AN ALGEBRAIC THEORY ABOUT SEMICLASSICAL AND CLASSICAL MATRIX ORTHOGONAL POLYNOMIALS

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Abstract. In this paper we introduce an algebraic theory of classical matrix orthogonal polynomials as a particular case of the semi-classical ones, defined by a distributional equation for the corresponding orthogonality functional. This leads to several properties that characterize the classical matrix families, among them, a structure relation and a second order differo-differential equation. In the particular case of Hermite type matrix polynomials we obtain all the parameters associated with the family and we prove that they satisfy, not only a differo-differential equation, but a second order differential one, as it can be seen in the scalar case.

Keywords: Orthogonal matrix polynomials, Semi-classical functionals, Differential equation, Structure relation.

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§1. Introduction

It is known that the most important and useful families $(P_n)_{n\geq 0}$ of orthogonal polynomials (O P) satisfy a second order differential equation $\Phi(x)P_n''(x) + \Psi(x)P_n'(x) = \lambda P_n(x)$ being the polynomial coefficients Φ, Ψ of degree not greater than 2 and 1 respectively. In fact, all the families that satisfy this property, called classical O P families, are known and have been exhaustively studied due to their numerous applications. Among other results, several equivalent characterizations of these families have already been discovered, such as a structure relation for the O P or a distributional equation $D(u\Phi) = u\Psi$ for the corresponding orthogonality functional u, called Pearson type equation.

As an immediate generalization of the classical families, Shohat started the study of the, so called, semi-classical O P [12], defined by a Pearson type equation $D(u\Phi) = u\Psi$ for the corresponding orthogonality functional u, being Φ, Ψ now polynomials with arbitrary degree. In [1, 7], it was given another approach to such polynomials, and Maroni presented in [9, 10] an algebraic theory of semi-classical O P which provide them with characterizations that generalize the known ones for the classical case. In particular, the differential equation for the classical O P becomes a differo-differential equation in the general semi-classical case.

In the last years, the study of matrix O P has attracted a great interest (see [4, 6, 8]). Durán proved in [5] that the matrix O P which satisfy a symmetric second order differential equation with polynomial coefficients are diagonal (up to a factor) with classical scalar O P in the diagonal. In spite of the variety of applications of matrix polynomials [4, 5, 6], there are not too many known families of matrix O P out of the diagonal case.

One way to study many families of matrix OP is to extend the analysis of differential properties of matrix O P started by Durán. A natural way to do this is to generalize the theory of semi-classical scalar O P to the matricial case. In a previous paper (see [2]) we defined the semi-classical matrix OP by a Pearson type equation for the related matrix functionals, obtaining analogous characterizations to the scalar case, among them, a structure relation and a differo-differential equation (in general, non-symmetric) for the matrix O P.

The following natural step is to define the classical matrix OP as a particular case of the semi-classical ones when restricting the degrees of the polynomials in the Pearson type equation. Then, we might obtain what would be the matrix generalizations of the known classical families of OP: Hermite, Laguerre, Jacobi and Bessel. It would be desirable a deeper study of the classical matrix families, obtaininig all the related elements for each family, such as the coefficients of the recurrence relation and the strucutre relation.

Concerning the differo-differential equation and contrary to the scalar case, we will see that for semi-classical matrix O P it does not trivially becomes a differential equation in the classical case. Therefore, in spite of the general study of semi-classical matrix O P, it remains open the problem of finding a characterization of classical matrix O P in terms of a second order differential equation.

The pourpose of this paper is to study the classical matrix O P as a particular case of the semi-classical ones. We will see how all the classical matrix families can be reduced to the Hermite, Laguerre, Jacobi and Bessel type. A detailed analysis of the Hermite family will be presented, obtaining all the related parameters and also a second order differential equation that characterizes them. We let for future papers the study of the remaining classical families, (see [3]).

