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# CONSTRUCTION OF QUASI-INTERPOLANTS ON UNIFORM PARTITIONS

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**Abstract.** We propose a new general method for constructing standard quasi-interpolation operators into the space spanned by the integer translates of a B-spline defined on a uniform partition of $\mathbb{R}^s$. The key tool is an appropriate error estimate with a leader term that contains and expression measuring the quality of the approximation. It is a function on the sequence defining the quasi-interpolating operator, and therefore, we define and solve a minimization problem in such a way that their solutions are characterized in terms of some splines that do not depend on the linear form defining the operator.

*Keywords:* B-splines, box splines, discrete quasi-interpolants, differential quasi-interpolants, integral quasi-interpolants, approximation power, error estimates.

*AMS classification:* 41A05, 41A15, 65D05, 65D07.

## §1. Introduction

We propose a new general method for constructing quasi-interpolation operators based on B-splines defined on uniform partitions $\tau$ of $\mathbb{R}^s$, $s \geq 1$. Let $\phi$ be such a B-spline on $\tau$, normalized by $\sum_{i\in\mathbb{Z}^s} \phi(\cdot - i) = 1$. Let $\mathcal{S} := \operatorname{span}(\phi(\cdot - i))_{i\in\mathbb{Z}^s}$ be the cardinal spline space spanned by the shifts of $\phi$.

The classical structure for a quasi-interpolant is given by the expression

$$Q(f) := \sum_{i\in\mathbb{Z}^s} \lambda f(\cdot + i)\,\phi(\cdot - i),$$

$\lambda$ being a linear functional (see e.g. [3], [2], [5]). Usually, $\lambda f$ is a linear combination of values of $f$ and some of its derivatives at some points in some open set containing the support of $\phi$; or a linear combination of values of $f$ at some points in this set; or a linear combination of weighted mean values of the function to be approximated, i.e. $\lambda f$ is given by

$$\sum_{j\in J} c_j f(-j), \quad \sum_{|i|\leq\ell} \sum_{j\in J_i} c_{i,j} D^{(i)} f(-j), \text{ or } \sum_{j\in J} c_j \langle f, \psi(\cdot - j)\rangle,$$

$J$ and $J_i$, $|i| \leq \ell$ for $0 \leq \ell \leq \deg\psi$, being finite subsets of $\mathbb{Z}^s$, and $\langle\cdot,\cdot\rangle$ and $\psi$ standing for the usual inner product and another B-spline.

That linear functional is defined to produce a quasi-interpolant $Q$ exact on a polynomial space included in $\mathcal{S}$. We will restrict our attention to these cases. More precisely, we will demand the exactness of $Q$ on $\mathbb{P}_n$, with $n$ such that $\mathbb{P}_n \subset \mathcal{S}$ and $\mathbb{P}_{n+1} \not\subseteq \mathcal{S}$, i.e. $Q$ realizes the approximation power of $\mathcal{S}$.

## §2. Estimating the quasi-interpolation error

For the scaled quasi-interpolant

$$Q_h f := \sum_{i\in\mathbb{Z}^s} \lambda f\left(h\left(\cdot+i\right)\right)\phi\left(\frac{\cdot}{h}-i\right)$$

considered here, we have the following result concerning the error $E_h f := f - Q_h f$. The notation $m_\alpha$ is used for the normalized monomial of order $\alpha$: $m_\alpha(x) = x^\alpha/\alpha!$.

**Proposition 1.** *Let $f \in C^{n+2}\left(\mathbb{R}^s\right)$. For every triangle $T$ in $h\tau$, there exist both a neighborhood $V = V(T)$, independent of $f$, and a constant $C > 0$, independent of $h$ and $T$, such that*

$$\|E_h f\|_{\infty,T} \le T_{n,Q} h^{n+1} \left|f\right|_{\infty,n+1,V} + C\, h^{n+2} \left|f\right|_{\infty,n+2,V},$$

*where*

$$T_{n,Q} := \max_{\alpha\in\mathbb{N}_0^s,\ |\alpha|=n+1} \|Qm_\alpha - m_\alpha\|_{\infty,[0,1]^s}.$$

