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Axial Couette flow of second grade fluid due to a longitudinal time dependent shear stress

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Abstract. The axial flow of a second grade fluid through an infinite straight circular cylinder is considered. The flow of the fluid is due to the longitudinal shear stress that is prescribed on the boundary of the cylinder. The velocity field and the resulting shear stress are determined by means of the finite Hankel and Laplace transforms. The corresponding solutions for Newtonian fluids, performing the same motion, are obtained as limiting case from our general solutions. Graphical illustrations are presented for the velocity field and the shear stress for both the second grade and Newtonian fluids.

Keywords: Second grade fluids, velocity field, longitudinal shear stress.

AMS classification: 53B25, 53C15.

§1. Introduction

In many engineering fields, such as oil exploitation, polymer chemical industry, bio-engineering is necessary to study the non-Newtonian fluid flows. The second grade fluid is the common viscoelastic fluid in industrial fields, such as polymer solutions. The most exact solutions in this field correspond to the case when the velocity is given by the boundary. The first exact solutions for second grade fluids, in which a constant shear stress is given on the boundary, seem to be those of Bandelli and Rajagopal [2].

The aim of this paper is to study the flow of a second grade fluid in a circular infinite cylinder due to a longitudinal time dependent shear stress. We establish both the velocity field and the resulting shear stress corresponding to the motion of the fluid. These solutions can be easily specialized to give the solutions to the Newtonian fluids performing the same motion. Finally, for comparison, the profiles of the velocity $v(r,t)$ and the shear stress $\tau(r,t)$, for the Newtonian and second grade fluids are plotted as functions of r for different values of the time t.

§2. Governing equations

The constitutive equation of the second grade fluids is given by [4, 5, 6, 10]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor, p is the pressure, $\mathbf{I}$ is the unit tensor, μ is the dynamic viscosity, α_1 and α_2 are the normal stress moduli and $\mathbf{A_1}$, $\mathbf{A_2}$ are the kinematic tensors. $\mathbf{A_2}$ is defined by

$$\mathbf{A_2} = \dot{\mathbf{A}}_\mathbf{1} + \mathbf{A_1}(\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T\mathbf{A_1}, \tag{2}$$

where $\mathbf{v}$ is the velocity field, $\mathbf{A_1} = \mathrm{grad}\,\mathbf{v} + (\mathrm{grad}\,\mathbf{v})^T$ and the superscript T denotes the transpose operator. In cylindrical coordinates (r, θ, z), the velocity of the axial flow is given by [4]

$$\mathbf{v} = \mathbf{v}(r,t) = v(r,t)\mathbf{e_z}, \tag{3}$$

where $\mathbf{e_z}$ is the unit vector in the z-direction. For such flows the constraint of incompressibility is automatically satisfied. Since the velocity field is independent of θ and z, we also assume that the extra-stress tensor $\mathbf{S}$ is independent of these variables. Furthermore, if the fluid is assumed to be at rest at the moment $t = 0$, then

$$\mathbf{S}(r,0) = \mathbf{0}. \tag{4}$$

Equalities (1), (2) and (3) lead to the constitutive relationship [2]

$$\tau(r,t) = (\mu + \alpha_1 \partial_t)\frac{\partial v(r,t)}{\partial r}, \tag{5}$$

where $\tau(r,t) = S_{rz}(r,t)$ is the shear stress which is different of zero.

In the absence of body forces and a pressure gradient in the $z-$direction, the balance of the linear momentum leads to the relevant equation

$$\rho\frac{\partial v(r,t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\tau(r,t), \tag{6}$$

where ρ is the constant density of the fluid.

Eliminating $\tau(r,t)$ among Eqs. (5) and (6), we attain to the governing equation

$$\frac{\partial v(r,t)}{\partial t} = (\nu + \alpha\partial_t)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t), \tag{7}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid and $\alpha = \alpha_1/\rho$.

