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SIMULATION OF RAINFALL EVENTS AND OVERLAND FLOW

Olivier Delestre and François James

Abstract. We are interested in simulating overland flow on agricultural fields during rainfall events. The model considered is the shallow water system (or Saint-Venant equations) without infiltration, complemented with a friction term. In this context, we definitely have to cope with dry/wet interfaces and water inflow on dry soil. We present a simplified one-dimensional model, discretized with a well-balanced finite volume method, and we describe the specific additional features needed to deal with dry/wet transitions and steady-state solutions due to topography and friction. The method as well as the choice of the friction term are tested and discussed both on analytical solutions and experimental results.

Keywords: Shallow water equations, finite volume schemes, well-balanced schemes, hydrostatic reconstruction, friction laws, rainfall hydrograph, analytical solution, dry/wet transitions.

AMS classification: 76M12, 74G05, 74G75, 35L65, 20C20.

Introduction

Rain on agricultural fields can yield to overland flow. This flow may have some undesirable effects. At the field scale, we can have soil erosion and pollutant transport. Downstream the fields, roads and houses may be damaged. To prevent these effects, control measures can be taken, such as grass strips. But one must know how the water is flowing in order to place efficiently these developments. In the spirit of [6, 7], we try to model these phenomenon by using the shallow water (or Saint-Venant) equations. Efficient numerical simulations are of great help in this context, because field measurements, such as velocities or water heights, are very difficult to obtain, especially during the rain event, which is quite unpredictible.

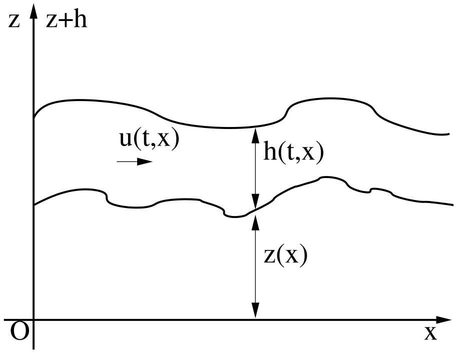
The aim of this paper is not to give a complete account on the problem, which has to be thaught of as a multi-scale problem: one has to deal with roughness induced at the decimeter scale (e.g. by furrows on agricultural surfaces), flows at the scale of ten square meters, which is the scale of the numerical topography data, and also the agricultural field itself, whose surface is of the order of the hectare. We give here a short review of the shallow water equations, with emphasis on some specific aspects in this context. Namely, since the rain is an intermittent phenomenon, we definitely have to cope with dry/wet transitions, a problem analogous to the vacuum apparition in gas dynamics. More classically in shallow water problems, we have to take into account carefully the interactions between the soil topography and the friction of water on the soil, which eventually lead to steady-state solutions that have to be computed accurately.

For this introduction to the topic, we deliberately use a simplified model, firstly by considering one-dimensional flows. This is enough to understand the ideas of the numerical methods, which can be developed in two space dimensions on a rectangular mesh. Next, from a more practical viewpoint, we neglect importants phenomena, which deserve a complete modelling: infiltration and soil erosion. Infiltration appears as a supplementary source term in the shallow water equations, and can be treated quite easily, when a relevant model is chosen. Erosion is a much more complex problem.

We begin by a short review of the shallow water system, recalling a few basic properties. Next, we describe numerical methods adapted to the situation, in particular we discuss briefly the discretization of the friction terms. Finally, we give several illustrations of the results. First we justify the choice of the method by comparison with analytical solutions. Next, we show an attempt of recovering experimental results, with a clear evidence that the choice of the friction laws is not obvious. The last section is devoted to an unstability phenomenon wich occurs when perturbating steady-state solutions (with rain for instance): the so-called roll-waves.