*Proof.* Suppose that $Q$ is an integral quasi-interpolation operator. Then, we have

$$Qf = \sum_{i\in\mathbb{Z}^s} \lambda f\left(\cdot+i\right)\phi\left(\cdot-i\right)$$

with

$$\lambda f\left(\cdot+i\right) = \sum_{j\in J} c_j \left\langle f\left(\cdot+i\right), \psi\left(\cdot-j\right)\right\rangle = \int_{\mathbb{R}^s} f(t)\, H(t-i)\, dt,$$

where

$$H := \sum_{j\in J} c_j \psi\left(\cdot-j\right).$$

Thus,

$$Qf = \int_{\mathbb{R}^s} f(t)\, K(t,\cdot)\, dt,$$

with

$$K(t,\cdot) := \sum_{i\in\mathbb{Z}^s} H(t-i)\,\phi\left(\cdot-i\right).$$

Taking into account that the scaled quasi-interpolant $Q_h$ in is equal to $\sigma_h Q \sigma_{1/h}$ where the scaling operator $\sigma_h$ is defined as

$$\sigma_h f = f\left(\frac{\cdot}{h}\right),$$

we get

$$Q_h f = \int_{\mathbb{R}^s} f(t)\, \frac{1}{h^s} K\left(\frac{t}{h}, \frac{\cdot}{h}\right) dt.$$

The kernel in this integral representation of $Q_h$ is $\mathbb{P}_{n-1}$-reproducing and shift-invariant, and has sufficient decay. Then, the next error estimate for the integral quasi-interpolation operator considered here follows from [4].

The proof for discrete quasi-interpolants is given in [1]. A similar method can be used to prove the result in the differential case. □

The constant $T_{n,Q}$ in the leading term of $E_h f$ is determined by how well $Q_h$ approximates the monomials of order $n+1$.

## §3. Achieving the required exactness

The operator $Q$ is exact on $\mathbb{P}_n$ if for all $\alpha \in \mathbb{N}_0^s$ such that $|\alpha| \leq n$ one gets

$$\lambda(m_\alpha) = g_\alpha(0),$$

where the polynomials $g_\alpha$ can be recursively computed as follows (see e.g. [3]):

$$g_0 = m_0, \quad g_\alpha = m_\alpha - \sum_{j\in\mathbb{Z}^s} \phi(j) \sum_{\beta \lneqq \alpha} m_{\alpha-\beta}(-j)\, g_\beta, \ |\alpha| > 0.$$

They are only sufficient conditions to guarantee the exactness of $Q$ on $\mathbb{P}_n$.

## §4. A minimization problem

It is natural to construct $Q$ by solving this minimization problem:

**Problem 1.** Minimize $T_{n,Q}$ subject to the exactness conditions $\lambda(m_\alpha) = g_\alpha(0), \ |\alpha| \leq n$.

The solutions of this problem (and the corresponding quasi-interpolants $Q$) can be easily characterized using the well-known Schoenberg operator

$$Sf := \textstyle\sum_{i\in\mathbb{Z}^s} f(i)\,\phi(\cdot - i).$$

**Proposition 2.** *Let $Q$ be one of the quasi-interpolants considered here defined from the linear functional $\lambda$. Let us suppose that $Q$ is exact on $\mathbb{P}_n$. If*

$$\lambda m_\alpha = g_\alpha(0) + \frac{1}{2}\left(\max_{[0,1]^s} G_\alpha + \min_{[0,1]^s} G_\alpha\right)$$

*for all $\alpha \in \mathbb{N}_0^s$ such that $|\alpha| = n+1$, where*

$$G_\alpha := m_\alpha - S g_\alpha,$$

*then $T_{n,Q}$ attains its minimum value.*

Note that $G_\alpha$ does not depend on $\lambda$.

## §5. A differential example

Let $\phi$ be the quadratic box-spline on the criss-cross triangulation $\tau_2$, centered at the origin (see e.g. [3]). Then $n = 2$, i.e. we can construct differential quasi-interpolants exact on $\mathbb{P}_2$ by minimizing the errors associated with the cubic monomials. We will restrict our attention to the case $\ell = 1$, i.e. we will suppose that the values of $f$ and its first order partial derivatives at the grid points are known.

