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SOME METHODS BASED ON CUBIC SPLINES TO SOLVE A REACTION-DIFFUSION PROBLEM: UNIFORM CONVERGENCE FOR GLOBAL SOLUTION AND NORMALIZED FLUX

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Abstract. In this paper we combine the classical cubic spline with two different finite difference schemes to find an approximation to the global solution and the global normalized flux of a singularly perturbed boundary value problem of reaction-diffusion type. We prove that if the schemes are constructed on a slight modification of a piecewise uniform Shishkin mesh, then the numerical solutions are uniformly convergent for both the global solution and the global normalized flux. We give theoretical error bounds showing the order of uniform convergence of the methods and we display some numerical examples corroborating in practice these orders of convergence.

Keywords: Reaction-diffusion problems, modified Shishkin mesh, cubic spline, global solution, global normalized flux.
AMS classification: 65L10, 65L12, 65L20.

§1. Introduction

We consider the singularly perturbed reaction–diffusion two-point boundary-value problem

$$\begin{aligned} &Lu(x) \equiv -\varepsilon u''(x) + b(x)u(x) = f(x), \quad x \in D = (0,1), \\ &u(0) = A, \quad u(1) = B, \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is a small parameter and b, f are sufficiently smooth functions such that $b(x) \geq \beta > 0$ on $\overline{D} = [0,1]$. Under these assumptions it is well known (see [4]) that (1) has an unique solution satisfying

$$|u^{(k)}(x)| \leq C\left(1 + \varepsilon^{-k/2} e(x, x, \beta, \varepsilon)\right), \quad 0 \leq k \leq j+1. \tag{2}$$

where $e(\xi_1, \xi_2, \beta, \varepsilon) = \exp(-\sqrt{\beta}\xi_1/\sqrt{\varepsilon}) + \exp(-\sqrt{\beta}(1-\xi_2)/\sqrt{\varepsilon})$, and the value of j depends on the smoothness of data b and f. Bounds (2) give the asymptotic behavior of the exact solution of (1) with respect to the diffusion parameter ε, showing the presence of boundary layers at both end points on $\overline{D}$.

To approximate the solution of (1) it is essential to devise efficient methods, giving good approximations for any value of the diffusion parameter ε, i.e., uniformly convergent methods. Many numerical methods having this property are developed in last years (see for instance [3, 5, 7, 8]), showing in some cases uniform convergence only at the nodal points and in other cases also uniform convergence for the global solution on $\overline{D}$. In this paper we extend the results of [6] by modifying the original piecewise uniform Shishkin mesh. We construct some methods giving good approximations for the global solution and the global normalized flux, by using a classical cubic spline based on the numerical solutions at mesh points.

The paper is organized as follows: in Section 2, we present the numerical methods used to solve (1) and we define the numerical cubic spline associated to the numerical solutions at mesh points. In Section 3 we prove the uniform convergence of the cubic spline in the approximation of both the global solution and the global normalized flux. Finally, in Section 4 we show some results obtained by the numerical methods in a particular example, corroborating in practice the theoretical results. Henceforth, C denotes any positive constant independent of the diffusion parameter ε and the discretization parameter N. C can take different values at different places.

§2. The finite difference schemes

The first step to define the finite difference scheme is to construct the mesh. Then, following [4], the domain $\overline{D}$ is divided into three subintervals as $\overline{D} = [0,\sigma) \cup [\sigma, 1-\sigma] \cup (1-\sigma, 1]$, where σ is the transition parameter given by

$$\sigma = \min\left\{1/4, \sigma_0 \sqrt{\varepsilon} \ln N\right\}, \tag{3}$$

and σ_0 is a positive constant. On the subintervals $[0,\sigma]$ and $[1-\sigma, 1]$ an uniform mesh with $N/4$ mesh intervals are placed, while $[\sigma, 1-\sigma]$ has an uniform mesh with $N/2$ mesh intervals. Obviously the mesh is uniform when $\sigma = 1/4$. The mesh size in $[\sigma, 1-\sigma]$ is $H = 2(1-2\sigma)/N$, and in $[0,\sigma]\bigcup[1-\sigma, 1]$ it is $h = 4\sigma/N$. Let $\overline{D}^N \equiv \{x_i : 0 = x_0 < \cdots < x_N = 1\}$ be the mesh and we denote by $h_{i+1} = x_{i+1} - x_i,\ i = 0, 1, \ldots, N-1$.

