\|^2 - \beta|u|^2.$$

Given $u_0 \in H$, $f_1 \in L^1(0,T;H')$ and $f_2 \in L^2(0,T;V')$, there exists a unique $u \in C([0,T];H) \cap L^2(0,T;V)$, solution, for any $v \in V$ and t a.e. in $]0,T[$, of

$$\begin{cases} \dfrac{d}{dt}(u,v) + \langle a(\cdot,u),v\rangle_{V',V} = \langle f_1,v\rangle_{H',H} + \langle f_2,v\rangle_{V',V}\,, \\ u(0) = u_0, \end{cases} \tag{1}$$

and the bilinear application $(f_1+f_2,u_0) \mapsto u$ is continuous from $(L^2(0,T;V')+L^1(0,T;H'))\times H$ to $L^2(0,T;V) \cap C([0,T];H)$. Moreover,

$$\frac{dJ_H u}{dt} \in L^1(0,T;H') + L^2(0,T;V')$$

and the first energy equality holds

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \langle a(\cdot,u),u\rangle_{V',V} = \langle f_1,u\rangle_{H',H} + \langle f_2,u\rangle_{V',V}\,.$$

Lemma 3 (J. Simon [4, 5])**.** *With the same hypothesis than the previous lemma, unless $a \in L^2(0,T,\mathcal{L}(V,V'))$ (instead of L^∞), there exists a unique u in $L^2(0,T;V) \cap L^\infty(0,T;H) \cap C_w([0,T];H)$ solution of* (1)*. Moreover,*

$$\frac{dJ_H u}{dt} \in L^1(0,T;V').$$

Remark 2.

1. J.-L. Lions considered $f \in L^2(0,T;V')$ which gives $dJ_H u/dt \in L^2(0,T;V')$, *i.e.*, $u \in W_{(H,V)(0,T)}$.
2. Assume for example that $V = H^1(\mathbb{R}^d)$, $H = H^s(\mathbb{R}^d)$ with $s \in [0,1]$, that $\langle a(\cdot,u),v\rangle_{V',V} = \int_{\mathbb{R}^d} \nabla u.\nabla v\,dx$ and denote by u_s the solution of Lions-Tartar's lemma. Then, if $s=0$, u_s is the solution of the heat equation; if $s=1$, u_s is the solution of the pseudoparabolic Sobolev equation; else, u_s is the solution of intermediate evolution problems, hard to characterize in term of PDE's since $(I-\Delta)^s$ is a non local fractional operator.

§5. Second energy equality

Theorem 4. *Consider $T>0$, $Q =]0,T[\times\Omega$, $u_0 \in V$, $g \in L^2(0,T,H)$ and u the solution of the lemma of Lions-Tartar. If a is independent of time, symmetric and coercive (i.e. $\beta = 0$) bilinear form, then $u \in H^1(0,T;H) \cap C_w([0,T],V)$. Moreover, $u \in C([0,T],V)$ and for any $t \in [0,T]$,*

$$\int_{]0,t[}\left|\frac{du}{dt}\right|^2 d\sigma + \frac{1}{2}a\,(u(t),u(t)) = \frac{1}{2}a\,(u(0),u(0)) + \int_{]0,t[}\Big(g(\sigma),\frac{du}{dt}(\sigma)\Big)\,d\sigma. \tag{2}$$

Proof. Since u is a mild solution, *i.e.* obtained by an implicit time-discretization scheme, it is a classic exercise to prove that $u \in H^1(0,T;H) \cap L^\infty(0,T;V)$. Then,

$$u \in C([0,T];H) \cap L^\infty(0,T;V) = C_w([0,T];V)$$

(cf. [2]). Moreover, since $u \in H^1(0,T;H)$, the time differentiation is understood in the space H, without any embeddings. Then, we will denote it by du/dt.

