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STABILITY OF EQUATORIAL AND HALO ORBITS AROUND A NON-SPHERICAL MAGNETIC PLANET

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Abstract. The presence of micron size dust particles is frequent in the solar system and their dynamics have attracted the attention of researchers from the very beginning. Indeed, dusty rings of giant planets can be modeled by very simple models that take into account the movement of a single particle. One of these models is the so-called generalized Störmer problem, where a charged particle is supposed to orbit a spherical planet with magnetosphere. In this case, it is known the presence of equatorial and circular halo orbits as well as their stability. However, planets are not perfect spheres, and their oblateness must be taken into account. The aim of this paper is to show how the oblateness of the body affects the existence and stability of equatorial and halo orbits.

Keywords: Planetary magnetosphere, Störmer problem, equatorial and halo orbits, oblateness, equilibria, stability.

AMS classification: 70F05, 70F15, 70H08, 70H12, 70H14.

§1. Introduction

The understanding of the dynamics of planetary tiny dusty rings is usually studied by means of a single particle model named, in the literature, the generalized Störmer problem [1]. This model describes the dynamics of a dust particle orbiting a rotating magnetic planet and it takes into account the gravitational and magnetic effects. In this work we will consider that the magnetic field is a perfect magnetic dipole aligned along the north–south poles of the planet and the planet's magnetosphere is a rigid conducting plasma which rotates with the same angular velocity as the planet, which entails that the charge is subject to a corotational electric field. Finally we will suppose that the planet is not spherical. This last assumption introduces an additional perturbation to the previous works of Howard et al. [4, 3], Dullin et al. [1], Grotta–Ragazzo et al. [2], where the spherical case is considered. The aim of this paper is to study the influence of the oblateness coefficient in the existence and stability of circular orbits paralell to the equator or lying in it, that is to say, halo orbits and equatorial orbits respectively.

The paper is structured in three sections. The first one includes the Hamiltonian formulation of the problem. Second section is devoted to analyze the existence of equatorial and halo orbits. Some results about the stability of circular orbits appear in the third section.

Now, we start with the formulation of the problem. After using dimensionless cylindrical coordinates and momenta $(\rho, z, \phi, P_\rho, P_z, P_\phi)$ and adding the influence of the oblateness to the generalized Störmer problem, the system can be modeled by the following two degrees of freedom Hamiltonian function (see [1, 5, 6] for details)

$$\mathcal{H} = \frac{1}{2}\left(P_\rho^2 + P_z^2 + \frac{P_\phi^2}{\rho^2}\right) - \frac{1}{r} - \delta\frac{P_\phi}{r^3} + \frac{\delta^2}{2}\frac{\rho^2}{r^6} + \delta\beta\frac{\rho^2}{r^3} + 3J_2\frac{z^2}{2r^5} - \frac{J_2}{2r^3}, \tag{1}$$

where r is the distance of the particle to the center of the mass of the planet and lengths and time are expressed, respectively, in units of the planetary radius R and the Keplerian frequency $w_K = \sqrt{M/R^2}$. The problem depends on five parameters. Three external parameters: δ, the ratio between magnetic and Keplerian interactions (charge–mass ratio); β, the ratio between electrostatic and Keplerian interactions; and J_2, the oblateness coefficient of the planet. The sign of J_2 indicates if the planet is oblate ($J_2 > 0$) or prolate ($J_2 < 0$). The other two parameters are internal ones: P_ϕ, the angular momentum and $\mathcal{H} = \mathcal{E}$, the energy of the system.

