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A REMARK ABOUT SYMMETRY OF SOLUTIONS TO SINGULAR EQUATIONS AND APPLICATIONS

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Abstract. In this article we will use the moving plane method to discuss the symmetry of solution to an elliptic equation with singularity. Moreover by choosing a particular type of nonlinearity we will show some a priori estimates with the help of moving plane method.

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§1. Introduction

Suppose that Ω is a bounded domain in $\mathbb{R}^n$. Consider the equation

$$\begin{aligned} -\Delta u &= \frac{1}{u^\delta} + f(u) \ \text{ in } \Omega, \\ u = 0 \ &\text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega, \end{aligned}$$

where $\delta > 0$ given and f is a locally lipchitz in $\mathbb{R}$. Extensive studies have been done on this equation in the past by many authors [1], [2], [5], [10], [12] and [13]. This kind of problem arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids as well as chemical heterogeneous chemical reactions.

In a famous paper [2] it was proved that equations of this kind admits a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$. Moreover there exists positive constants R and Q s.t

$$Rp(d(x)) \leq u(x) \leq Qp(d(x))$$

near $\partial\Omega$, where $d(x) = \text{dist}(x, \partial\Omega)$ and $p \in C([0, a]) \cap C^2((0, a])$ is the local solution of the problem

$$-p^{''} = g(p(s)), \quad p(s) > 0, \quad 0 < s < a, \quad p(0) = 0,$$

where $a > 0$ and g is a monotone decreasing continuous function.

In another famous paper [7] it was proved by the help of the moving plane method that if $u \in C(\overline{B}) \cap C^2(B)$ is a positive solution of

$$\begin{aligned} \Delta u + f(u) &= 0 \ \text{ in } B \\ u &= 0 \ \text{ on } \partial B \end{aligned}$$

where B is the unit ball and f is a locally lipchitz in $\mathbb{R}$. Then u is radially symmetric in B and $\frac{\partial u}{\partial r}(x) < 0$.

The original proof requires that solutions be C^2 up to the boundary. The main feature of our paper is to find the symmetry of the solution to the problem with singularity without any assumptions on the smoothness of the solutions up to the boundary. We also prove the existence of universal bounds for superlinear and singular problems following the idea of [9].

§2. Main results and preliminaries

Our main result is the following:

Theorem 1. *Suppose that Ω is a bounded domain which is convex in x_1 direction and symmetric with respect to the plane $x_1 = 0$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a positive solution of*

$$\begin{aligned} &\Delta u + \frac{1}{u^\delta} + f(u) = 0 \ \ in\ \Omega \\ &u = 0 \ \ on\ \partial\Omega,\ \ u > 0\ in\ \Omega \end{aligned}$$

where $\delta > 0$ given and f is a locally lipchitz in $\mathbb{R}$. Then u is symmetric w.r.t x_1 and $D_{x_1}(x) < 0$ for any $x \in \Omega$ with $x_1 > 0$.

To proof the main theorem we need preliminary which we are going to state now. Let Ω be a bounded domain in $\mathbb{R}^n$. Consider the operator L in Ω

$$Lu = \sum_{i,j}^{n} a_{ij}(x)D_{ij}(x)u + \sum_{i}^{n} b_i(x)D_i u + c(x)u$$

for $u \in C^2(\Omega) \cap C(\overline{\Omega})$. We assume that a_{ij}, b_i and c are continuous in Ω. The coefficient matrix $A = (a_{ij})$ is positive definite everywhere in Ω. Likewise, we denote $D^* := (\det(A))^{1/n}$ as the geometric mean of the eigenvalues of A.

Definition 1. Define for every $u \in C^2(\Omega)$,

$$\Gamma^+(u) = \{y \in \Omega; u(x) \le u(y) + Du(y).(x - y), x \in \Omega\}.$$

The set $\Gamma^+(u)$ is called the upper contact set of u and the Hessian matrix (D^2u) is nonpositive on $\Gamma^+(u)$.

Let us state a lemma from [11] (see Lemma 2.24) required to the proof of Alexandroff Maximum Principle.