§3. Axial flow through an infinite circular cylinder

Let us consider an incompressible second grade fluid at rest in an infinite circular cylinder of radius R. At time $t = 0^+$, the cylinder is suddenly pulled with a time dependent shear stress. Due to the shear, the fluid is gradually moved. It's velocity being of the form (3) and imposed initial and boundary conditions are

$$v(r,0) = 0\ ; \quad r \in [0, R)\ , \tag{8}$$

$$\tau(R,t) = (\mu + \alpha_1\partial_t)\frac{\partial v(R,t)}{\partial r} = ft, \quad t > 0. \tag{9}$$

Applying Laplace transform to Eqs. (7), (9) and using (8), we obtain

$$q\bar{v}(r,q) = (\nu + \alpha q)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,q), \tag{10}$$

$$\left.\frac{\partial \bar{v}(r,q)}{\partial r}\right|_{r=R} = \frac{f}{q^2(\mu + \alpha_1 q)}. \tag{11}$$

In order to obtain an analytical solution of the problem (10) and (11), the finite Hankel transform method is used. We define the Hankel transform of the function $\bar{v}(r,q)$ by [3]

$$\bar{v}_H(r_n,q) = \int_0^R r\bar{v}(r,q)J_0(rr_n)\,dr, \tag{12}$$

where r_n, $n = 1,2,3, ldots$, are the positive roots of the equation

$$J_1(Rr) = 0. \tag{13}$$

In the above relation, $J_\nu(\cdot)$ is the Bessel function of the first kind of order ν [7]. Multiplying now both sides of Eq. (10) by $rJ_0(rr_n)$, integrating then with respect to r from 0 to R and taking into account the condition (11) and the equality

$$\int_0^R r\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\Big)\bar{v}(r,q)J_0(rr_n)dr = -r_n^2\bar{v}_H(r_n,q) + RJ_0(Rr_n)\frac{\partial\bar{v}(R,q)}{\partial r}\ ,$$

we find that

$$\bar{v}_H(r_n,q) = \frac{Rf}{\rho}J_0(Rr_n)\,\frac{1}{q^2(q+\alpha r_n^2 q + \nu r_n^2)}. \tag{14}$$

We rewrite Eq. (14) as

$$\bar{v}_H(r_n,q) = \bar{v}_{1H}(r_n,q) + \bar{v}_{2H}(r_n,q), \tag{15}$$

where

$$\bar{v}_{1H}(r_n,q) = \frac{RfJ_0(Rr_n)}{r_n^2}\,\frac{1}{q^2(\mu+\alpha_1 q)} \tag{16}$$

and

$$\bar{v}_{2H}(r_n,q) = -\frac{RfJ_0(Rr_n)}{q}\,\frac{1}{r_n^2(\mu+\alpha_1 q)(q+\alpha r_n^2 q+\nu r_n^2)}. \tag{17}$$

The inverse Hankel transform of the function $\bar{v}_{1H}(r_n,q)$ and $\bar{v}_{2H}(r_n,q)$ are

$$\bar{v}_1(r,q) = \frac{r^2 f}{2R}\,\frac{1}{q^2(\mu+\alpha_1 q)}, \quad \bar{v}_2(r,q) = \frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{J_0^2(Rr_n)}\,\bar{v}_{2H}(r_n,q). \tag{18}$$

From (15)-(18) we find that the Laplace transform of the velocity $v(r,t)$, has the form

$$\bar{v}(r,q) = \frac{r^2 f}{2R}\,\frac{1}{q^2(\mu+\alpha_1 q)} - \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}\,\frac{1}{q(\mu+\alpha_1 q)(q+\alpha r_n^2 q+\nu r_n^2)}. \tag{19}$$

Applying the discrete inverse Laplace transform to Eq. (19), using the expansion

$$\frac{1}{q(q+\alpha r_n^2 q+\nu r_n^2)} = \frac{q^{-2}}{(1+\alpha r_n^2)+\nu r_n^2 q^{-1}} = \sum_{k=0}^{\infty}(-\nu r_n^2)^k\frac{q^{-k-2}}{(1+\alpha r_n^2)^{k+1}}, \tag{20}$$

the convolution theorem and the formulae

$$L^{-1}\Big\{\frac{1}{q^a}\Big\} = \frac{t^{a-1}}{\Gamma(a)},\ a>0, \quad L^{-1}\Big\{\frac{q^b}{(q^a-d)^c}\Big\} = G_{a,b,c}(d,t),\ \mathrm{Re}(ac-b)>0, \tag{21}$$

where $G_{a,b,c}(d,t)$ are the generalized G-functions defined as [8]