We have

$$\lambda_\mu f = f(0) + \frac{1}{16}\left(D^{(1,0)}f(1,0) - D^{(1,0)}f(-1,0)\right) - \mu\left(D^{(1,0)}f(0,1) - D^{(1,0)}f(0,-1)\right)$$
$$- \mu\left(D^{(0,1)}f(1,0) - D^{(1,0)}f(-1,0)\right) + \frac{1}{16}\left(D^{(0,1)}f(0,-1) - D^{(0,1)}f(0,1)\right).$$

The exactness of $Q$ on $\mathbb{P}_2$ is guaranteed by the conditions

$$\lambda m_\alpha = g_\alpha(0), \ |\alpha| \leq 2.$$

Since $\max_{[0,1]^2} G_\alpha = -\min_{[0,1]^2} G_\alpha$ when $|\alpha| = 3$, the new linear equations yielding the minimum of $T_{2,Q}$ are given by

$$\lambda m_\alpha = g_\alpha(0), \ |\alpha| = 3.$$

When $J_{0,0} = \{(0,0)\}$ and $J_{1,0} = J_{0,1} = \{(0,0), (\pm 1, 0), (0, \pm 1)\}$, the solution of this linear system depends on a parameter $\mu$, and provides the linear functional

$$\begin{aligned}\lambda_\mu f = f(0) &+ \frac{1}{16}\left(D^{(1,0)} f(1,0) - D^{(1,0)} f(-1,0)\right) - \mu\left(D^{(1,0)} f(0,1) - D^{(1,0)} f(0,-1)\right) \\ &- \mu\left(D^{(0,1)} f(1,0) - D^{(1,0)} f(-1,0)\right) + \frac{1}{16}\left(D^{(0,1)} f(0,-1) - D^{(0,1)} f(0,1)\right).\end{aligned}$$

The value $\mu = 0$ gives a differential quasi-interpolant $Q^*$ having minimally supported fundamental functions. We have the following result concerning its associated error.

**Proposition 3.** *Let $f \in C^3\left(\mathbb{R}^2\right)$. For every triangle $T$ in $h\tau_2$, there exist both a neighborhood $V_T$, independent of $f$, and constants $C_\alpha > 0$, independent of $h$ and $T$, such that*

$$\left\| D^\alpha \left(Q_h^* f - f\right)\right\|_{\infty,T} \leq C_\alpha h^{3-|\alpha|} \left\| D^3 f\right\|_{\infty,V_T}.$$

*Moreover,*

$$\begin{aligned} C_{0,0} &= \frac{153 + 15\sqrt{10} + 13\sqrt{13}}{648} \simeq 0.381646, \\ C_{1,0} = C_{0,1} &= \frac{198 + 10\sqrt{10} + 13\sqrt{13}}{324} \simeq 0.853379. \end{aligned}$$

We consider the test function, whose graphic is given in Figure 1.

$$f(x,y) = 3(1-x)^2 e^{-x^2-(y+1)^2} - 10\left(\frac{x}{5} - x^3 - y^5\right) e^{-x^2-y^2} - \frac{1}{3} e^{-(x+1)^2-y^2}.$$

Figure 2 shows the errors associated with the new differential quasi-interpolation operator $Q_h^*$ for some different values of the steplength $h$.

In order to show the performance of $Q_h^*$, we also give in Figure 3 the plots of the errors associated with the classical differential quasi-interpolant $\widetilde{Q}_h$ that uses the partial derivatives up to the order two, for the same values of $h$:

$$\widetilde{Q}_h f = \sum_{i \in \mathbb{Z}^2} \left( f(ih) - \frac{h^2}{8}\left(D^{(2,0)} f(ih) + D^{(0,2)} f(ih)\right)\right) \phi\left(\frac{\cdot}{h} - i\right).$$

The operator $Q_h^* f$ obtained solving the minimization problem gives good results when compared with $\widetilde{Q}_h f$, although the latter uses second order partial derivatives.

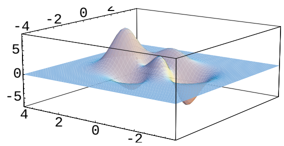

Figure 1: The test function $f$.