§1. Model

The model we consider here are the so-called shallow-water equations, which are convenient for small heights of water, according to the following scheme



The unknowns are here the velocity of the water $u(t,x)$, and its height $h(t,x)$. The shape of the bottom is also called the topography, it is a given function z. For our specific application, the model has to be complemented by taking int account friction on the soil and rain. Therefore the equations are

$$\partial_t h + \partial_x(hu) = R(t), \qquad \partial_t(hu) + \partial_x\Big(hu^2 + \frac{gh^2}{2}\Big) = -gh\Big(\partial_x z + S_f\Big), \tag{1}$$

where g is the gravity constant, $R(t)$ the rain intensity, assumed constant in space, and $S_f(h,u)$ the friction term. Notice that infiltration in the soil can be accounted by a source term in the first equation like $R(t) - I(t,x)$, where I is a given function. We shall denote by $q = hu$ the water flow, or discharge. The typical practical configuration we consider is a channel with finite length L, so that the system must be set on the interval $]0,L[$, and complemented with boundary conditions at inflow and outflow we do not detail here, see an example in Section 3.

Concerning the friction term, it is a given function of h and u, two examples widely used in hydrology (see for instance [6, 7, 8, 9]) are the Manning and the Darcy-Weisbach friction laws, which are given respectively by

$$S_f = -\frac{k^2 u|u|}{h^{4/3}} = -\frac{k^2 q|q|}{h^{10/3}}, \qquad S_f = -\frac{ku|u|}{8gh} = -\frac{kq|q|}{8gh^3}, \tag{2}$$

where $k > 0$ stands for the roughness coefficient. Both laws are derived from empirical considerations, in particular in the context of pipelines. The problem of their relevance in the present context of overland flow is difficult.

The system can be rewritten in a more compact form by setting

$$U = \begin{pmatrix} h \\ q \end{pmatrix}, \quad F(U) = \begin{pmatrix} q \\ q^2/h + gh^2/2 \end{pmatrix}, \quad B = \begin{pmatrix} R \\ -gh\left(\partial_x z + S_f\right) \end{pmatrix}.$$

We obtain therefore

$$\partial_t U + \partial_x F(U) = \partial_t U + F'(U)\partial_x U = B.$$

The system is by definition hyperbolic if the matrix $F'(U)$ admits a basis of eigenvectors with real eigenvalues, strictly hyperbolic if the eigenvalues are distinct. An easy computation shows that the shallow water system is strictly hyperbolic provided $h > 0$, with eigenvalues $\lambda_-(U) = u - \sqrt{gh}$, $\lambda_+(U) = u + \sqrt{gh}$. When $h = 0$, the system is no longer hyperbolic, actually it is rather meaningless, since $h = 0$ means that there is no water, so that the velocity u cannot be defined. This is exactly the problem of the vacuum in the Euler equations of fluid mechanics, and leads to severe numerical problems, which cannot be avoided in our context since we consider rain on dry soils.

At this point, we introduce an important quantity, the so-called Froude number

$$Fr = \frac{u}{\sqrt{gh}}. \tag{3}$$

This dimensionless number plays the same role as the Mach number in fluid mechanics, and allows to classify the flows:

- $Fr < 1$ subcritical flow, as in a river (corresponding to subsonic flow in fluid mechanics);
- $Fr > 1$ supercritical flow, as in a torrent (subsonic flow);
- $Fr = 1$ critical flow (transonic flow).

The differences between these flows can be easily experimented by observing the surface waves obtained by throwing a stone in a river.

§2. Numerical method

The shallow water system is discretized by a finite volume method on a fixed time-space grid. A time step $\Delta t > 0$ and a space step $\Delta x > 0$ are fixed, we set $x_i = i\Delta x$, and the interval $]x_i - \Delta x/2, x_i + \Delta x/2[$ will be referred to as the cell i. The finite volume scheme can be written in a compact form as

$$\frac{d}{dt}U_i + \frac{1}{\Delta t}(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}) = S_i, \tag{4}$$

where the vector U_i is an approximation of the conservative variables in the cell i, $\mathcal{F}_{i+1/2}$ is the numerical flux at the interface between cells i and $i+1$, and S_i a discretization of the source term. Boundary conditions are treated by the method of characteristics (see [4]). The scheme is completely determined once the numerical flux and the source term discretization have been fixed. These choices are not independant one from the other.