Circular periodic trajectories appear as equilibria of the Hamiltonian system

$$\dot{\rho} = \frac{\partial\mathcal{H}}{\partial P_\rho}, \qquad \dot{z} = \frac{\partial\mathcal{H}}{\partial P_z}, \qquad \dot{P}_\rho = -\frac{\partial\mathcal{H}}{\partial \rho}, \qquad \dot{P}_z = -\frac{\partial\mathcal{H}}{\partial z},$$

or equivalently, as critical points of the generalized potential energy function (called effective potential) that can be written as

$$U_{eff} = \frac{P_\phi^2}{2\rho^2} - \frac{1}{r} - \delta\,\frac{P_\phi}{r^3} + \frac{\delta^2}{2}\frac{\rho^2}{r^6} + \delta\beta\frac{\rho^2}{r^3} + 3J_2\frac{z^2}{2r^5} - \frac{J_2}{2r^3}.$$

As it is usual in the literature, we introduce the particle angular velocity,

$$\omega = \dot{\phi} = \frac{\partial\mathcal{H}}{\partial P_\phi} = \frac{P_\phi}{\rho^2} - \frac{\delta}{r^3},$$

to eliminate P_ϕ, because ω is a more interesting parameter from the point of view of applications. To simplify the calculations, we also move to spherical variables (r, θ, ϕ) given by

$$\rho = r\sin\theta, \quad z = r\cos\theta, \quad \theta \in [0, \pi/2].$$

With these changes, circular orbits are obtained as the solutions of the nonlinear system of equations

$$\begin{cases} -6J_2 + 2r^2 + (9J_2 - 2\delta(\beta - \omega)r^2 - 2r^5\omega^2)\sin^2\theta = 0, \\ (-3J_2 + 2\delta(\beta - \omega)r^2 - r^5\omega^2)\sin 2\theta = 0. \end{cases} \tag{2}$$

Two types of equilibria, or circular orbits, appear depending on whether $\sin 2\theta$ is equal to zero or not. The first one occurs for $\sin 2\theta = 0$ and then $\theta = 0$ or $\theta = \pi/2$. If $\theta = 0$ then $\rho = 0$ and it constitutes a degenerate case, only meaningful for $J_2 > 1/3$. If $\theta = \pi/2$ we find the equatorial orbits. The second case takes place for $\sin 2\theta \neq 0$, and it gives rise to circular orbits parallel to the equator, also called halo orbits.

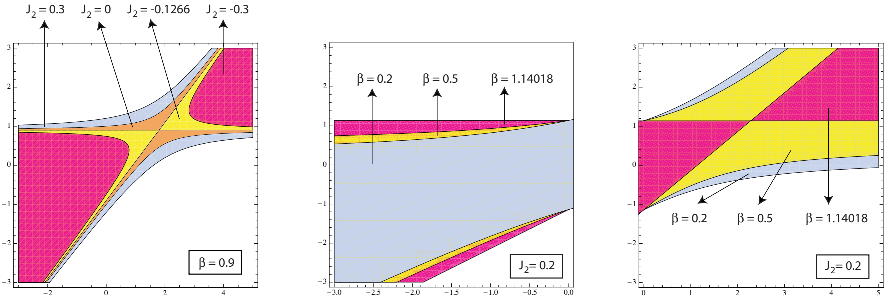

Figure 1: Regions of existence of equatorial orbits fixed β (left figure) and fixed J_2 (central and right figure) in the δ–ω plane.

§2. Circular orbits

An important question is to establish the conditions under which each type of circular orbit exists. We start discussing the case of equatorial orbits.

2.1. Equatorial orbits

As $\theta = \pi/2$, the second equation of the nonlinear system (2) is always verified and, for the first equation to be satisfied, r must be a positive real root of the following polynomial equation in the variable r

$$3\,J_2 + 2(1 - \beta\delta + \delta\omega)\,r^2 - 2\,\omega^2\,r^5 = 0. \tag{3}$$

Each positive real root of (3) corresponds to an equatorial orbit. Langbort [7] and Dullin et al. [1] have studied, respectively, some particular cases, when the particle is not charged ($\delta = 0$) and when the planet is spherical ($J_2 = 0$). Assuming $\delta \neq 0$ and $J_2 \neq 0$, some different results about the existence of equatorial orbits are obtained. They can be summarized in the following propositions (for details the reader is referred to [6]).

