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TESTING NUMERICAL METHODS FOR SOLVING INTEGRAL EQUATIONS

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Abstract. Many modeling problems in physics and in a variety of engineering fields lead to integral equations. We briefly describe the main classical techniques to obtain approximated solutions of them: Nyström methods and projection methods. Moreover, we introduce a new method to approximate the solution of integral equations based in a variational scheme. We test these techniques with numerical examples and we show several tables in order to measure the error obtained by the presented methods.

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§1. Introduction

Integral equations are equations involving an unknown function which appears under an integral sign. The theory of integral equations has close contacts with many different areas of mathematics.

We consider the Fredholm integral equation of the second kind

$$f(t) = x(t) - \int_0^1 k(t,s)x(s)\,ds,\ 0 \le t \le 1. \tag{1}$$

It is known that the expression (1) in operator form can be written

$$f = x - \mathcal{K}x = (I - \mathcal{K})X.$$

Such equations occur widely in diverse areas of applied mathematics and physics, such as potential theory and radiation heat transfer but also some other equations reducible to it, and, in particular, the Lippman-Schwinger equation in potential scattering. In addition, many problems in the fields of differential equations can be recast as integral equations.

It is usually to impose to the operator $I - \mathcal{K}$ certain assumptions in order to establish the existence and uniqueness of solution of (1).

§2. Solving Fredholm integral equations of the second kind

The main numerical methods for solve these type of integral equations are Nyström methods or quadrature methods and the projection methods, based in approximate the numerical integral.

The projection methods with the collocation and Galerkin methods as special case, are a general tool that can also solve equations of the first kind. They are known as spectral methods and pseudospectral methods respectively.

The collocation method seeks an approximate solution from a finite dimensional space by requiring that equation (1) to satisfy only at a finite number of points, called collocation points. In collocation methods one can use e. g. interpolation functions in polynomial or spline spaces. Because of the better convergence properties of splines, spline collocation is superior to polynomial collocation.

The Galerkin method and Petrov–Galerkin method, with many variants [3], consists in finding a best approximation to the exact solution of (1) in a finite dimensional space by the minimizing of the so called energy functional. One of the advantages of the Petrov-Galerkin method is that it allows to achieve the same order of convergence as the Galerkin method with much less computational cost by choosing the test spaces to be spaces of piecewise polynomials of lower degree.

Another method which is employed for the solution of integral equations on smooth closed curves is *the qualocation method*. Qualocation method is a Petrov–Galerkin method in which the outer integrals are performed numerically by special quadrature rules.

We have been developed another method in order to approximate the solution of (1). The method is based in the minimization of a functional that involves (1) and that it is similar to the Petrov–Galerkin method.

Our aim in this work is to detail the computational part of distinct solving methods and to show different examples in order to compare them. Throughout this overview, we will assume that the integral equations have a unique solution to be determined.

§3. Preliminaries and notations

The Euclidean norm and inner product in $\mathbb{R}^n$ will be denoted respectively by $\langle\cdot\rangle$ and $\langle\cdot,\cdot\rangle$. Moreover, we designate by $H^k(0,1)$ the Sobolev space of order k, which is equipped with the inner product and norm

$$((u,v))_k = \sum_{i=0}^{k} \int_0^1 u^{(i)}(t)v^{(i)}(t)\,dt, \quad \|u\|_k = (u,u)_k^{1/2},$$

the semi–inner products and semi-norms

$$(u,v)_j = \int_0^1 u^{(j)}(x)v^{(j)}(x)\,dx,\ |u|_j = (u,u)_j^{1/2},\ \forall\, j = 0,\ldots,k.$$

Let $k(t,s) \in H^3(0,1)\times L^2(0,1)$ be a given function, and we designate by $\mathcal{K}$ the integral operator associated with $k(t,s)$,

$$\mathcal{K}u(t) = \int_0^1 \int_0^1 k(t,s)u(s)\,ds\,dt, \quad \forall u \in H^3(0,1).$$

Finally, we assume that $f \in C(0,1)$ and $\mathcal{K}$ is a compact operator on $C(0,1)$, and that 1 is not an eigenvalue of $\mathcal{K}$.

§4. Mixed Variational Method

This method can be briefly described as follows. For each $h \in \mathbb{R}_+$ and $N \in \mathbb{N}$ fixed, with $h = 1/N$, let

$$\Delta_h = \{0 = t_0 < \cdots < t_N = 1\},\ t_i = i\,h,$$

be a subset of distinct points of $[0, 1]$

We denote by $S(3, 2; \Delta_h)$ the space of the splines of degree 3 and class 2 associated with Δ_h, i.e.

$$S(3, 2; \Delta_h) = \{s \in C^2(0, 1) : s|_{[t_{i-1},t_i]} \in \mathbb{P}_3[t_{i-1}, t_i],\ i = 1, \ldots, N\}.$$

A basis of this finite dimensional space is given by B-splines functions.