Indeed it is well-known that source terms in hyperbolic systems of conservation laws give rise to serious problems. The main difficulty is to find schemes that preserve equilibria (steady-states solutions). In system (1), the main problems are due to

– topography: pools, lakes;

– friction terms: balance between kinematics and friction.

The rain source term can be treated by a second-order accurate Strang type splitting.

Schemes that preserve equilibria are known as well-balanced schemes. The strategy to obtain such schemes consists in choosing first a consistent numerical flux for the system without source terms. Next, a correction is given to take into account equilibria. The reader can find all the details and a large bibliography in the book [3]. We merely give a sketch of the method here, with emphasis on the problem of friction. The numerical flux is the so-called HLL flux, and the order 2 is obtained in space by a MUSCL type reconstruction, in time by Runge Kutta (Heun) (see [3] for details). Notice that dry/wet transitions imply a specific reconstruction for the water height, not only for the velocity as usual (see [1]).

First we consider the equilibria for topography. They are given by

$$hu = Cst, \qquad u^2/2 + g(h+z) = Cst.$$

However a complete resolution of these equations would lead to a far too time consuming scheme. Thus, following [3, 1, 2], we limit ourselves to the equibria at rest:

$$u = 0, \qquad g(h+z) = Cst.$$

This procedure is known as the (second order) hydrostatic reconstruction, and it turns out to give good results at an acceptable numerical cost. We refer the reader interested into details to the preceding references.

Now we turn to friction terms, which can be treated by two different means. The first one aims at building a well-balanced scheme for friction as well as topography, is the apparent topography method, introduced by [3]. It consists in building an modified topography z_{app} which takes into account the friction, as follows:

$$z_{app} = z - b, \qquad \text{with } \partial_x b = S_f.$$

We proceed then exactly as before, with this new topography (detailed computations for the friction laws (2) can be found in [5]). This gives rise to a scheme which computes neatly equilibrium states, but is not completely satisfactory on transition solutions, as we shall see in the next section.

Therefore we turned to a splitting method, and we chose the semi-implicit treatment proposed in [4], not only because it preserves steady states at rest, but also for its stability. For the Darcy-Weisbach friction law (2)-right, it writes

$$q_i^{n+1} + \frac{f|q_i^n|q_i^{n+1}}{8h_i^n h_i^{n+1}}\Delta t = q_i^n + \frac{\Delta t}{\Delta x_i}(\mathcal{F}_{i+1/2G} - \mathcal{F}_{i-1/2D}),$$

where the right-hand side is nothing more than the discharge obtained at each step of the second order in time Runge-Kutta reconstruction. Notice also the simplicity of the method, which gives an explicit value for q_i^{n+1}. Now we illustrate these ideas on a set of analytical solutions.

§3. Analytical solutions

Here we present briefly an adaptation to the $1-d$ case and our friction laws of an idea presented in [8, 9] for pseudo two dimensional cases. At steady states, we have $\partial_t h = \partial_t u = \partial_t q = 0$, thus the mass-conservation equation gives $q = cst$ and we get the equation

$$\partial_x z = \left(\frac{q^2}{gh^3} - 1\right)\partial_x h + S_f(q, h) \tag{5}$$

where $S_f(q, h)$ depends on the friction law chosen, for instance (2). For any given value of the constants k and q, once we are given an explicit expression for $h(x)$, then formula (5) allows us to compute the topography corresponding to this steady state and this water height. Other friction laws can of course be chosen.

As an example, we consider a channel of length 1000 m, with a specified water height $h(x)$ given by

$$h(x) = \left(\frac{4}{g}\right)^{1/3}\left(1 + \frac{1}{2}\exp\left(-16\left(\frac{x}{1000} - \frac{1}{2}\right)^2\right)\right).$$

The friction model is the Manning law, with roughness coefficient $k = 0.033$. The topography is calculated iteratively thanks to (5). To make use of the shallow water system, we have now to impose boundary conditions. Since the flow is subcritical both at inflow $x = 0$ and outflow $x = 1000$, we have to impose the value of one quantity at inflow and one at outflow. We choose to put a discharge of $q = 2$ m^2/s at inflow and a water height corresponding to the value of $h(1000)$ downstream.