Proposition 1. *The region of existence of equatorial orbits enlarges for increasing values of J_2, fixed β. If J_2 is fixed, the region of existence enlarges or diminishes with β depending on the sign of the charge of the particle.*

Proof. The proof of this proposition, as well as the subsequent ones, is based on the analysis of the discriminant of the polynomial in equation (3), and on the fact that the radius of the orbit must be greater than one to be meaningful. Therefore, two curves appear delimiting the region of existence of equatorial orbits:

$$3125 J_2^3 \omega^4 + 32(1 - \beta\delta + \delta\omega)^5 = 0, \tag{4}$$

$$2 - 2\beta\delta + 3J_2 + 2\delta\omega - 2\omega^2 = 0. \tag{5}$$

A detailed discussion of (4) and (5), in terms of the parameters β and J_2, yields the desired result. An illustration is given in Figure 1. □

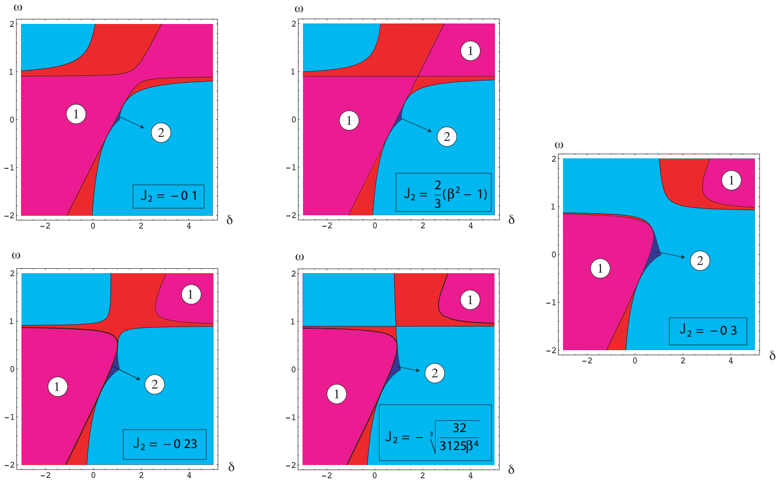

Figure 2: Number of equatorial orbits for $\beta = 0.9$ and different values of $J_2 < 0$.

In the proof of Proposition 1 it can be seen that for a prolate planet there may exist two positive real roots of the equation (3). In this case, it is interesting to know when these two roots give rise to two meaningful equatorial orbits, with radius greater than one. In this sense we obtain the following result.

Proposition 2. *If $J_2 < 0$, there is a region where two equatorial orbits with $r > 1$ exist at the same time.*

Proof. The region is defined by the contact points of the limiting curves (4) and (5). As the contact points are function of β and J_2, this region varies as the parameters change, as it is showed in Figure 2. □

It is worth noting that the region with two equatorial orbits is, in general, small in comparison with the region with only one equatorial orbit.

2.2. Halo orbits

The discussion about the existence of halo orbits is more difficult. Now, as $\theta \neq \pi/2$, none of the equations (2) vanishes identically and we are left to the equivalent system:

$$-3J_2 + 2\delta(\beta - \omega)r^2 - r^5\omega^2 = 0, \tag{6}$$

$$\sin^2\theta = \frac{6J_2 - 2r^2}{9J_2 - 2\delta(\beta - \omega)r^2 - 2r^5\omega^2)}. \tag{7}$$

The influence of J_2 in the region of existence of halo orbits can be summarized in the following two propositions.

Proposition 3. *The region of existence of halo orbits diminishes as J_2 increases for fixed β. Besides, if $J_2 > 0$, there is a range of charge-mass ratio not allowed for a particle to be in*

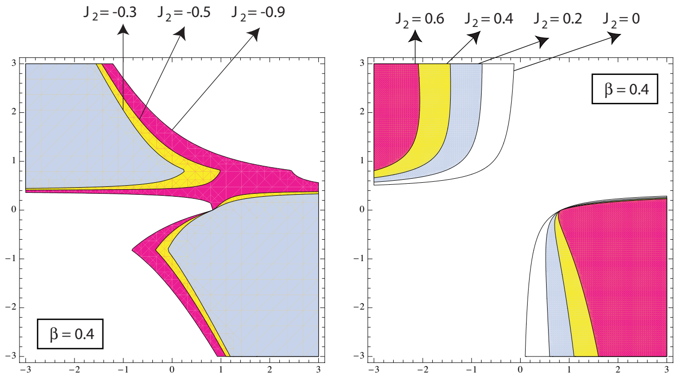

Figure 3: Region of existence of halo orbits for fixed β.