For the exact values $u(x_i)$ $i = 0, \ldots, N$, of the function u at the nodal points, it s well known that there exists an interpolating cubic spline $s(x)$ given by

$$\begin{aligned} s(x) = {} & \frac{(x_{i+1}-x)^3}{6h_{i+1}} M_i + \frac{(x-x_i)^3}{6h_{i+1}} M_{i+1} + \left(u_i - \frac{h_{i+1}^2}{6} M_i\right)\left(\frac{x_{i+1}-x}{h_{i+1}}\right) + \\ & + \left(u_{i+1} - \frac{h_{i+1}^2}{6} M_{i+1}\right)\left(\frac{x-x_i}{h_{i+1}}\right),\ x_i \le x \le x_{i+1},\ i = 0, \ldots, N-1, \end{aligned} \tag{4}$$

where $u_i = u(x_i)$, $M_i = u''(x_i)$, $i = 0, \ldots, N$. From this cubic spline the approximation to the global normalized flux is obtained by $\sqrt{\varepsilon} s'(x)$.

To calculate a numerical cubic spline, we can use the discrete solution $U_i,\ i = 0, \ldots N$, given by a finite difference scheme at mesh points, and then, defining $\overline{M}_i = (b_i U_i - f_i)/\varepsilon$, $i = 0, \ldots N$, the numerical cubic spline is defined as

$$\begin{aligned} S(x) = {} & \frac{(x_{i+1}-x)^3}{6h_{i+1}} \overline{M}_i + \frac{(x-x_i)^3}{6h_{i+1}} \overline{M}_{i+1} + \left(U_i - \frac{h_{i+1}^2}{6} \overline{M}_i\right)\left(\frac{x_{i+1}-x}{h_{i+1}}\right) + \\ & + \left(U_{i+1} - \frac{h_{i+1}^2}{6} \overline{M}_{i+1}\right)\left(\frac{x-x_i}{h_{i+1}}\right),\ x_i \le x \le x_{i+1},\ i = 0, \ldots, N-1. \end{aligned} \tag{5}$$

This spline gives an approximation to the exact solution of the boundary value problem (1) at the whole domain $\overline{D}$ and also an approximation to the normalized flux by using $\sqrt{\varepsilon}S'(x)$.

To obtain the uniform convergence for the global solution and the global normalized flux, it will be necessary to use a slight modification of the original Shsihkin mesh. Following the original idea of Surla (see [8]), we define a new parameter $\overline{H} = \sqrt{\varepsilon/\beta}N\ln N$ and we construct a modified Shishkin mesh as follows. If $H/2 \leq \overline{H}$ the mesh is the same that the original Shishkin mesh; on the other hand, when $H/2 > \overline{H}$, we introduce two new points, $\overline{x}_{N/4} = x_{N/4} + \overline{H}$ and $\overline{x}_{3N/4} = x_{3N/4} - \overline{H}$. So, in this case the number of mesh points is $N_1 = N + 2$, and they are given by

$$x_i = \begin{cases} ih, \quad i = 0, 1, \ldots, N/4, \\ \sigma + \overline{H}, \quad i = N/4 + 1, \\ \sigma + (i - 1 - N/4)H, \quad i = N/4 + 2, \ldots, 3N/4, \\ 1 - \sigma - \overline{H}, \quad i = 3N/4 + 1, \\ 1 - \sigma, \quad i = 3N/4 + 2, \\ 1 - \sigma + (i - 3N/4 + 2)h, \quad i = 3N/4 + 3, \ldots, N_1 \end{cases} \tag{6}$$