We first compare the results obtained by the apparent topography and the semi-implicit scheme in preserving the equilibrium state. It turns out that both methods preserve correctly the steady state along time, as is evidenced by fig. 1.

Since for our application we are particularly interested in non-stationary solutions, we have considered an initially dry soil and the upstream discharge $q = 2$ m^2/s, and computed the unsteady solution up to equilibrium. Both methods (apparent topography and semi-implicit treatment) converge towards the steady state, with slightly better results with the apparent topography method. However, before the steady state is reached, we have a wet/dry transition (fig. 2). We note that the apparent topography method is not adapted to this transition: we have a peak in the velocity profile (fig. 2-left), which appears also in the water height profile. With the semi-implicit treatment, the water height profile is very clean (fig. 2-right).

In figure 3, two more examples of computation of steady states are displayed, both with sub- and supercritical inflows and outflows, and using the semi-implicit method. The numerical scheme deals in particular with transition from one regime to the other, including hydraulic jumps (fig. 3-right).

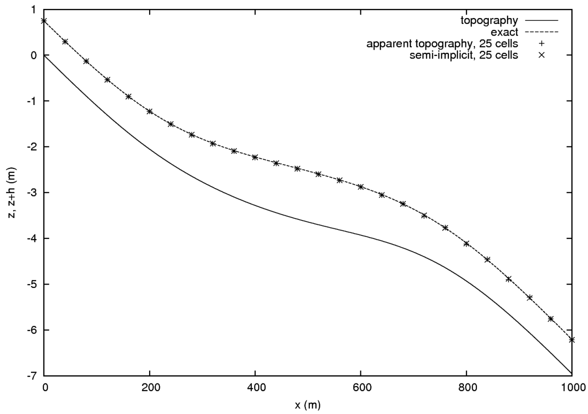


Figure 1: Steady state solution, subcritical inflow and outflow: apparent topography +, semi-implicit ×, analytical −.

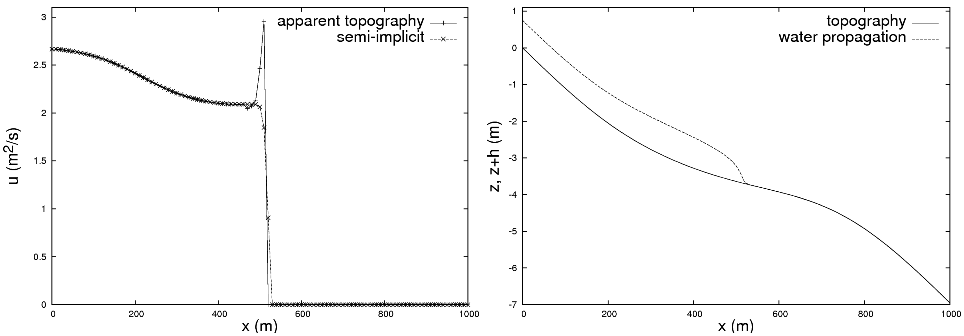


Figure 2: Left: water front velocities at $t = 200$ s: apparent topography (+), semi-implicit treatment (×). Right: water front height at $t = 200s.$, semi-implicit