$$G_{a,b,c}(d,t) = \sum_{j=0}^{\infty} \frac{d^j\,\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\,\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}, \tag{22}$$

we find that

$$\begin{aligned} v(r,t) = {} & \frac{fr^2}{2R}\left\{\frac{\alpha_1}{\mu^2}[\exp(-\frac{\mu t}{\alpha_1})-1]+\frac{t}{\mu}\right\} - \frac{2f}{\alpha_1 R}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^2 J_0(Rr_n)}\sum_{k=0}^{\infty}(-\nu r_n^2)^k \\ & \times \int_0^t G_{1,0,1}(-\mu/\alpha_1, s)G_{0,-k-2,k+1}(-\alpha r_n^2, t-s)\,ds, \end{aligned} \tag{23}$$

which can be simplified by using the following relations

$$G_{0,-k-2,k+1}(-\alpha r_n^2, t) = \sum_{j=0}^{\infty}(-\alpha r_n^2)^j \frac{\Gamma(k+j+1)}{\Gamma(k+1)\Gamma(j+1)}\,\frac{t^{k+1}}{\Gamma(k+2)} = \frac{t^{k+1}}{(k+1)!}\,\frac{1}{(1+\alpha r_n^2)^{k+1}}, \tag{24}$$

$$\begin{aligned} \sum_{k=0}^{\infty}(-\nu r_n^2)^k G_{0,-k-2,k+1}(-\alpha r_n^2, t) &= \frac{-1}{\nu r_n^2}\sum_{k=0}^{\infty}\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)^{k+1}\frac{1}{(k+1)!} \\ &= \frac{1}{\nu r_n^2}\left[1-\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right], \end{aligned} \tag{25}$$

and

$$G_{1,0,1}(-\mu/\alpha_1, t) = \exp(-\frac{\mu t}{\alpha_1}). \tag{26}$$

Now the velocity field $v(r,t)$ has form

$$v(r,t) = \frac{fr^2}{2\mu R}(t-\frac{\alpha_1}{\mu}) - \frac{2f}{\mu\nu R}\sum_{n=1}^{\infty}\left[1-(1+\alpha r_n^2)\exp\left(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\right)\right]\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}. \tag{27}$$

§4. Calculation of the shear stress

Applying the Laplace transform to Eq. (5), we find that

$$\bar{\tau}(r,q) = (\mu+\alpha_1 q)\frac{\partial\bar{v}(r,t)}{\partial r}. \tag{28}$$

Differentiating Eq. (19) with respect to r and using the identity

$$\frac{d}{dr}J_0(rr_n) = -r_n J_1(rr_n),$$

we find $\bar{\tau}(r,q)$, after using Eq. (28)

$$\bar{\tau}(r,q) = \frac{fr}{Rq^2} + \frac{2f}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n J_0(Rr_n)}\,\frac{1}{q(q+\alpha r_n^2 q+\nu r_n^2)}. \tag{29}$$

Applying inverse Laplace transform to Eq. (29) by using (21) and (25), we get the shear stress $\tau(r,t)$

$$\tau(r,t) = \frac{frt}{R} + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}\Big[1 - \exp\Big(-\frac{\nu r_n^2 t}{1+\alpha r_n^2}\Big)\Big]. \tag{30}$$

§5. Limiting case $\alpha_1 \to 0$

Making $\alpha_1 \to 0$ into Eqs. (27) and (30), we obtain the velocity field

$$v(r,t) = \frac{fr^2 t}{2R\mu} - \frac{2f}{R\nu\mu}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n^4 J_0(Rr_n)}\Big(1 - e^{-\nu r_n^2 t}\Big), \tag{31}$$

and the associated shear stress

$$\tau(r,t) = \frac{frt}{R} + \frac{2f}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)}{r_n^3 J_0(Rr_n)}\Big(1 - e^{-\nu r_n^2 t}\Big), \tag{32}$$

corresponding to a Newtonian fluid, performing the same motion.

Eqs. (31) and (32) are identical with those found by W. Akhtar *et al.* [1].

§6. Conclusions

In this paper, the velocity field and the associated shear stress corresponding to the axial flow of second grade fluids through a circular cylinder are determined. The motion is due to a longitudinal shear stress which is prescribed on the boundary of the cylinder. More exactly, at the moment $t = 0^+$ the cylinder is pulled with a time dependent shear stress along its axis.