On this modified Shishkin mesh we consider two different finite difference schemes. The first one, constructed in [6], is a hybrid scheme defined as

$$\begin{aligned} &L^N U_i^N \equiv r_i^- U_{i-1}^N + r_i^c U_i^N + r_i^+ U_{i+1}^N = q_i^- f_{i-1} + q_i^c f_i + q_i^+ f_{i+1}, 1 \leq i \leq N-1, \\ &U_0^N = A, \quad U_N^N = B, \end{aligned} \tag{7}$$

where for indices $i = 1, \ldots, N/4 - 1$ and also $3N/4 + 1, \ldots, N - 1$, the coefficients of the scheme are given by

$$\begin{aligned} r_i^- &= \frac{-3\varepsilon}{h_i(h_i + h_{i+1})} + \frac{h_i}{2(h_i + h_{i+1})} b_{i-1}, \quad r_i^c = \frac{3\varepsilon}{h_i h_{i+1}} + b_i, \\ r_i^+ &= \frac{-3\varepsilon}{h_{i+1}(h_i + h_{i+1})} + \frac{h_{i+1}}{2(h_i + h_{i+1})} b_{i+1}, \\ q_i^- &= \frac{h_i}{2(h_i + h_{i+1})}, \quad q_i^c = 1, \quad q_i^+ = \frac{h_{i+1}}{2(h_i + h_{i+1})}, \end{aligned} \tag{8}$$

and for indices $i = N/4, \ldots, 3N/4$, the coefficients are now given by

$$\begin{aligned} &r_i^- = \frac{-2\varepsilon}{h_i(h_i + h_{i+1})}, \quad r_i^c = \frac{2\varepsilon}{h_i h_{i+1}} + b_i, \quad r_i^+ = \frac{-2\varepsilon}{h_{i+1}(h_i + h_{i+1})}, \\ &q_i^- = 0, \quad q_i^c = 1, \quad q_i^+ = 0. \end{aligned} \tag{9}$$

The second method is the HOC (High Order Compact) scheme constructed in [3], which is defined as

$$\begin{gathered} L_{\varepsilon,N}[U_i] \equiv r_i^- U_{i-1} + r_i^c U_i + r_i^+ U_{i+1} = Q_N(f_i), \; 1 \leq i \leq N-1, \\ U_0 = A, \quad U_N = B, \end{gathered} \tag{10}$$

where the coefficients are given by

$$\begin{aligned} r_i^- &= \frac{-2\varepsilon}{(h_i+h_{i+1})h_i} - \delta_{i,N/4}\frac{(h_{i+1}-h_i)b_i}{3h_i} - \frac{(h_{i+1}^3+h_i^3)b_i'}{6(h_i+h_{i+1})h_i}\,\mathrm{sgn}\, b_i', \\ r_i^+ &= \frac{-2\varepsilon}{(h_i+h_{i+1})h_{i+1}} + \delta_{i,3N/4}\frac{(h_{i+1}-h_i)b_i}{3h_{i+1}} + \frac{(h_{i+1}^3+h_i^3)b_i'}{6(h_i+h_{i+1})h_i}(1-\ \mathrm{sgn}\, b_i'), \\ r_i^c &= -r_i^- - r_i^+ + Q_N^2(b_i), \end{aligned} \tag{11}$$

whit sgn $z_i = 1$, if $z_i \geq 0$ and sgn $z_i = 0$, if $z_i < 0$, $\delta_{il} = 1$ if $i = l$, $\delta_{il} = 0$ if $i \neq l$ and

$$\begin{aligned} Q_N(z_i) \equiv z_i &+ \frac{h_{i+1}-h_i}{3}\left(z_i' + \frac{b_i z_i}{2\varepsilon}\left(\delta_{i,N/4}h_i - \delta_{i,3N/4}h_{i+1}\right)\right) + \\ &+ \frac{h_{i+1}^3+h_i^3}{12(h_i+h_{i+1})}\left(z_i'' + \frac{b_i z_i}{\varepsilon} + \frac{b_i' z_i}{\varepsilon}(h_i \mathrm{sgn}\, b_i' - (1-\mathrm{sgn}\, b_i')h_{i+1})\right). \end{aligned} \tag{12}$$

§3. Uniform convergence for the global solution and the normalized flux

In this section we give the main results showing the uniform convergence for the global solution and for the global normalized flux, using the cubic spline together with the two finite difference schemes previously defined.