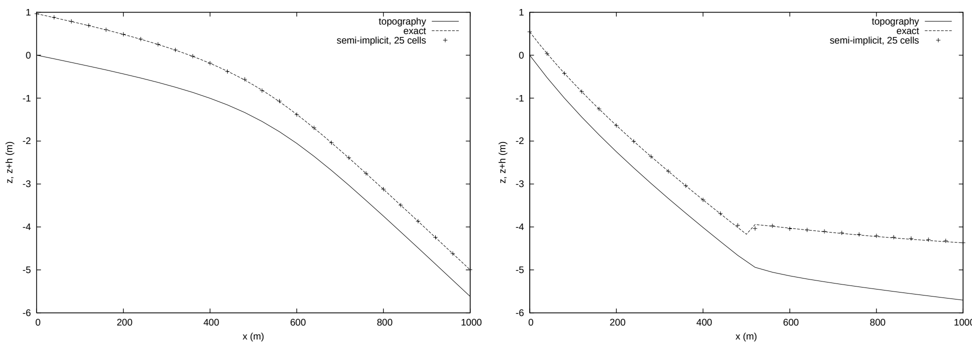


Figure 3: Steady state solution, numerical (symbols) vs analytical (lines). Left: subcritical inflow and supercritical outflow, right: supercritical inflow and subcritical outflow.

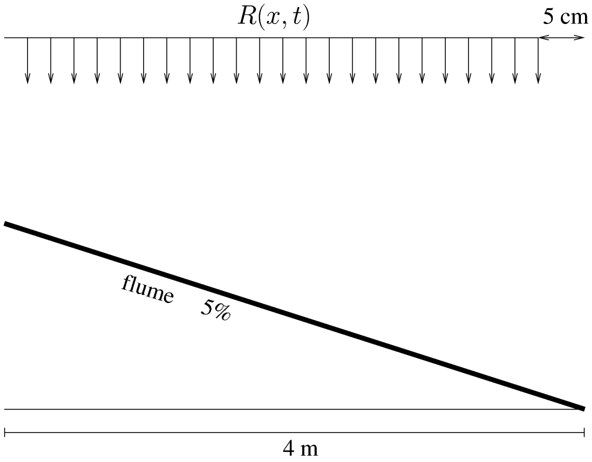


Figure 4: Experimental configuration.

§4. Rainfall hydrograph test

In this section we present another test case, based on experimental measurements realized thanks to the ANR project METHODE in a flume at the rain simulation facility at INRA-Orléans. The flume is 4 m long with a slope of 5% (fig. 4). The simulation duration is 250 s. The rainfall intensity $R(x,t)$ is described by

$$R(x,t) = \begin{cases} 50 \text{ mm/h} & \text{if } (x,t) \in [0, 3.95 \text{ m}] \times [5, 125 \text{ s}], \\ 0 & \text{otherwise.} \end{cases}$$

For this test, dry/wet transitions are involved, since on the one hand there is no rain on the last 5 cm of the flume, on the other hand rain falls on a dry soil. The measured output is an hydrograph, that is a plot of the discharge versus time (see fig. 5).

The mathematical model for this ideal overland flow is the following. We consider a uniform plane catchment whose overall length in the direction of flow is L. The surface roughness and slope are assumed to be constant in space and time. The friction law is the Darcy-Weisbach one. We consider a constant rainfall excess such that

$$R(x,t) = \begin{cases} I & \text{for } 0 \le t \le t_d, 0 \le x \le L, \\ 0 & \text{otherwise,} \end{cases}$$

where I is the rainfall intensity and t_d is the duration of the rainfall excess. First we compute some explicit "naive" analytical solution to the problem. We notice that three phases can clearly be identified on the hydrograph: a first non-steady step at the beginning of the rainfall event, then a steady-state and lastly another non-steady step when rain stops. The first and the second step solutions can be computed explicitly, and the "naive" solution is obtained by assuming a simple concatenation of the two parts (we refer to [5] for the detailed computations).

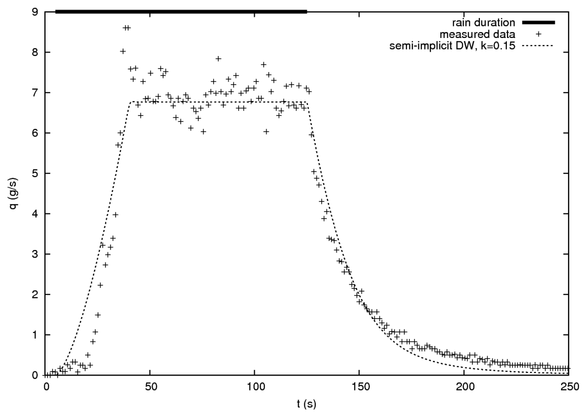


Figure 5: Comparison between experimental measures (+) and numerical results (−).