The paper has been organized as follows: In Section 2 we introduce the basic definitions and notations and some previous results (see [2]) about semi-classical matrix O P needed to define and study the classical case. In Section 3 we introduce the classical matrix O P. In it, it is included the specific study of the Hermite case.

§2. Semi-classical functionals

In the following we shall denote by $\mathbb{P}^{(m)}$ the $\mathbb{C}^{(m,m)}-$left-module:

$$\mathbb{P}^{(m)} := \left\{ \sum_{k=0}^{n} \alpha_k x^k | \alpha_k \in \mathbb{C}^{(m,m)}; n \geq 0 \right\}$$

and by means of $\mathbb{P}^{(m)\prime}$ the $\mathbb{C}^{(m,m)}$-right-module $\mathrm{Hom}\big(\mathbb{P}^{(m)}, \mathbb{C}^{(m,m)}\big)$.

Forall $P \in \mathbb{P}^{(m)}$ and $u \in \mathbb{P}^{(m)\prime}$ the duality bracket is defined by $\langle P, u\rangle := u(P)$.

For $k \geq 0$ and $u \in \mathbb{P}^{(m)\prime}$ the linear functional $ux^kI \in \mathbb{P}^{(m')}$ is given by

$$\langle P, ux^kI\rangle := \langle x^kP, u\rangle,$$

where I denotes the $m \times m$ identity matrix. A linear extension gives the right-product $uP \in$

$\mathbb{P}^{(m)}$ for $u \in \mathbb{P}^{(m)\prime}$, $P \in \mathbb{P}^{(m)}$, with $P(x) = \sum_{k=0}^{n} p_k x^k$, in the following way

$$\langle Q, uP\rangle = \sum_{k=0}^{n} \langle x^k Q, u\rangle p_k.$$

With this notation we have that if $P \in \mathbb{P}^{(m)}$ and $u \in \mathbb{P}^{(m)\prime}$, it is $\langle \alpha P, u\beta\rangle = \alpha\langle P, u\rangle\beta$, $\forall \alpha, \beta \in \mathbb{C}^{(m,m)}$.

The inner product associated with $u \in \mathbb{P}^{(m)\prime}$ is defined by

$$\langle P, Q\rangle_u := \langle P, uQ^*\rangle$$

where Q^* denotes the trasposed conjugated of Q. This inner product satisfies

$$\langle \alpha P, Q\beta\rangle_u = \alpha\langle P, Q\rangle_u \beta^* \qquad \forall \alpha, \beta \in \mathbb{C}^{(m,m)}$$

for $P, Q \in \mathbb{P}^{(m)}$ and $u \in \mathbb{P}^{(m)\prime}$.

We denote by $C_k := \langle x^k I, u\rangle$ the k-th moment with respect to $u \in \mathbb{P}^{(m)\prime}$.

Given $u \in \mathbb{P}^{(m)\prime}$ with moments $(C_k)_{k\geq 0}$, we say that u is quasi-definite (or non-singular) if det $[(C_{k+j})_{k,j=0}^{n}] \neq 0$, $\forall n \geq 0$, where $(C_{k+j})_{k,j=0}^{n}$ is the Hankel-block matrix

$$\begin{pmatrix} C_0 & C_1 & \dots & C_n \\ C_1 & C_2 & \dots & C_{n+1} \\ \dots & \dots & \dots & \dots \\ C_n & C_{n+1} & \dots & C_{2n} \end{pmatrix}.$$

Finally, we consider the derivative operator on the space $\mathbb{P}^{(m)\prime}$ as the linear operator $D : \mathbb{P}^{(m)\prime} \to \mathbb{P}^{(m)\prime}$ such that

$$\langle P, Du\rangle = -\langle P', u\rangle.$$

From the latter and the definition of the right-product it is straightforward to prove that $D(uA) = (Du)A + uA'$, for all $u \in \mathbb{P}^{(m)\prime}$ and $A \in \mathbb{P}^{(m)}$. (See [2]).