Lemma 2. *Suppose $g \in L^1_{loc}(\Omega)$ is nonnegative. Then for any $u \in C^2(\Omega) \cap C(\overline{\Omega})$, there holds*

$$\int_{B_k(0)} g \le \int_{\Gamma^+(u)} g(Du)\left|\det D^2u\right|,$$

where $\Gamma^+(u)$ is the upper contact set of u, $B_k(0)$ is the ball with radius k and center 0 and $k = (1/d)(\sup_\Omega u - \sup_{\partial\Omega} u^+)$, where d is the diameter of Ω.

Now we give the Alexandroff Maximum Principle

Theorem 3. *Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq f$ in Ω with the following conditions*

$$\frac{|b|}{D^*}, \ \frac{f}{D^*} \in L^n(\Omega) \ \text{ and } \ c \leq 0 \ \text{ in } \ \Omega.$$

Then there holds

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \left\| \frac{f^-}{D^*} \right\|_{L^n(\Gamma^+(u))},$$

where C is a constant depend only on n, diam(Ω) and $\|f^-/D\|_{L^n(\Gamma^+(u))}$.

Note here that $c(x)$ is assumed to be only measurable and no assumption on the boundedness is required. We are providing the sketch of the proof for the convenience of the reader.

Proof. Without loss of generality we assume $u < 0$ on $\partial\Omega$. Set $\Omega^+ = \{u > 0\}$. Take $g(p) = (|p|^n + \mu^n)^{-1}$ and then let $\mu \to 0^+$.

Recall the area-formula for Du in $\Gamma^+ \cap \Omega^+ \subset \Omega$ gives

$$\int_{Du(\Gamma^+ \cap \Omega^+)} \leq \int_{\Gamma^+ \cap \Omega^+} g(Du) \left| \det(D^2(u)) \right|,$$

where $D^2(u)$ is the Jacobian of the map $Du : \Omega \to \mathbb{R}^n$.

First we have,

$$\begin{gathered} -a_{ij} D_{ij} u \leq b_i D_i u + cu - f, \\ -a_{ij} D_{ij} u \leq b_i D_i u - f \ \text{ in } \ \Omega^+ = \{x; u(x) > 0\}, \\ -a_{ij} D_{ij} u \leq |b||Du| + f^-. \end{gathered}$$

Then by Cauchy inequality we have,

$$-a_{ij} D_{ij} u \leq 2 \left(|b|^n + \frac{(f^-)^n}{\mu^n} \right)^{1/n} . (|Du|^n + \mu^n)^{1/n}.$$

So, by Lemma 2 and recalling that

$$\det(-D^2 u) \leq \frac{1}{D} \left(\frac{-a_{ij} D_{ij} u}{n} \right)^n \ \text{ on } \ \Gamma^+,$$

where $D = \det(A)$, we have

$$\int_{B_k(0)} g \leq \frac{2^n}{n^n} \int_{\Gamma^+ \cap \Omega^+} \frac{|b|^n + \mu^{-n} (f^-)^n}{D}.$$

Now evaluating the integral in the left-hand side we have,

$$\int_{B_k(0)} g = \frac{\omega_n}{n} \log \left(\frac{k^n}{\mu^n} + 1 \right),$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Therefore we obtain

$$k^n \leq \mu^n \left\{ \exp \left\{ \frac{2^n}{\omega_n n^n} \left[\left\| \frac{b}{D^*} \right\|^n_{L^n(\Gamma^+ \cap \Omega^+)} + \mu^{-n} \left\| \frac{f^-}{D^*} \right\|^n_{L^n(\Gamma^+ \cap \Omega^+)} \right] \right\} - 1 \right\}.$$

If $f \not\equiv 0$ then choose any $\mu > 0$ and then let $\mu \to 0$. This completes the proof. □

Next we give a statement of Hopf Maximum Principle and a Strong Maximum Principle adapted to our situation (see [11]). Let us assume the operator L as described above with the assumption that a_{ij}, b_i are continuous and hence bounded in $\overline{\Omega}$ and $c(x)$ is bounded below.