Given $a_1 = 0$, $a_2 = p$, $a_3 = 1$, being p a knot of Δ_h, we define the operator $\rho : H^3(0, 1) \to \mathbb{R}^3$ by

$$\rho v = ((I - \mathcal{K})v(a_i))_{i=1,2,3}$$

and let $\beta = (f(a_i))_{i=1,2,3}$.

For each $h \in \mathbb{R}^+$, we define

$$G_h = \{u \in X_h : \rho u = \beta\}$$

and the vectorial space

$$G_h^0 = \{u \in X_h : \rho u = \mathbf{0}\}.$$

It is said that x_h is an approximated solution of (1) if x_h is a solution of the problem

$$\begin{cases} x_h \in G_h, \\ \forall v \in G_h, \quad J(x_h) \le J(v), \end{cases} \tag{2}$$

where J is the functional defined on $H^3(0, 1)$ by

$$J(v) = |(I - \mathcal{K})v - f|_3^2\,.$$

The next result guarantees the existence and the uniqueness of the solution of Problem (2).

Theorem 1. *Problem (2) has a unique solution characterized as the unique solution of the following variational problem: find x_h such that*

$$\begin{cases} x_h \in G_h, \\ \forall v \in G_h^0, ((I - \mathcal{K})\sigma_N, (I - \mathcal{K})v)_3 = ((I - \mathcal{K})v, f)_3. \end{cases}$$

Proof. It is clear that G_N is a nonempty closed convex subset of $S(3, 2; \Delta_N)$. Now, we consider the form $a : S(3, 2; \Delta_N) \times S(3, 2; \Delta_N) \to \mathbb{R}$ given by $a(u, v) = 2\,(((u, v)))$.
Note that the application a is bilinear, symmetric, continuous and coercive since $S(3, 2; \Delta_N)$ is a finite dimensional space. Let $\varphi(v) = ((I - \mathcal{K})v, f)_3$ be a linear form, which is clearly continuous. Now, Stampacchia's Theorem (see [2]) can be applied and we conclude the proof. □

As a consequence of this result we can obtain that there exists a unique $(x_h, \tau) \in X_h \times \mathbb{R}^3$ such that for all $v \in X_h$

$$((I - \mathcal{K})x_h, (I - \mathcal{K})v)_3 + \langle \tau, \rho v \rangle = (f, (I - \mathcal{K})v)_3, \tag{3}$$

where x_h is the unique solution of Problem (2).

§5. Computation of the methods

We detail the computation of the methods described above. In all cases, we seek an approximate function $x_h(t) \in X_h$ where X_h is a finite dimensional subspace.

The approximated solution of the integral equation (1) can be written as

$$x_h(t) = \sum_{i=1}^{n} \alpha_i \phi_i(t), \quad t \in D,$$

where $\{\phi_1, \ldots, \phi_h\}$ is a basis of X_h. The question is how we can determinate the unknown coefficients α_i, $i = 1, \ldots, n$, by using the above methods.

Collocation method. We choose distinct node points $t_1, \ldots, t_n \in [0, 1]$ and we impose that

$$f(t_i) - \lambda x_h(t_i) - \int_0^1 k(t_i, s) x_h(s)\, ds = 0, \quad i = 1, \ldots, n.$$

Then, we have now a linear system of n equations with unknown α_i.

Galerkin method. We impose that

$$\langle (\lambda I - \mathcal{K}) x_h, \phi_j \rangle = \langle f, \phi_j \rangle, \quad \forall\, j = 1, \ldots, n.$$

The coefficients α_i are determined by solving

$$\sum_{i=1}^{n} \alpha_i \left(\langle \lambda\, \phi_i, \phi_j \rangle - \langle \mathcal{K}\phi_i, \phi_j \rangle \right) = \langle f, \phi_j \rangle, \quad \forall\, j = 1, \ldots, n.$$

Nyström method. It requires the choice of some approximate quadrature rule. By using a numerical scheme, from

$$\int_0^1 k(t, s) x_h(s)\, ds \approx \sum_{j=1}^{n} \omega_j k(t, t_j) x_h(t_j), \quad 0 \le j \le n,$$

we obtain the linear system

$$\lambda x_h(t_i) + k(t_i, t_j)\, x_h(t_j) = f(t_i), \quad i = 1, \ldots, n.$$

Here the set ω_j are the weights of the quadrature rule, while the n points t_j are the abscissas. An interesting approach of this method can be found in [1].