Theorem 1. *Let $u(x)$ be the solution of* (1) *and $S(x)$ be the numerical spline given in* (5)*, based on the solution of the finite difference scheme* (7)–(9) *constructed on the modified Shishkin mesh* (6)*. Then, the error satisfies*

$$|S(x) - u(x)| \leq \left(N^{-2}\ln^2 N + N^{3-\sqrt{\beta}\sigma_0}\ln^3 N\right), \quad \forall x \in \overline{D}. \tag{13}$$

Proof. We only give the main ideas of the proof; for full details see [2]. Let $x_i \leq x \leq x_{i+1}$, $i = 0, \ldots, N-1$, be; then, using (4), (5), Taylor expansions and the bounds (2) for the derivatives of the exact solution u, we can prove that

$$\begin{aligned} |s(x) - u(x)| &\leq Ch_{i+1}^3\left(1 + \varepsilon^{-3/2}e(x_i, x_i, \beta, \varepsilon)\right), \quad \text{if } x \leq 1/2, \\ |s(x) - u(x)| &\leq Ch_{i+1}^3\left(1 + \varepsilon^{-3/2}e(x_{i+1}, x_{i+1}, \beta, \varepsilon)\right), \quad \text{if } x \geq 1/2, \\ |s(x) - S(x)| &\leq C\left(1 + b^* h_{i+1}^2/\varepsilon\right)\max\{|u_i - U_i|, |u_{i+1} - U_{i+1}|\}, \end{aligned} \tag{14}$$

where $b^* = \max_{x\in D} b(x)$. Then, if $\sigma = 1/4$ and $\varepsilon^{-1/2} \leq C\ln N$, it is straightforward to obtain that

$$|S(x) - u(x)| \leq C\left(N^{-3}\ln^3 N + N^{-\sqrt{\beta}\sigma_0}\right). \tag{15}$$

On the other hand, when $1/4 > \sigma_0\sqrt{\varepsilon}\ln N$, we distinguish several cases depending on the location of the mesh point x_i, concretely when x_i is inside the boundary layer, outside the layer or x_i is one of the transition points σ or $1-\sigma$. From the uniform stability of the hybrid scheme, easily we have $|s(x)-S(x)| \leq C\left(1 + b^* h_{i+1}^2/\varepsilon\right)|\tau_i|$, where the local error at x_i satisfies (see [6])

$\tau_i = (\varepsilon/H^2)\,(R_3(x_i, x_{i+1}, u) + R_3(x_i, x_{i-1}, u))$, where $R_n(a, p, g) = (1/n!)\int_a^p (p-\xi)g^{(n+1)}(\xi)d\xi$ denotes the remainder of the Taylor expansion. Using the integral form for the remainder, integrating by parts and taking into account that $e(x_j, x_j, \beta, \varepsilon) \le N^{-\sqrt{\beta}\sigma_0}$, $j = i-1, i, i+1$, it is possible to prove the required result. □

Theorem 2. *Let* $\sqrt{\varepsilon}u'(x)$ *be the normalized flux of* (1) *and* $\sqrt{\varepsilon}S'(x)$ *be the normalized flux obtained from the cubic spline based on the numerical solution of the finite difference scheme* (7)–(9) *constructed on the modified Shishkin mesh* (6)*. Then, for any* $x \in \overline{D}$*, it holds*

$$\sqrt{\varepsilon}\left|S'(x) - u'(x)\right| \le \begin{cases} C(N^{-2}\ln^3 N + N^{3-\sqrt{\beta}\sigma_0}\ln^3 N), & \text{if } N^{-1} > \sqrt{\varepsilon}, \\ C(N^{-1}\sqrt{\varepsilon}\ln^2 N + N^{1-\sqrt{\beta}\sigma_0}), & \text{if } N^{-1} \le \sqrt{\varepsilon}. \end{cases} \tag{16}$$