Then we have the following results:

Lemma 4 (Hopf Lemma)**.** *Let B an open ball in $\mathbb{R}^n$ with $x_0 \in \partial B$. Suppose $u \in C^2(B) \cap C(B \cup \{x_0\})$ satisfies $Lu \geq 0$ in B with $c(x) \leq 0$ and uniformly bounded in B. Assume in addition that*

$$u(x) < u(x_0) \quad \textit{for any } \; x \in B \textit{ and } u(x_0) \geq 0.$$

Then for each outward direction $\overline{\nu}$ and an outward normal direction $\overline{n}$ at x_0 with $\overline{\nu}.\overline{n} > 0$ there holds:

$$\liminf_{t \to 0^+} \frac{1}{t}[u(x_0) - u(x_0 - t\nu)] > 0.$$

Remark 1. If in addition $u \in C^2(\Omega) \cap C^1(\Omega \cup \{x_0\})$ then we have

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

The proof of Lemma 4 can be found in [11]. From Lemma 4 we can prove the following strong maximum principle:

Theorem 5 (Strong Maximum Principle)**.** *Let Ω be a bounded and connected domain in $\mathbb{R}^n$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ in Ω with $c(x) \leq 0$. Then, the nonnegative maximum of u can be assumed only on $\partial\Omega$ unless u is constant in $\overline{\Omega}$.*

We adapt the proof given in [11].

Proof. Let M be the nonnegative maximum of u in $\overline{\Omega}$. Set $\Sigma := \{x \in \Omega;\; u(x) = M\}$. It is relatively closed in Ω. We want to show $\Sigma = \Omega$.

We prove by contradiction. If Σ is a proper set of Ω, then we may find an open ball $B \subset \Omega \setminus \Sigma$ with a point on its boundary belonging to Σ. (In fact, we may choose a point $p \in \Omega \setminus \Sigma$ such that $\mathrm{d}(p, \Sigma) < \mathrm{d}(p, \partial\Omega)$ first and then extend the ball. It hits Σ before hitting $\partial\Omega$). Suppose $x_0 \in \partial B \cap \Sigma$. Obviously we have $Lu \geq 0$ in B and

$$u(x) < u(x_0) \quad \text{for any } \; x \in B \text{ and } u(x_0) = M \geq 0.$$

Lemma 4 (note that c is bounded in B since by construction, $\overline{B} \subset \Omega$) implies $\frac{\partial u}{\partial \nu} > 0$ where ν is the outward normal direction at x_0 to the ball B. While x_0 is the interior maximal point of Ω, hence $Du(x_0) = 0$. This leads to a contradiction. □

A straightforward consequence of Theorem 5 is the following result:

Corollary 6 (Comparison Principle)**.** *Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ in Ω with $c(x) \leq 0$ in Ω. If $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω. In fact, either $u < 0$ in Ω or $u \equiv 0$ in Ω.*

§3. Proof of the main result

Write $x = (x_1, y) \in \Omega$ for $y \in \mathbb{R}^{n-1}$. We will prove

$$u(x_1, y) < u(x_1^*, y) \text{ for any } x_1 > 0 \text{ and } x_1^* < x_1 \text{ with } x_1^* + x_1 > 0.$$

Then letting $x_1^* \to -x_1$, we get $u(x_1, y) \leq u(-x_1, y)$ for any x_1. Then by changing the direction $x_1 \to -x_1$,we get the symmetry.

We let a = sup x_1 for $(x_1, y) \in \Omega$ and for $0 < \lambda < a$, we define

$$\begin{aligned}
\Sigma_\lambda &= \{(x_1, \ldots, x_n) \in \Omega \mid x_1 > \lambda\},\\
T_\lambda &= \{(x_1, \ldots, x_n) \in \Omega \mid x_1 = \lambda\},\\
\Sigma'_\lambda &= \{(2\lambda - x_1, ..., x_n) \in \Omega \mid (x_1, \ldots, x_n) \in \Sigma_\lambda\}.
\end{aligned}$$

Notice that Σ'_λ is the reflection of Σ_λ with respect to T_λ. In the following we denote by x_λ the image of x with respect to T_λ.

In Σ_λ, we define $w_\lambda(x) = u(x) - u(x_\lambda)$ for $x \in \Sigma_\lambda$. Then by Mean Value Theorem we have

$$\begin{aligned}
&\Delta w_\lambda + c(x, \lambda) w_\lambda - \frac{\delta w_\lambda}{u_\gamma^{\delta+1}} = 0 \text{ in } \Sigma_\lambda.\\
&w_\lambda \leq 0 \textit{ and } w_\lambda \neq 0 \text{ on } \partial\Sigma_\lambda.
\end{aligned} \tag{1}$$

where $u_\gamma(x) = u(x_\gamma)$ with x_γ is a suitable convex combination of x and x_λ and $c(x, \lambda)$ is a bounded function in Σ_λ .