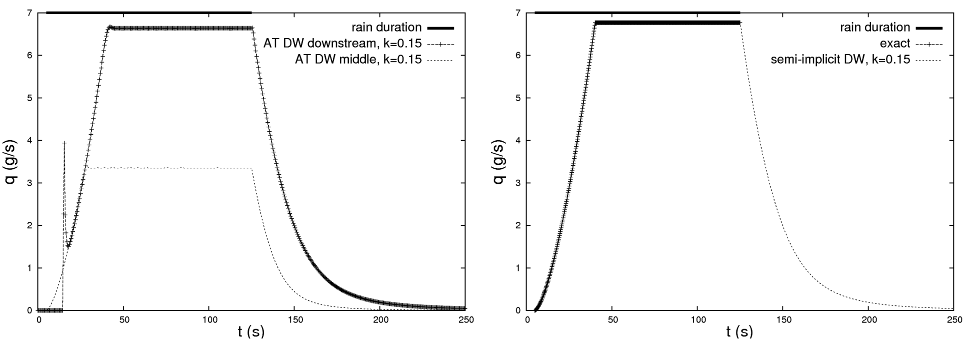


Figure 6: Computed rainfall hydrographs for Darcy-Weisbach's law (DW). Left: apparent topography method (AT). Right: semi-implicit scheme.

At first we compare numerical results with the analytical "naive" solution. Once again, with (fig. 6-a) we show that with the apparent topography method, we get a peak on the discharge downstream that we do not get far from this transition. With the semi-implicit method, we do not have this peak (fig. 6-b). This treatment gives good results close to the "naive" exact solution. The hydrograph is well calculated (fig. 6-b), notice here the computed hydrograph at the middle of the flume, a quantity hardly accessible by experiment.

Next, we propose a comparison between experimental measurements and numerical simulation (fig. 5), obtained with the Darcy-Weisbach friction law. We obtain a reasonable agreement, but it turns out that it is impossible to fit correctly the shape of both the increasing and decreasing parts of the hydrograph. This indicates clearly that the model has to be modified, for instance by choosing alternative friction laws, but this is beyond the scope of this paper.