Proof. The proof follows similar ideas to these ones of Theorem 1. Again we take $x_i \le x \le x_{i+1}$, $i = 0, \ldots, N-1$. Now it is possible to obtain that

$$\begin{aligned} &\sqrt{\varepsilon}\left|s'(x) - u'(x)\right| \le C\sqrt{\varepsilon}h_{i+1}^3\left(1 + \varepsilon^{-2}e(x_i, x_i, \beta, \varepsilon)\right), \quad \text{if } x \le 1/2, \\ &\sqrt{\varepsilon}\left|s'(x) - u'(x)\right| \le C\sqrt{\varepsilon}h_{i+1}^3\left(1 + \varepsilon^{-2}e(x_{i+1}, x_{i+1}, \beta, \varepsilon)\right), \quad \text{if } x \ge 1/2, \\ &\sqrt{\varepsilon}\left|s'(x) - S'(x)\right| \le C\left(\sqrt{\varepsilon}/h_{i+1} + b^* h_{i+1}/\sqrt{\varepsilon}\right)\max\{|u_i - U_i|, |u_{i+1} - U_{i+1}|\}. \end{aligned} \tag{17}$$

Then, using Taylor expansions with the remainder in integral form and distinguishing the cases when the mesh is or non uniform and depending on the location of the mesh point in the domain (inside the layer, outside the layer or the transition points), it is not difficult to prove the required result. □

Remark 1. From Theorems 1 and 2 we see that if $\sqrt{\beta}\sigma_0 \ge 5$, then the global solution has order of uniform convergence $O(N^{-2}\ln^3 N)$ and the global normalized flux has almost second order of uniform convergence except for $N^{-1} \le \sqrt{\varepsilon}$, which is less interesting in practice. Our computational results in the next section show that even when $N^{-1} \le \sqrt{\varepsilon}$ the results show the same orders of convergence than for $N^{-1} > \sqrt{\varepsilon}$.

The two following theorems prove the uniform convergence for the global solution and for the global normalized flux, when the HOC scheme is used. Their proof is similar to this one of the two previous theorems for the hybrid scheme; full details can be found in [1].

Theorem 3. *Let u the solution of* (1) *and S the numerical spline given in* (5)*, based on the solution of the finite difference scheme* (10)–(12) *constructed on the modified Shishkin mesh* (6)*. Then, the error satisfies*

$$|S(x) - u(x)| \le \left(N^{-4}\ln^4 N + N^{4-\sqrt{\beta}\sigma_0}\ln^4 N\right), \quad \forall x \in \overline{D}. \tag{18}$$

Theorem 4. *Let* $\sqrt{\varepsilon}u'$ *be the normalized flux of* (1) *and* $\sqrt{\varepsilon}S'$ *be the normalized flux obtained from the cubic spline approximations, based on the numerical solution of the finite difference scheme* (10)–(12) *constructed on the modified Shishkin mesh* (6)*. Then, for any* $x \in \overline{D}$*, it holds*

$$\sqrt{\varepsilon}\left|S'(x) - u'(x))\right| \le \begin{cases} C(N^{-4}\ln^4 N + N^{4-\sqrt{\beta}\sigma_0}), & \text{if } x = (x_i + x_{i+1})/2,\ N^{-1} > \sqrt{\varepsilon}, \\ (N^{-3}\sqrt{\varepsilon}\ln^4 N + N^{1-\sqrt{\beta}\sigma_0}), & \text{if } x = (x_i + x_{i+1})/2,\ N^{-1} \le \sqrt{\varepsilon}, \\ C(N^{-3}\sqrt{\varepsilon}\ln^4 N + N^{3-\sqrt{\beta}\sigma_0}\ln^3 N), & \text{in other case}. \end{cases} \tag{19}$$