We need to show $w_\lambda < 0$ in Σ_λ for any $\lambda \in (0, a)$. We divide the proof in three steps.

Step 1. For any λ close to a, we first show $w_\lambda \leq 0$, i.e we can actually start the moving plane. For λ close to a, we are rearranging (1) as:

$$\begin{aligned}
&\Delta w_\lambda - \left[c^-(x, \lambda) + \frac{\delta}{u_\gamma^{\delta+1}}\right] w_\lambda = -c^+(x, \lambda) w_\lambda \text{ in } \Sigma_\lambda,\\
&w_\lambda \leq 0 \text{ and } w_\lambda \neq 0 \text{ on } \partial\Sigma_\lambda.
\end{aligned}$$

Now, since $\sup_{\partial\Sigma_\lambda} w_\lambda = 0$, we have by Theorem 3 that for λ close to a,

$$\begin{aligned}
&\sup_{\Sigma_\lambda} w_\lambda \leq C(n, d)\, \|c^+ w_\lambda^+\|_{L^n(\Sigma_\lambda)},\\
&\sup_{\Sigma_\lambda} w_\lambda \leq C(n, d)\, \|c^+\|_{L^\infty(\Sigma_\lambda)} |\Sigma_\lambda|^{1/n} \sup_{\Sigma_\lambda} w_\lambda \leq \frac{1}{2} \sup_{\Sigma_\lambda} w_\lambda,
\end{aligned}$$

where d denotes the diameter of Ω. So we have $w_\lambda \leq 0$ for λ close to a.

Applying Corollary 6, we get $w_\lambda < 0$ in Σ_λ for λ close to a.

Step 2. Let (λ_0, a) be the largest interval of values of λ such that $w_\lambda < 0$ in Σ_λ. We want to show $\lambda_0 = 0$. If $\lambda_0 > 0$ by continuity $w_{\lambda_0} \leq 0$ in Σ_{λ_0} and $w_{\lambda_0} \neq 0$ on $\partial\Sigma_{\lambda_0}$. Now by Theorem 5 we have $w_\lambda < 0$ in Σ_{λ_0}. We will show that for a small $\epsilon > 0$ we have $w_{\lambda_0-\epsilon} < 0$ in $\Sigma_{\lambda_0-\epsilon}$, thus getting a contradiction that (λ_0, a) is the largest interval of values of λ such that $w_\lambda < 0$ in Σ_λ.

Fix $\theta > 0$ (to be determined). Let K be a closed subset in Σ_{λ_0} such that $|\Sigma_{\lambda_0-\epsilon} \setminus K| < \theta/2$. The fact $w_{\lambda_0} < 0$ in Σ_{λ_0} implies $w_{\lambda_0}(x) \leq -p < 0$ for any $x \in K$ and some $p > 0$. By continuity we have $w_{\lambda_0-\epsilon} < 0$ in K. For $\epsilon > 0$ small, $|\Sigma_{\lambda_0-\epsilon} \setminus K| < \delta$.

We choose δ in such a way that we may apply Theorem 3 to $w_{\lambda_0-\epsilon}$ in $\Sigma_{\lambda_0-\epsilon} \setminus K$. Hence we get $w_{\lambda_0-\epsilon} \leq 0$ in $\Sigma_{\lambda_0-\epsilon} \setminus K$.

Therefore we obtain that for any $\epsilon > 0$ small enough, we have $w_{\lambda_0-\epsilon}(x) \leq 0$ in $\Sigma_{\lambda_0-\epsilon}$. Again, using corollary 6, we get $w_{\lambda_0-\epsilon}(x) < 0$ in $\Sigma_{\lambda_0-\epsilon}$. Therefore, $\lambda_0 = 0$.