Let us fix $s \in [0,T[$ and for any positive ϵ, denote by v_ϵ the solution of the differential equation (see section 6 for further informations)

$$\epsilon\frac{dv_\epsilon}{dt} + v_\epsilon = u, \text{ for } t > s, \quad \text{with } v_\epsilon(s,.) = u(s).$$

Then, testing the evolution equation with $u - v_\epsilon$ leads us to

$$\epsilon\int_{]s,t[}\Big(\frac{du}{dt},\frac{dv_\epsilon}{dt}\Big)d\sigma + \int_{]s,t[} a(u,u-v_\epsilon)\,d\sigma = \epsilon\int_{]s,t[}\Big(g,\frac{dv_\epsilon}{dt}\Big)d\sigma.$$

Thus, by monotonicity of a,

$$\epsilon\int_{]s,t[}\Big(\frac{du}{dt},\frac{dv_\epsilon}{dt}\Big)d\sigma + \int_{]s,t[} a(v_\epsilon,u-v_\epsilon)\,d\sigma \leq \epsilon\int_{]s,t[}\Big(g,\frac{dv_\epsilon}{dt}\Big)d\sigma,$$

i.e., by using the differential equation, we get

$$\epsilon\int_{]s,t[}\Big(\frac{du}{dt},\frac{dv_\epsilon}{dt}\Big)d\sigma + \epsilon\int_{]s,t[} a\Big(v_\epsilon,\frac{dv_\epsilon}{dt}\Big)d\sigma \leq \epsilon\int_{]s,t[}\Big(g,\frac{dv_\epsilon}{dt}\Big)d\sigma,$$

and, by integration,

$$\int_{]s,t[}\Big(\frac{du}{dt},\frac{dv_\epsilon}{dt}\Big)d\sigma + \frac{1}{2}a\left(v_\epsilon(t),v_\epsilon(t)\right) \leq \int_{]s,t[}\Big(g,\frac{dv_\epsilon}{dt}\Big)d\sigma + \frac{1}{2}a(u(s),u(s)). \tag{3}$$

Since by construction (see annex) v_ϵ converges to u in $H^1(s,T;H)\cap L^2(s,T;V)$ and, for any t, $v_\epsilon(t)$ converges weakly to $u(t)$ in V,

$$\int_{]s,t[}\Big|\frac{du}{dt}\Big|^2 d\sigma + \frac{1}{2}a(u(t),u(t)) \leq \int_{]s,t[}\Big(g,\frac{du}{dt}\Big)d\sigma + \frac{1}{2}a(u(s),u(s)).$$

Moreover, $u \in C_w([0,T],V)$ and $\limsup_{t\to s^+} a(u(t),u(t)) \leq a(u(s),u(s))$. Then, u is continuous from the right from $[0,T[$ to V.

Consider now $0 < t < t+\Delta t \leq T$. Then,

$$\int_0^t\Big(\frac{du}{dt},\frac{u(\sigma+\Delta t)-u(\sigma)}{\Delta t}\Big)d\sigma + \int_0^t a\Big(u(\sigma),\frac{u(\sigma+\Delta t)-u(\sigma)}{\Delta t}\Big)d\sigma$$
$$= \int_0^t\Big(g(\sigma),\frac{u(\sigma+\Delta t)-u(\sigma)}{\Delta t}\Big)d\sigma.$$

Thus,

$$\int_0^t\Big(\frac{du}{dt}(\sigma),\frac{u(\sigma+\Delta t)-u(\sigma)}{\Delta t}\Big)d\sigma + \frac{1}{2\Delta t}\int_t^{t+\Delta t} a(u(\sigma),u(\sigma))\,d\sigma$$
$$\geq \frac{1}{2\Delta t}\int_0^{\Delta t} a(u(\sigma),u(\sigma))\,d\sigma + \int_0^t\Big(g(\sigma),\frac{u(\sigma+\Delta t)-u(\sigma)}{\Delta t}\Big)d\sigma.$$

Therefore, the above remark yields

$$\int_0^t \left|\frac{du}{dt}(\sigma)\right|^2 d\sigma + \frac{1}{2}a(u(t),u(t)) \geq \frac{1}{2}a(u_0,u_0) + \int_0^t \left(g(\sigma), \frac{du}{dt}\right) d\sigma.$$

Adding this to (3) with $s = 0$, we get (2) for any $t \in [0,T[$. Then, $u \in C_w([0,T],V)$ and $\lim_{t\to s} a(u(t),u(t)) = a(u(s),u(s))$ yield $u \in C([0,T[,V)$. We conclude the proof by remarking that the same result holds for time $T+1$ instead of T. □