Method	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
hybrid	7.2314E−1	2.0974E−1	5.1976E−2	1.4784E−2	4.5414E−3	1.4013E−3	4.2698E−4	1.2885E−4
scheme	1.7857	2.0127	1.8138	1.7028	1.6964	1.7145	1.7285	
HOC	3.0110E+0	8.8361E−1	1.9046E−1	3.1506E−2	4.2628E−3	4.9632E−4	5.1766E−5	4.9917E−6
scheme	1.7687	2.2139	2.5958	2.8858	3.1025	3.2612	3.3744	

Table 1: Uniform errors and uniform orders for the global solution

Remark 2. From Theorems 3 and 4 it follows that if $\sqrt{\beta}\sigma_0 \geq 8$, then the approximation to the global solution has order of uniform convergence $O(N^{-4}\ln^4 N)$, and the approximation to the global normalized flux has almost fourth order of uniform convergence at midpoints and almost third order in the rest, for all cases except for $N^{-1} \leq \sqrt{\varepsilon}$. Again, the computational results show that if $N^{-1} \leq \sqrt{\varepsilon}$, the same orders of convergence than in the case $N^{-1} > \sqrt{\varepsilon}$ are obtained.

§4. Numerical Experiments

To illustrate the efficacy of our numerical methods, we solve the problem

$$-\varepsilon u''(x) + (1 + x^2 + \cos(\pi x))u(x) = 1 + x^{4.5} + \sin(\pi x),\ x \in (0, 1),\ u(0) = 0, \quad u(1) = 0,$$

for which the exact solution is unknown. We are only interested in the errors outside the mesh points; then, in the tables we will show the errors at midpoints, $x = (x_i + x_{i+1})/2$, of the corresponding modified Shishkin mesh. To approximate the maximum errors at midpoints we use a variant of the double mesh principle (see [4]). The idea is to calculate the numerical solution U^N on the modified Shishkin mesh $\overline{D}^N$ and also the numerical solution $\widetilde{U}^N$ on a new mesh $\widetilde{D}^N$, for which the transition parameter is now given by $\widetilde{\sigma} = \min\left\{1/4, \sigma_0\sqrt{\varepsilon}\ln(N/2)\right\}$.

This slightly altered value of σ will ensure that the positions of transition points remain the same in meshes $\overline{D}^N$ and $\widetilde{D}^{2N}$ and the midpoints $x = (x_i + x_{i+1})/2$ of the mesh $\overline{D}^N$ are also mesh points of the mesh $\widetilde{D}^{2N}$. Then the errors at midpoints are obtained by $E_\varepsilon^N = \max_x |S^N(x) - \widetilde{S}^{2N}(x)|$, $E^N = \max_\varepsilon E_\varepsilon^N$, where S_N and $\widetilde{S}_{2N}$ are the splines defined by (5) on the meshes $\overline{D}^N$ and $\widetilde{D}^{2N}$ respectively. From these errors, the numerical orders of convergence and the uniform orders of convergence are given by $p_\varepsilon^N = \log_2\left(E_\varepsilon^N/E_\varepsilon^{2N}\right)$, $p^N = \log_2\left(E^N/E^{2N}\right)$. We show the results on the range of values $\varepsilon = 2^0, 2^{-2}, 2^{-4}, \ldots, 2^{-48}$.

Table 1 displays the results for the hybrid and the HOC scheme. For each scheme, the first row gives the uniform maximum errors E^N and the second one the uniform orders of convergence p^N. From this table we deduce the almost second order of uniform convergence for the hybrid scheme and the fourth order of uniform convergence, except by the logarithmic factor, for the HOC scheme, in agreements with Theorems 1 and 3 respectively.