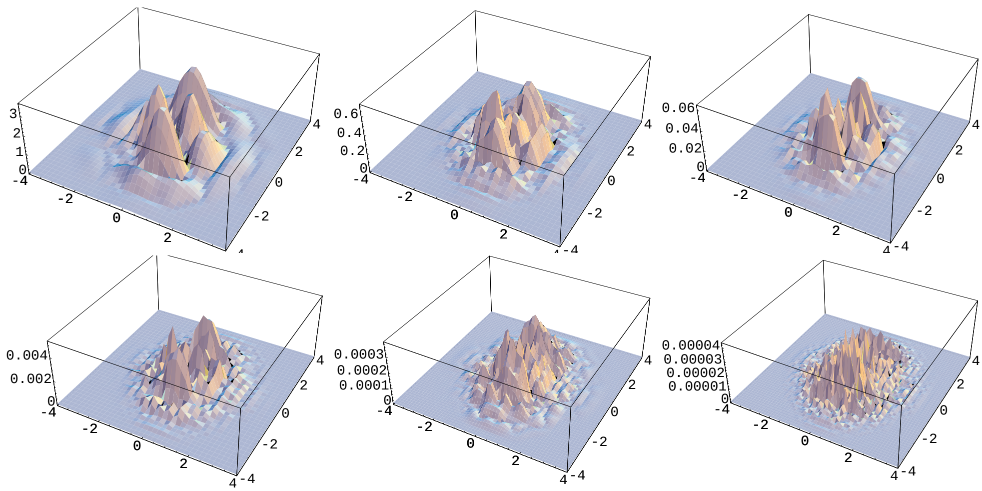

Figure 2: Quasi-interpolation errors $Q_h^* f$ for the test functions for $h = \frac{1}{2^n}$, $0 \le n \le 5$.

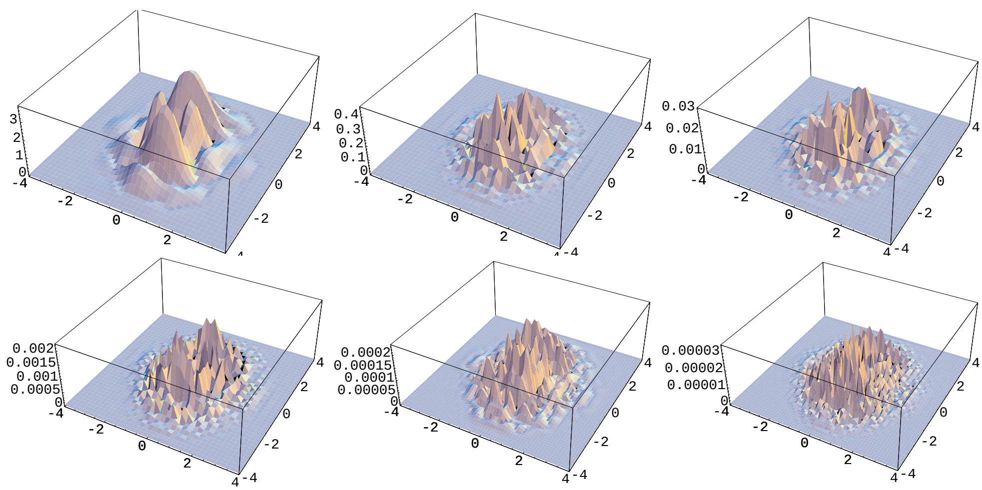

Figure 3: Quasi-interpolation errors $\widetilde{Q}_h f$ for the test functions for $h = \frac{1}{2^n}$, $0 \le n \le 5$.

## §6. An integral example

Let $\tau$ be the uniform mesh of the plane generated by the directions $d_1 := (1,0)$, $d_2 := (0,1)$, $d_3 := d_1 + d_2$ and $d_4 := -d_1 + d_2$. Let $\phi$ be the box spline associated to the direction set $X = \{d_1, d_1, d_2, d_2, d_3, d_4\}$, centered at the origin (cf. [3]). It is one of the two box splines in $\mathbb{P}_4^2(\tau_2)$. It is well known (cf. [2]) that $\mathbb{P}_3$ is the space of maximal total degree included in $\mathcal{S}(\phi)$, that is the construction we have given runs with $n = 3$. It can be easily verified that the unique nonzero values of $\phi$ at the integers are

$$\begin{aligned}
&\phi(0,0) = \frac{5}{12},\\
&\phi(1,0) = \phi(-1,0) = \phi(0,1) = \phi(0,-1) = \frac{1}{8},\\
&\phi(1,1) = \phi(-1,1) = \phi(-1,-1) = \phi(1,-1) = \frac{1}{48}.
\end{aligned}$$