Corollary 5. *The same result holds if $\beta \neq 0$.*

Proof. If u is a solution, then it is also the solution, for any $v \in V$ and t a.e. in $]0,T[$, of

$$\frac{d}{dt}(u,v) + a(u,v) + \beta(u,v) = (g + \beta u, v), \quad \text{with } u(0) = u_0. \tag{4}$$

Then, the result is just a consequence of the theorem. □

§6. Annex

Let us fix $s \in [0,T[$ and, for any positive ϵ, denote by v_ϵ the solution of the differential equation

$$\epsilon\frac{dv_\epsilon}{dt} + v_\epsilon = u, \text{ for } t > s, \quad \text{with } v_\epsilon(s,\cdot) = u(s), \tag{5}$$

where $u \in H^1(s,T,H) \cap C_w([s,T],V)$.

Lemma 6. *As ϵ goes to 0^+, v_ϵ converges to u in $H^1(s,T;H) \cap L^2(s,T;V)$ and $v_\epsilon(t)$ converges weakly to $u(t)$ in V, for any t.*

Proof. If v_ϵ is the solution of (5), then,

$$v_\epsilon(t) = u(s)e^{(s-t)/\epsilon} + \int_s^t \frac{u(\sigma)}{\epsilon}e^{(\sigma-t)/\epsilon}\, d\sigma$$

and $v_\epsilon(t)$ is bounded in V, independently of t. Thus, by "multiplying in V" equation (5) by v_ϵ, we get that

$$\epsilon\frac{d}{dt}\|v_\epsilon\|^2 + \|v_\epsilon\|^2 \leq \|u\|^2,$$

i.e.

$$\epsilon\|v_\epsilon(t)\|^2 + \int_s^t \|v_\epsilon\|^2\, d\sigma \leq \int_s^t \|u\|^2\, d\sigma + \epsilon\|u(s)\|^2. \tag{6}$$

Moreover, dv_ϵ/dt satisfies

$$\epsilon\frac{d^2v_\epsilon}{dt^2} + \frac{dv_\epsilon}{dt} = \frac{du}{dt}, \text{ for } t > s, \quad \text{with } \frac{dv_\epsilon}{dt}(s) = 0, \tag{7}$$

where $du/dt \in L^2(s,T,H)$. Thus, by "multiplying in H" the above equation by dv_ϵ/dt, we get that

$$\epsilon\frac{d}{dt}\left|\frac{dv_\epsilon}{dt}\right|^2 + \left|\frac{dv_\epsilon}{dt}\right|^2 \leq \left|\frac{du}{dt}\right|^2,$$

i.e.

$$\epsilon\left|\frac{dv_\epsilon}{dt}(t)\right|^2 + \int_s^t \left|\frac{dv_\epsilon}{dt}\right|^2 d\sigma \le \int_s^t \left|\frac{du}{dt}\right|^2 d\sigma. \tag{8}$$

As a first conclusion, there exists a positive constant C such that

$$\left|\frac{dv_\epsilon}{dt}\right|_{L^2(s,T,H)} \le C; \quad \forall t,\ \sqrt{\epsilon}\left|\frac{dv_\epsilon}{dt}(t)\right| \le C, \quad |v_\epsilon(t) - u(t)| \le C\sqrt{\epsilon},$$

and v_ϵ converges weakly to u in $H^1(s, T, H)$ and strongly in $C([s, T], H)$.

Adding that $v_\epsilon(t)$ is bounded in V for any t, $v_\epsilon(t)$ converges weakly to $u(t)$ in V for any t and v_ϵ converges weakly to u in $L^2(s, T, V)$ (note that u is the only possible limit-point).

Then, on the one hand, (6) yields

$$\limsup_{\epsilon\to 0^+} \int_s^t \|v_\epsilon\|^2\ d\sigma \le \int_s^t \|u\|^2\ d\sigma$$

and v_ϵ converges to u in $L^2(s, T, V)$. On the other hand, (8) yields

$$\limsup_{\epsilon\to 0^+} \int_s^t \left|\frac{dv_\epsilon}{dt}\right|^2\ d\sigma \le \int_s^t \left|\frac{du}{dt}\right|^2\ d\sigma$$

and v_ϵ converges to u in $H^1(s, T, H)$. □

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