To compare the efficacy of the methods, we show the results obtained by using the scheme developed in [8], based on a spline collocation method. Table 2 displays the results obtained in this case; from it we see that if the diffusion parameter ε is not very small then the results are good confirming the almost second order of uniform convergence. Nevertheless, for ε sufficiently small the maximum errors do not stabilize for any value of the discretization parameter N, and therefore the method does not show the uniform convergence for the

Method	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
$\varepsilon = 2^{-8}$	$4.4875E{-2}$	$1.1582E{-2}$	$2.9268E{-3}$	$7.3339E{-4}$	$1.8337E{-4}$	$4.5846E{-5}$	$1.1461E{-5}$	$2.8654E{-6}$
	1.9540	1.9846	1.9966	1.9998	1.9999	2.0000	2.0000	
$\varepsilon = 2^{-16}$	$4.9999E{-1}$	$1.9545E{-1}$	$6.2321E{-2}$	$2.6428E{-2}$	$8.6093E{-3}$	$2.7160E{-3}$	$8.4123E{-4}$	$2.5446E{-4}$
	1.3551	1.6490	1.2377	1.6181	1.6644	1.6909	1.7250	
$\varepsilon = 2^{-24}$	$5.0940E{-1}$	$1.9767E{-1}$	$6.2740E{-2}$	$2.6489E{-2}$	$8.6265E{-3}$	$2.7211E{-3}$	$8.4278E{-4}$	$2.5493E{-4}$
	1.3657	1.6556	1.2440	1.6185	1.6646	1.6909	1.7251	
$\varepsilon = 2^{-32}$	$1.2115E{+0}$	$3.9149E{-1}$	$6.2766E{-2}$	$2.6493E{-2}$	$8.6276E{-3}$	$2.7214E{-3}$	$8.4288E{-4}$	$2.5496E{-4}$
	1.6297	2.6409	1.2444	1.6186	1.6646	1.6909	1.7251	
$\varepsilon = 2^{-40}$	$1.5892E{+1}$	$5.3076E{+0}$	$4.4122E{-1}$	$4.7986E{-2}$	$8.6277E{-3}$	$2.7214E{-3}$	$8.4288E{-4}$	$2.5496E{-4}$
	1.5822	3.5885	3.2008	2.4756	1.6646	1.6909	1.7251	
$\varepsilon = 2^{-44}$	$6.2879E{+1}$	$2.1047E{+1}$	$1.7229E{+0}$	$1.8829E{-1}$	$2.1247E{-2}$	$2.7214E{-3}$	$8.4288E{-4}$	$2.5496E{-4}$
	1.5789	3.6108	3.1938	3.1476	2.9649	1.6909	1.7251	
$\varepsilon = 2^{-48}$	$2.5082E{+2}$	$8.4008E{+1}$	$6.8499E{+0}$	$7.4981E{-1}$	$8.9050E{-2}$	$9.8585E{-3}$	$8.4288E{-4}$	$2.5675E{-4}$
	1.5781	3.6164	3.1915	3.0738	3.1752	3.5480	1.7150	

Table 2: Uniform errors and uniform orders for the global solution using the Surla scheme

Method	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
hybrid	$4.8352E{-2}$	$4.5769E{-2}$	$3.3284E{-2}$	$1.8024E{-2}$	$7.8996E{-3}$	$2.9894E{-3}$	$1.0135E{-3}$	$3.2552E{-4}$
scheme	0.0792	0.4595	0.8849	1.1900	1.4019	1.5605	1.6386	
HOC	$2.7893E{-1}$	$1.1928E{-1}$	$4.2171E{-2}$	$1.2092E{-2}$	$2.7963E{-3}$	$5.5628E{-4}$	$1.0096E{-4}$	$1.7301E{-5}$
scheme	1.2256	1.5000	1.8022	2.1125	2.3296	2.4620	2.5449	

Table 3: Uniform errors and uniform orders for the global normalized flux

global solution. So, we can conclude that this method is considerably worse than these ones developed in this paper.