Step 3. We have $w_\lambda \leq 0$ for all $\lambda \in (0, a)$. Applying now Corollary 6 and Lemma 4 to the equation

$$\begin{aligned} &\Delta w_\lambda - \left[c^-(x,\lambda) + \frac{\delta}{u_\gamma^{\delta+1}}\right] w_\lambda = c^+(x,\lambda) w_\lambda \ \text{ in } \Sigma_\lambda, \\ &w_\lambda \leq 0 \ \text{ and } \ w_\lambda \neq 0 \ \text{ on } \ \partial\Sigma_\lambda, \end{aligned}$$

we have $w_\lambda < 0$ for $\lambda \in (0, a)$.
Note that w_λ admits its maximum along $\Sigma_\lambda \cap \Omega$. Again applying the next part of Lemma 4 we have

$$D_{x_1} w_\lambda|_{x_1=\lambda} = 2 D_{x_1} u_\lambda|_{x_1=\lambda} < 0.$$

The proof of Theorem 1 is now complete.

§4. Some a priori estimates

In this section we will produce some a priori results for (1) with the function f being replaced by a specific type of non-linearity. The equation is given by:

$$\begin{aligned} &-\Delta u - \frac{1}{u^\delta} = R(x) u^\alpha \ \text{ in } \Omega, \\ &u = 0 \ \text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega, \end{aligned} \tag{2}$$

where R is continuous and strictly positive function in $\overline{\Omega}$ and $1 < \alpha < \frac{n+2}{n-2}$ with $\delta > 0$ is given.

We want to find some a priori estimates on the solutions of the above equation i.e., we show a uniform bound for the solutions and we achieve that goal with the help of a blow-up technique in a compact subset of Ω. For the rest of the domain, we apply Theorem 1 for deriving a uniform bound of solutions in a neighborhood of $\partial\Omega$.

We start by a lemma which is a global result of Liouville type (see [8]).

Lemma 7. *Let $u(x)$ be a non-negative C^2 solution of*

$$\Delta u + u^\alpha = 0 \ \textit{in } \mathbb{R}^n \tag{3}$$

with $1 < \alpha < (n+2)/(n-2)$. Then $u(x) \equiv 0$.

Remark 2. Our main result is for f depending only on u but the same thing holds for $f(x,u)$ with f is a locally lipchitz w.r.t the second variable and continuous w.r.t the first variable.

To prove the result we need few lemmata. First we state here a result of [2].

Lemma 8. *Consider the equation given by*

$$\begin{aligned} -\Delta u &= \frac{1}{u^{\delta}} \ \text{ in } \Omega \\ u &= 0 \ \text{ on } \partial\Omega. \end{aligned}$$

Then there exists unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover we can find $0 < c_0 \le c_1$ such that

1. *For $0 < \delta < 1$, we have $c_0 d(x) \le u \le c_1 d(x)$.*
2. *For $\delta = 1$, we have $c_0 d(x) \ln (A/d(x))^{1/2} \le u \le c_1 d(x) \ln (A/d(x))^{1/2}$ where $A > 1$ is large enough.*
3. *For $\delta > 1$, we have $c_0\{d(x)\}^{2/(\delta+1)} \le u \le c_1\{d(x)\}^{2/(\delta+1)}$.*

The above result together with the comparison principle show that any non trivial solution u to (2) satisfies $u(x) \ge cd(x)$ with $c > 0$ independent of u. Next we state a strong comparison principle (see [6] for the extension in the case of quasilinear elliptic operators):

Lemma 9. *Let $u, v(\ge 0) \in C^2(\Omega) \cap C(\overline{\Omega})$ and satisfies*

$$\begin{aligned} -\Delta u - u^{-\delta} &= f, \\ -\Delta v - v^{-\delta} &= g, \end{aligned}$$

with $u = v = 0$ on $\partial\Omega$, $0 < \beta < 1$ with $f, g \in C(\overline{\Omega})$ such that $0 \le f \le g$ pointwise everywhere in Ω and $f \not\equiv g$. Then $0 < u < v$ in Ω.

Now we are ready to proceed to the main result of this section:

Theorem 10. *Suppose that Ω is a bounded domain which is strictly convex. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a positive solution of*

$$\begin{aligned} -\Delta u - \frac{1}{u^{\delta}} &= R(x)u^{\alpha} \ \text{ in } \Omega \\ u &= 0 \ \text{ on } \partial\Omega. \end{aligned} \tag{4}$$

where $\delta > 0$, $1 < \alpha < (n+2)/(n-2)$ and R is continuous and strictly positive function in $\overline{\Omega}$. Then $u(x) < C$ for some uniform constant C where C only depends α and Ω .