Mixed Variational method. By replacing in (3), we have for all $v \in X_h$

$$\sum_{i=1}^{n} \alpha_i ((I - \mathcal{K})\phi_i, (I - \mathcal{K})v)_3 + \langle \tau, \rho v \rangle = (f, (I - \mathcal{K})v)_3,$$

subject to the restrictions

$$x_h(a_j) - \int_0^1 k(a_j, s)\, x_h(s)\, ds = f(a_j),\ \ j = 1, 2, 3.$$

Taking $v = \phi_i$, for $i = 1, \ldots, n$, we obtain a linear system of order $n + 3$ with the unknown $\alpha_1, \ldots, \alpha_n, \tau_1, \tau_2, \tau_3$. The matrix form of such system is

$$\begin{pmatrix} A & D \\ D^t & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \tau \end{pmatrix} = \begin{pmatrix} \widehat{f_1} \\ \widehat{f_2} \end{pmatrix},$$

where

$$A = \Big(((I - \mathcal{K})\phi_i, (I - \mathcal{K})\phi_j)_3\Big)_{1 \le i, j \le n},$$

$$D = \left(\phi_i(a_j) - \int_0^1 k(a_j, s)\, \phi_i(s)\, ds\right)_{\substack{1 \le i \le n \\ 1 \le j \le 3}},$$

$$\widehat{f_1} = ((f, (I - \mathcal{K})\phi_i)_3)^t_{1 \le i \le n}, \qquad \widehat{f_2} = (f(a_i))^t_{1 \le i \le 3}.$$

§6. Numerical examples

We present several numerical experiments and we compare the results with the exact solution of the integral equation.

Now, we have chosen the space of cubic spline functions for the numerical experiments. For each $N \in \mathbb{N}$ let $h = 1/N$ and let

$$\Delta_N = \{0 = t_0 < \ldots < t_N = 1\},\ \ t_i = i\,h,\ \ i = 0, \ldots, N,$$

be a subset of distinct points of $[0, 1]$.

A basis of this $S(3, 2; \Delta_N)$ is given by B-splines functions. We denote this basis as $\mathcal{B} = \{B_i : 1 \le i \le N + 3\}$. The general expression of $B_i(t)$ is

$$B_i(t) = \frac{1}{6h^3} \begin{cases} (t - t_{i-2})^3, & \text{if } t \in [t_{i-2}, t_{i-1}], \\ h^3 + 3h^2(t - t_{i-1}) + 3h(t - t_{i-1})^2 - 3(t - t_{i-1})^3, & \text{if } t \in [t_{i-1}, t_i], \\ h^3 + 3h^2(t_{i+1} - t) + 3h(t_{i+1} - t)^2 - 3(t_{i+1} - t)^3, & \text{if } t \in [t_i, t_{i+1}], \\ (t_{i+2} - t)^3, & \text{if } t \in [t_{i+1}, t_{i+2}], \\ 0, & \text{otherwise.} \end{cases}$$

We take $\phi_i(t) = B_i(t)$. In order to show the efficiency, we have computed an estimation in several spaces of the error $\|x_h(t) - x(t)\|$, where x is the exact solution of the integral equation (1) and x_h is the approximated solution.

Example 1. We consider the simple test equation

$$x(t) - \int_0^1 x(s)(t^2 - t - s^2 + s)\, ds = -2t^3 + 3t^2 - t, \quad t \in [0, 1].$$

with exact solution $x(t) = -2t^3 + 3t^2 - t$. See Table 1.

	Method		
Computed error in	Collocation	Galerkin	Variational
$H^0(0,1)$	0	0	0
$H^1(0,1)$	3.17(−14)	7.65(−10)	0
$H^2(0,1)$	8.16(−13)	2.47(−8)	2.45(−15)
$H^3(0,1)$	1.27(−11)	4.56(−7)	6.32(−15)
$C^0(0,1)$	6.35(−15)	1.08(−10)	0
$C^1(0,1)$	1.73(−13)	4.69(−9)	2.33(−15)
$C^2(0,1)$	3.71(−12)	1.20(−7)	5.46(−15)

Table 1: Table of the computed relative error for Example 1 for $N = 4$ equidistant knots.

	Method		
Computed error in	Collocation	Galerkin	Variational
$H^0(0,1)$	4.34(−8)	1.05(−8)	3.71(−6)
$H^1(0,1)$	3.06(−6)	3.02(−6)	2.35(−5)
$H^2(0,1)$	2.97(−4)	3.07(−4)	3.17(−4)
$H^3(0,1)$	3.44(−2)	3.45(−2)	3.21(−2)
$C^0(0,1)$	1.25(−7)	1.61(−7)	5.22(−6)
$C^1(0,1)$	6.85(−6)	9.44(−6)	6.01(−5)
$C^2(0,1)$	1.02(−3)	1.54(−3)	7.31(−4)

Table 2: Table of the computed relative error for Example 2 for $N = 4$ equidistant knots.

Example 2. The following integral equation

$$x(t) - \int_0^1 x(s)e^{-t-s}\,ds = e^t - e^{-t}, \quad t \in [0,1].$$

has the exact solution $x(t) = e^t$. See Table 2.

6.1. Future work

We plan to do research in the following items:

1. The study of the mixed variational method for integral equations with Cauchy kernels.
2. Numerical experiments in distinct finite dimensional spaces as wavelets spaces.
3. The extension of the mixed variational method to the two dimensional case.

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