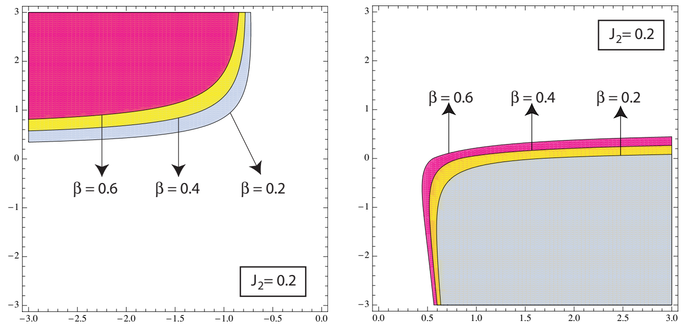

Figure 4: Region of existence of halo orbits for fixed J_2.

halo orbit. If J_2 is fixed, the region enlarges or diminishes with β depending on the sign of the charge.

Proof. The result follows from analysis of polynomial equation (6) and equation (7), taking into account that $0 \leq \sin^2\theta \leq 1$ and $r > 1$. As in the equatorial case, we find several limiting curves which depend on the parameters β and J_2. We arrive to the desired conclusion by the discussion of the limiting curves. Figures 3 and 4 illustrate the results, where Figure 3 shows the difference between oblate and prolate cases. Note how the gap of charge-mass ratios increases with the oblateness. □

Proposition 3 considers halo orbits with $r > 1$, which is a strong constrain for non-spherical bodies. Besides, it focuses on the existence of at least one halo orbit, but not in the number of them. Next proposition solves these aspects, which are illustrated in Figure 5.

Proposition 4. *There is a region where two halo orbits exist at the same time if $J_2 > 1/3$. This limit reduces to $J_2 > 1/8$ if the body is a homogeneous ellipsoid of revolution.*

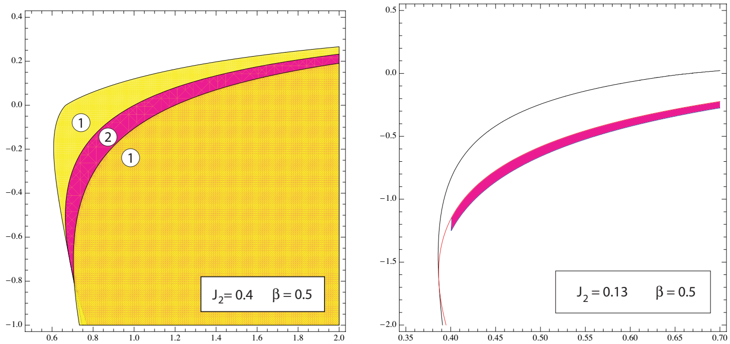

Figure 5: Region of existence of halo orbits for an oblate planet.

§3. Stability

Beyond the existence of circular orbits, their stability is an important question, as it determines the persistence of them with time. In this way, the stability follows from their character as critical points of the effective potential. Thus, if the Hessian matrix has two positive eigenvalues at the corresponding equilibrium, it is stable. The entries of the Hessian matrix are given by the second order partial derivatives of the effective potential

$$\frac{\partial^2 U_{eff}}{\partial r^2} = \frac{(\delta^2 + 2\beta\delta r^3 - 6\delta\omega r^3 + 3\omega^2 r^6)\sin^2\theta + 18J_2 r\cos^2\theta - 6J_2 r - 2r^3}{r^6},$$
$$\frac{\partial^2 U_{eff}}{\partial \theta^2} = \frac{2(\delta + \omega r^3)^2 + [2\delta^2 - 3J_2 r + \omega^2 r^6 + 2\delta(\beta + \omega)r^3]\cos 2\theta}{r^4},$$
$$\frac{\partial^2 U_{eff}}{\partial r\partial \theta} = \frac{-2\delta^2 + 9J_2 r - 2\delta(\beta - 2\omega)r^3 + 2\omega^2 r^6}{r^5}\sin\theta\cos\theta.$$