To approximate the errors associated to the normalized flux, again only at midpoints $x = (x_i + x_{i+1})/2$ of the modified Shishkin mesh, the first idea is to use the derivatives of the numerical splines S_N and $\widetilde{S}_{2N}$ defined on the meshes $\overline{D}^N$ and $\widetilde{D}^{2N}$ respectively; then, we calculate $F_\varepsilon^N = \max_x \sqrt{\varepsilon}\left|S'_N(x) - \widetilde{S}'_{2N}(x)\right|$, $F^N = \max_\varepsilon F_\varepsilon^N$. From these values, the order of convergence and the ε-uniform order of convergence for the flux are calculated by $q_\varepsilon^N = \log_2\left(F_\varepsilon^N / F_\varepsilon^{2N}\right)$, $q^N = \log_2\left(F^N / F^{2N}\right)$.

From Table 3 we cannot observe the predicted almost fourth order of convergence for the normalized flux at midpoints. The reason is related with the use of the double mesh principle, because the midpoint of one mesh is becoming the nodal point in doubling the mesh. Then, to find the errors for the normalized flux we use a second numerical idea. We consider a new mesh $\overline{\overline{D}}^N$ where the mesh points are $\overline{\overline{x}}_{3i} = x_i$, $\overline{\overline{x}}_{3i+1} = x_i + h_{i+1}/3$, $\overline{\overline{x}}_{3i+2} = x_i + 2h_{i+1}/3$, $i = 0, 1, \ldots, N-1$, and $\overline{\overline{x}}_{3N} = x_N$. We denote by $\overline{\overline{U}}^{3N}$ the numerical solution on this mesh and $\overline{\overline{S}}_{3N}$ the corresponding cubic spline. Then, the error associated to the normalized flux at any point x which is not a mesh point, is calculated by $\sqrt{\varepsilon}\left|S'_N(x) - \overline{\overline{S}}'_{3N}(x)\right|$. Table 4 displays the results obtained by using this idea; from it we deduce the almost fourth order of uniform convergence according with Theorem 4.

Method	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$
HOC	$2.2766E{-}1$	$7.7506E{-}2$	$2.0308E{-}2$	$4.2751E{-}3$	$7.6856E{-}4$	$1.0860E{-}4$	$1.2598E{-}5$	$1.2857E{-}6$
scheme	1.5545	1.9323	2.2480	2.4757	2.8231	3.1078	3.2925	

Table 4: Uniform errors and uniform orders for the global normalized flux

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References

[1] Bawa, R. K., and Clavero, C. Higher order global solution and normalized flux for singularly perturbed reaction-diffusion problems. *Appl. Math Comp. 216*, 7 (2010), 2058–2068.

[2] Clavero, C., Bawa, R. K., and Natesan, S. A robust second-order numerical method for global solution and global normalized flux of singularly perturbed self-adjoint boundary-value problems. *Int. J. Comput. Math. 86*, 10–11 (2009), 1731—-1745.

[3] Gracia, J., Lisbona, F., and Clavero, C. High order ε-uniform methods for singularly perturbed reaction-diffusion problems. *Lecture Notes in Computer Science 1988* (2001), 350–358.

[4] Miller, J. J. H., O'Riordan, E., and Shishkin, G. I. *Fitted numerical methods for singular perturbation problems*. World Scientific, Singapore, 1996.

[5] Natesan, S., and Bawa, R. K. Second-order numerical schemes for singularly perturbed reaction–diffusion robin problems. *Journal Numerical Analysis, Industrial and Applied Mathematics 2* (2007), 177–192.

[6] Natesan, S., Bawa, R. K., and Clavero, C. A uniformly convergent method for global solution and normalized flux of singular perturbation problems of reaction-diffusion type. *International Journal of Information and Systems Sciences 3* (2007), 207–221.

[7] Stajanovic, M. Global convergence method for singularly perturbed boundary value problems. *International J. Comp. Appl. Math. 181* (2005), 326–335.

[8] Surla, K., and Uzelac, Z. A uniformly accurate spline collocation method for a normalized flux. *International J. Comp. Appl. Math. 166* (2004), 291–305.

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