The solutions determined by means of the Laplace and finite Hankel transforms satisfy all imposed initial and boundary conditions. The corresponding solutions for Newtonian fluids, performing the same motion, are obtained as limiting case from our solutions. Finally, in Figs. 1 and 2, the profiles of the velocity and shear stress of the second grade fluid (curves $v(r)$ and $\tau(r)$) and Newtonian fluid (curves $vN(r)$ and $\tau N(r)$) are plotted as function of r for different values of the time t. From these figures we have that for low values of the time t the second grade fluid flows slower than the Newtonian fluid and this difference disappear when the values of the time increase.

In all figures we consider $R = 0.1$, $f = 2$, $\rho = 1260$, $\mu = 1.48$, $\alpha = 80$. The units of parameters in Figs. 1 and 2 are from SI units and the roots r_n have been approximated with [9] $r_n = (4n-1)\pi/(4R)$.

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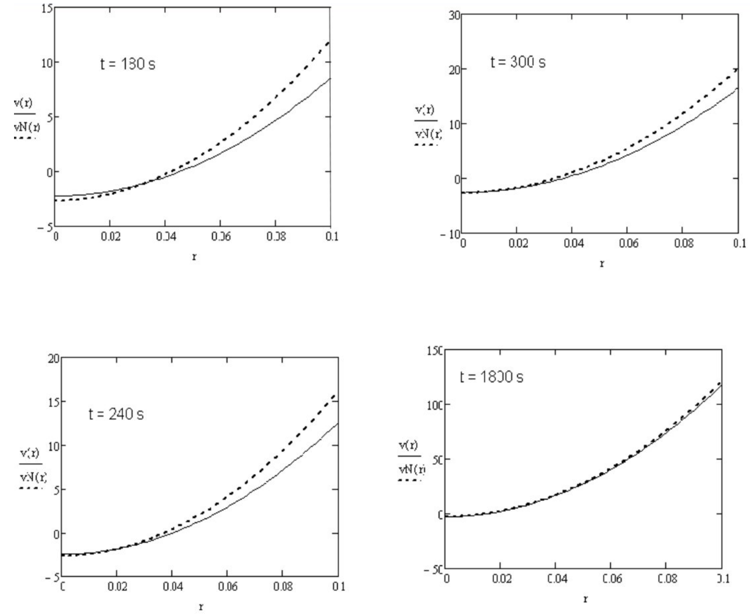

Figure 1: Velocity profiles $v(r)$ for different values of the time t: $v(r)$– the second grade fluid, $vN(r)$–the Newtonian fluid.

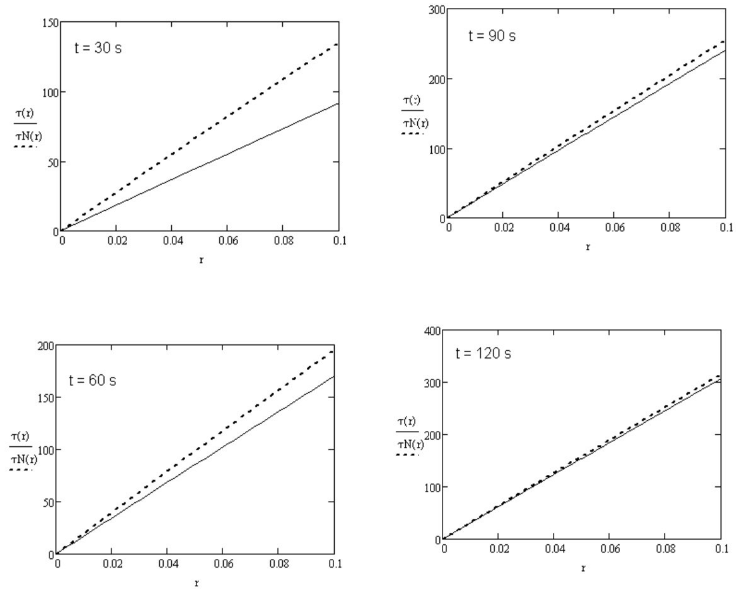

Figure 2: The profiles of the shear stress $\tau(r)$ for different values of the time t: $\tau(r)$– the second grade fluid, $\tau N(r)$– the Newtonian fluid.

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