Proof. We are going to divide the domain into two parts given by:

$$\begin{aligned} \Omega_\eta &= \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \ge \eta\}, \\ \Omega \setminus \Omega_\eta &= \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \eta\}, \end{aligned}$$

where $\eta > 0$ is small enough.

We proof the theorem by contradiction. Let on the contrary there exists a sequence of solutions $u^k(x)$ of (4) and a sequence of points $P_k \in \Omega_\eta$ such that $M_k = \sup_\Omega u^k(x) = u^k(P_k) \to +\infty$ as $k \to +\infty$.

We first prove that $P_k \to P \in \Omega_\eta$. For that, we apply the moving plane method as in the previous section . Applying the method used for the proof of Theorem 1 (see also [3]) and the convexity of Ω (precisely, we move the hyperplane in a direction close to the outward normal in a neighborhood of any point of the boundary), we have a $H > 0$ (depending on the domain and independent of k) and a $T > 0$ such that:

$u_k(x - t\gamma)$ is decreasing for $t \in [0, T]$ for $\gamma \in \mathbb{R}^n$ satisfying $|\gamma| = 1$ and $(\gamma.n(x)) \geq H$, $n(x)$ is the unit normal to $\partial\Omega$ at x and for $x \in \partial\Omega$.

The fact that $u_k(x - t\gamma)$ is non-decreasing in t for x, t and γ decribed above we have to positive numbers α_1 and α_2 both depending on Ω such that, for any x belonging to $\Omega \setminus \Omega_{\alpha_2} = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) < \alpha_2\}$, we have a measurable set I_x with

- $|I_x| \geq \alpha_1$,
- $I_x \subset \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \geq \alpha_2/2\}$,
- $u_k(\kappa) \geq u_k(x)$ for all κ in I_x.

Then, multiplying the equation satisfied by u_k by the L^1-normalised positive eigenfunction ϕ_1 associated to the first eigenvalue,

$$\lambda_1(\Omega) := \inf_{u \in H_0^1(\Omega),\, u \not\equiv 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2},$$

we get that

$$\lambda_1(\Omega) \int_\Omega u_k \phi_1 \mathrm{d}x = \int_\Omega \frac{\phi_1}{u_k^\delta}\, \mathrm{d}x + \int_\Omega R(x) u_k^\alpha \phi_1\, \mathrm{d}x.$$

Observing that, for any $\ell > \lambda_1(\Omega)$, there exists $C > 0$ such that

$$\frac{1}{t^\delta} + R(x) t^\alpha \geq \ell t - C \quad \text{for any } \ t \in \mathbb{R}^+ \ \text{ and uniformly for } x \in \Omega.$$

Then, fixing $\ell > \lambda_1(\Omega)$, it follows that

$$(\ell - \lambda_1(\Omega)) \int_\Omega u_k \phi_1 \leq C.$$

Thus, from above, we get for $x \in \Omega \setminus \Omega_{\alpha_2}$

$$u_k(x) \int_{I_x} \phi_1 \mathrm{d}x \leq \int_{I_x} u_k \phi_1 \leq C.$$

Then, $u_k(x) \leq C/|I_x|^{1/2} \leq C/\alpha_1$ for $x \in \Omega \setminus \Omega_{\alpha_2}$. Therefore, $\mathrm{dist}(M_k, \partial\Omega) \geq \alpha_2$. We now apply the blow-up analysis of [9].