3.1. Stability of equatorial orbits

Here we will only discuss the stability of equatorial orbits. In this case the crossed derivative vanishes and the stability decouples in the radial direction (along the equator) and the vertical direction (away the equator), given by the eigenvalues

$$\lambda_r = \delta^2 - 6J_2 r - 2r^3 + 2\beta\delta r^3 - 6\delta r^3\omega + 3r^6\omega^2,$$
$$\lambda_\theta = 3J_2 - 2\beta\delta r^2 + 2\delta r^2\omega + r^5\omega^2.$$

Exploiting the idea that if $\lambda_r = 0$ or $\lambda_\theta = 0$, a change in the stability occurs, we arrive to the following results which are illustrated in Figures 6, 7 and 8.

Proposition 5. *The area of radial stability enlarges when J_2 decreases, fixed β. The contrary for the stability away the equator.*

Proposition 6. *For fixed J_2, the region of stability, both radial and away the equator, enlarges if β increases and the charge is negative. For positive charged particles the region of stability away the equator diminishes for increasing β.*

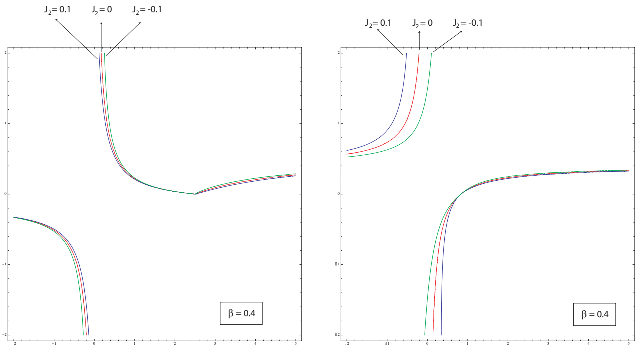

Figure 6: Regions of radial and vertical stability (respectively, left and right graphs) for equatorial orbits fixed β.

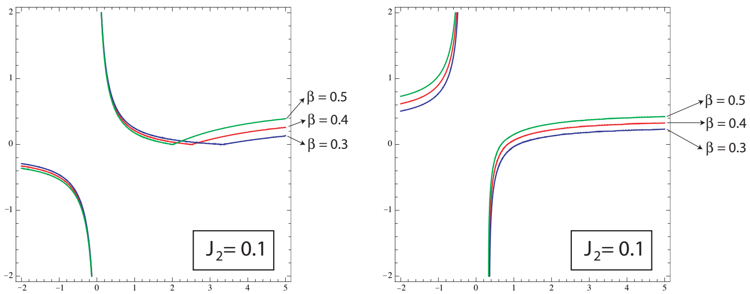

Figure 7: Regions of radial and vertical stability (respectively, left and right graphs) for equatorial orbits fixed J_2.

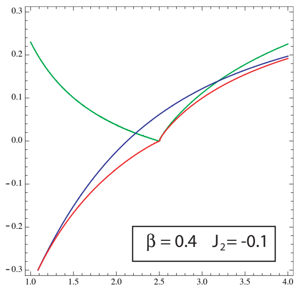

Figure 8: Changes of stability in the region with two equatorial orbits.

Proposition 7. *In the region with two equatorial orbits, the larger one suffers two changes of stability in the radial direction for positive charged particles.*

Proposition 8. *If $\lambda_r = 0$, a saddle-center bifurcation takes place, whereas, if $\lambda_\theta = 0$, there is a pitchfork bifurcation.*

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