From these values we obtain the following expressions for the polynomials in the Appell sequence associated to $\phi$:

$$\begin{aligned}
&g_{0,0} = 1,\ g_{1,0} = m_{1,0},\ g_{0,1} = m_{0,1},\ g_{2,0} = m_{2,0} - \frac{1}{6},\ g_{1,1} = m_{1,1},\ g_{0,2} = m_{0,2} - \frac{1}{6},\\
&g_{3,0} = m_{3,0} - \frac{1}{6}m_{1,0},\ g_{2,1} = m_{2,1} - \frac{1}{6}m_{0,1},\ g_{1,2} = m_{1,2} - \frac{1}{6}m_{1,0},\ g_{0,3} = m_{0,3} - \frac{1}{6}m_{0,1},\\
&g_{4,0} = m_{4,0} - \tfrac{1}{6}m_{2,0} + \frac{1}{72},\ g_{3,1} = m_{3,1} - \frac{1}{6}m_{1,1},\ g_{2,2} = m_{2,2} - \frac{1}{6}m_{2,0} - \tfrac{1}{6}m_{0,2} + \frac{5}{144},\\
&g_{1,3} = m_{1,3} - \frac{1}{6}m_{1,1},\ g_{0,4} = m_{0,4} - \frac{1}{6}m_{0,2} + \frac{1}{72}.
\end{aligned}$$

After some computations, we get $G_{3,1} = G_{1,3} = 0$, and

$$\begin{aligned}
&\max_{[0,1]^2} G_{4,0} = G_{4,0}\left(\frac{1}{2},0\right) = \frac{1}{384}, && \min_{[0,1]^2} G_{4,0} = G_{4,0}(0,0) = 0,\\
&\max_{[0,1]^2} G_{2,2} = G_{2,2}(0,0) = 0, && \min_{[0,1]^2} G_{2,2} = G_{2,2}\left(\frac{1}{2},\frac{1}{2}\right) = -\frac{1}{192},\\
&\max_{[0,1]^2} G_{0,4} = G_{0,4}\left(0,\frac{1}{2}\right) = \frac{1}{384}, && \min_{[0,1]^2} G_{0,4} = G_{0,0}(0,0) = 0.
\end{aligned}$$

Thus, given a discrete, differential or integral linear form $\lambda$, we obtain the following equations that characterize the solutions of the minimization problem:

$$\begin{aligned}
&\lambda m_{0,0} = 1,\ \lambda m_{1,0} = \lambda m_{0,1} = 0,\\
&\lambda m_{2,0} = \lambda m_{0,2} = -\frac{1}{6},\ \lambda m_{1,1} = 0,\ \lambda m_{3,0} = \lambda m_{2,1} = \lambda m_{1,2} = \lambda m_{0,3} = 0,\\
&\lambda m_{4,0} = \lambda m_{0,4} = \frac{35}{2304},\ \lambda m_{2,2} = \frac{37}{1159},\ \lambda m_{3,1} = \lambda m_{1,3} = 0.
\end{aligned}$$

As a integral linear functional uses a B-spline $\psi$ as weight function in the inner products, we choose $\psi = \phi$. Moreover, let $J$ be the set of the integer $i = (i_1, i_2)$ such that $|i_1|+|i_2| \le 2$.Taking into account that the nonzero moments of $\psi$ are

$$\mu_{0,0} = 1,\ \mu_{2,0} = \mu_{0,2} = \frac{1}{3},\ \mu_{4,0} = \mu_{0,4} = \frac{3}{10},\ \mu_{2,2} = \frac{17}{180},$$

the expansion of $\lambda m_\alpha$, $|a| \le 4$, results in a linear system on $c = (c_j)_{|j_1|+|j_2|\le 2}$ whose unique solution is

$$\begin{aligned} c_{0,0} &= \frac{11071}{2880}, c_{1,0} = c_{0,1} = c_{-1,0} = c_{0,-1} = -\frac{11}{12},\\ c_{2,0} &= c_{0,2} = c_{-2,0} = c_{0,-2} = \frac{991}{11520},\\ c_{1,1} &= c_{-1,1} = c_{-1,-1} = c_{1,-1} = \frac{689}{5760}. \end{aligned}$$

Note that $c$ is a lozenge sequence and so the fundamental function of the associated quasi-interpolant has the same symmetries than the box spline $\phi$.

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