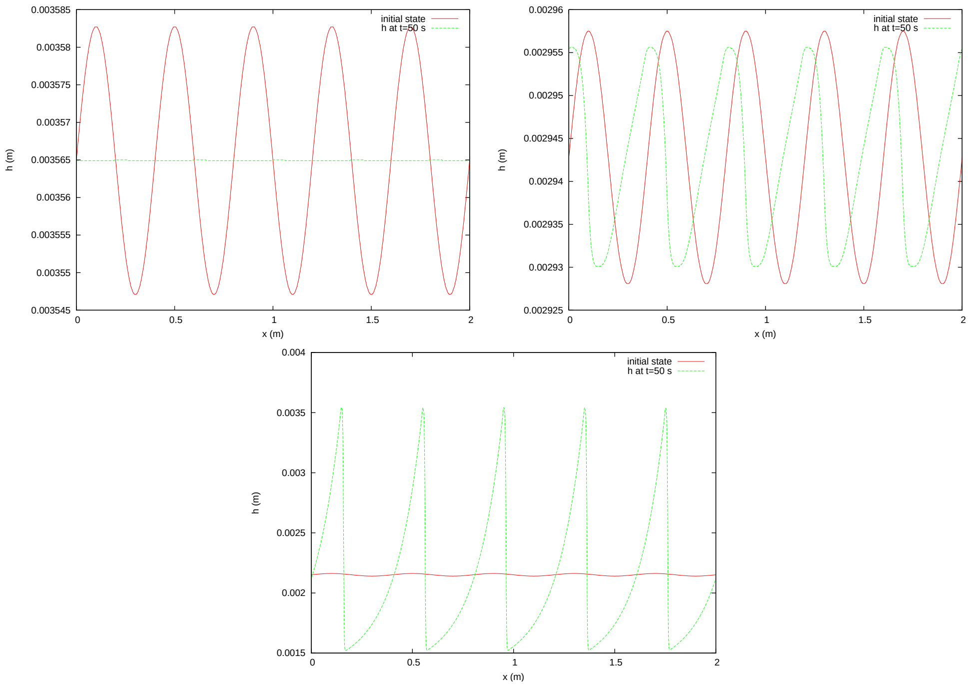


Figure 7: Perturbed initial and final ($t = 200$ s) states for Fr=1.5 (top left), Fr=2 (top right), Fr=3.7 (bottom).

§5. Roll waves

This section is devoted to some examples of the so-called “roll-waves”, a phenomenon which results from the competition between topography and friction. Several steady regimes turn out to be unstable, a slight perturbation generating a periodic travelling wave with shocks (hydraulic jumps). In ref [10], Que and Xu gather a set of explicit computations in the simple case of a constant steady states in inclined open channels with constant slope. They provide a precise analysis for the linear stability, proving in particular the following criterion: the initial constant state is linearly stable if and only if the Froude number (3) is smaller than 2.

We recover here these results, using the semi-implicit scheme described above, together with hydrostatic reconstruction. The initial height of water is different for each case, but the amplitude of the perturbation is the same. The “final states” showed here are computed at time $t = 200$ s, since it turns out that the solution is stabilized at this time. All cases are perfectly computed, the convergence rates for different values of the Froude number are given in figure 8.

Comparisons between the initial perturbation and the final state are displayed in fig. 7. For $Fr = 2$, the initial state is supposed to be exactly stable, the smaller amplitude of the final result is due to the numerical diffusion. Notice the nonlinear effects (fig. 7, top right). For $Fr < 2$ (top left), the initial perturbation completely disappears, for $Fr > 2$ (bottom), a

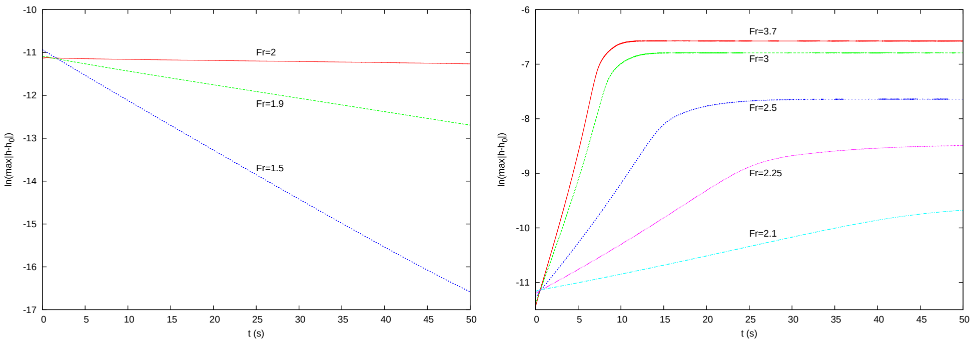


Figure 8: Convergence rate to the final state, Fr≤ 2 (left), Fr>2 (right).

roll-wave appears, whose amplitude depends on the initial state (see fig. 8).

Conclusion

This preliminary study of overland flow due to rainfall events clearly enlights several specific difficulties. First, from the numerical point of view, it seems that the apparent topography method, which was designed in order to catch steady states, is not adapted for wet/dry transitions. The semi-implicit treatment seems to be better in the problems we consider and gives good results compared to experimental data. Next, the model itself has to be improved, in particular regarding the empirical friction laws we used, which were not developed in this hydrological context. Finally, more realistic situations require infiltration and two-dimensional simulations, which are in progress and already validated on analytical solutions. This will be again compared with experimental data, as for the flume test.

Acknowledgements

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