Let $B_R(a)$ denote a ball with radius R and centre $a \in \mathbb{R}^n$. Let λ_k be a sequence of positive numbers(to be defined later) and $y = (x - P_k)/\lambda_k$. Define the scaled function

$$v_k(y) = \lambda_k^{2/(\alpha-2)} u_k(x).$$

We choose λ_k so that $\lambda_k^{2/(\alpha-2)} M_k = 1$. Since $M_k \to +\infty$, we have $\lambda_k \to 0$ as $k \to +\infty$. For large k, $v_k(y)$ is well-defined in $B_{\eta/\lambda_k}(0)$, and

$$\sup_{y \in B_{\eta/\lambda_k}(0)} v_k(y) = v_k(0) = 1.$$

Moreover, $v_k(y)$ satisfies in $B_{\eta/\lambda_k}(0)$ the following equations:

$$-\lambda_k^{-2\alpha/(\alpha-1)} \Delta u_k - \lambda_k^{2\delta/(\alpha-1)} [v_k]^{-\delta} = R(\lambda_k y + P_k) \lambda_k^{-2\alpha/(\alpha-1)} [v_k]^{\alpha},$$
$$-\Delta v_k = \lambda_k^{2(\alpha+\delta)/(\alpha-1)} [v_k]^{-\delta} + R(\lambda_k y + P_k)[v_k]^{\alpha}.$$

From Lemma 9, we have $u_k \geq c_0\{d(x)\}^{\alpha}$, where α depends on δ. Again by Lemma 10 we have $v_k \geq \lambda_k^{2/(\alpha-2)} u_k$. Combining these two results we have $v^k \geq p(> 0)$ in Ω_η with p depending upon η and δ

Therefore given any radius R such that $B_R(0) \subset B_{\frac{\eta}{\lambda_k}}(0)$ we can, by elliptic L^p estimates, find uniform bounds for $\|v_k\|_{W^{2,p}(B_R(0))}$. Choosing p large we obtain by Morrey's embedding theorem that $\|v_k\|_{C^{1,\beta}(B_R(0))}$ for $0 < \beta < 1$ is also uniformly bounded. So for any sequence $k \to +\infty$, there exists a subsequence $k_j \to +\infty$ such that $v^{k_j} \to v$ in $W^{2,p} \cap C^{1,\beta}$, $p > n$ on $B_R(0)$. By Holder Continuity $v(0) = 1$ again since $R(\lambda_k y + P_k) \to R(P)$ as $k \to +\infty$, we have that

$$-\Delta v = R(P) v^{\alpha},$$
$$v(0) = 1.$$

We claim that v is well-defined in all of $\mathbb{R}^n$ and $v_{k_j} \to v$ in $W^{2,p} \cap C^{1,\beta}$, $p > n$ on compact subsets. To show this we consider $B_R(0) \subset B'_R(0)$. Repeating the above argument with $B'_R(0)$, the subsequence $v^{k'_j}$ has a convergent subsequence $v^{k'_j} \to v'$ on $B'_R(0)$, v' satisfies $\lambda_k^{2/(\alpha-2)} M^k = 1$ and if restricted to $B_R(0)$ gives v. By unique continuation, the entire original sequence converges and v is well defined. By Lemma 4, we have $v = 0$ in $\mathbb{R}^n$, a contradiction since $v(0) = 1$.

This completes the proof. □

The existence of a priori bounds together with the theory of global bifurcation in the context of singular problems (see [12] and the extension for more singular nonlinearities [4]) can be used to prove existence of multiple solutions. Precisely, let us consider the following problem where $\lambda \in \mathbb{R}^+$ is a parameter:

$$\begin{aligned} -\Delta u = \lambda \left(\frac{1}{u^{\delta}} + R(x) u^{\alpha} \right) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \ u > 0 \text{ in } \Omega. \end{aligned} \tag{5}$$

In particular, we can prove the following result:

Theorem 11. *Let $\delta \in (0, 3)$ and $1 < \alpha < (n+2)/(n-2)$. Then, there exists an unbounded connected set $C \subset \mathbb{R}^+ \times \left(L^{\infty}(\Omega) \cap H_0^1(\Omega) \right)$ of solutions (λ, u) to (5) such that*

(i) there exists $\Lambda > 0$ such that $\Pi_{\mathbb{R}} C = [0, \Lambda]$;

(ii) for any $\lambda \in (0, \Lambda)$, there exists two solutions (λ, u_λ) and (λ, v_λ) belonging to C and such that $u_\lambda < v_\lambda$ in Ω.

The above theorem can be proved by showing that the conected component set of the minimal solutions curve admits a turning point at $\lambda = \Lambda$ and from the existence of universal bounds at $\lambda > 0$ bends back to $\lambda = 0$ where the branch admits an asymptotic bifurcation point.

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