Remark 1. Given the sequence $(C_k)_{k\geq 0} \subset \mathbb{C}^{(m,m)}$, there exists a unique $u \in \mathbb{P}^{(m)\prime}$ such that $\langle x^k I, u\rangle = C_k$. This establishes an isomorphism between $\mathbb{P}^{(m)\prime}$ and the $\mathbb{C}^{(m,m)}-$right-module of formal power series with coefficients in $\mathbb{C}^{(m,m)}$, $\sum_{k=0}^{n} C_k x^k$.

Definition 1. A functional $u \in \mathbb{P}^{(m)\prime}$ is hermitian if $C_k^* = C_k$ for all $k \geq 0$.

Theorem 1. *Let $u \in \mathbb{P}^{(m)\prime}$ be quasi-definite. Then there exists a unique (up to non-singular left matrix factors) sequence of left orthogonal matrix polynomials $(P_n)_{n\geq 0}$ with respect to u, that is:*

(i) $P_n \in \mathbb{P}^{(m)}$, $dgP_n = n$.
(ii) The leading coefficient of P_n is non-singular.
(iii) $\langle P_n, P_m\rangle_u = K_n\delta_{nm}$, where K_n is non-singular.
This sequence verifies a recurrence relation:

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$$

with $P_0 \in \mathbb{C}^{(m,m)}$ non-singular, $P_{-1} = \theta$ (where θ denotes the zero matrix), and α_n, β_n, $\gamma_n \in \mathbb{C}^{(m,m)}$, α_n, γ_n non-singular.

Proof. See [4, 11] □

The last result of this theorem has a converse (Favard's Theorem): for any sequence $(P_n)_{n\geq 0}$ verifying the above recurrence relation there exists a unique (up to factors) quasi-definite functional u so that $(P_n)_{n\geq 0}$ is its sequence of matrix orthogonal polynomials (see [4, 11]).

Now, we will introduce the semi-classical character for a functional $u \in \mathbb{P}^{(m)'}$.

Definition 2. Let $u \in \mathbb{P}^{(m)'}$ be quasi-definite. We will say that u is semi-classical if there exist $A, B \in \mathbb{P}^{(m)}$, with det $A \neq 0$, so that it is verified the distributional equation

$$D\,(uA) = uB \tag{1}$$

called Pearson type equation. We will also say that the corresponding sequence of left orthogonal matrix polynomials $(P_n)_{n\geq 0}$ is semi-classical.

If $u \in \mathbb{P}^{(m)'}$ is semiclassical, for every $C \in \mathbb{P}^{(m)}$,

$$D\,(uAC) = u\,(AC' + BC)$$

holds. It is just a consequence of the rule for the derivation of the right product. This result implies that, for every $u \in \mathbb{P}^{(m)'}$, the set

$$\mathcal{M}_u = \{A \in \mathbb{P}^{(m)} \mid D(uA) = uB,\ B \in \mathbb{P}^{(m)}\}$$

is a right-ideal of $\mathbb{P}^{(m)}$. However, contrary to the scalar case, it is not necessarily principal because the euclidian division algorithm is not valid in $\mathbb{P}^{(m)'}$ and, therefore, we cannot use $\mathcal{M}_u$ for the classification of semi-classical functionals.

We can solve the classification problem in the following way: If $u \in \mathbb{P}^{(m)'}$ is semi-classical then there exists $\alpha \in \mathbb{P} \setminus \{0\}$ so that $\alpha I \in \mathcal{M}_u$ (i.e., if $D(uA) = uB$, just choosing $\alpha I = A(\text{adj}A) = (\text{det}A)I$). Moreover,

$$\widetilde{\mathcal{M}}_u = \{\alpha \in \mathbb{P} \mid D(u\alpha I) = uB,\ B \in \mathbb{P}^{(m)}\}$$

is a bilateral ideal of $\mathbb{P}$ that is principal. So, there exists an $\alpha \in \mathbb{P}$, unique up non-trivial factors in $\mathbb{C}$, that is generator of $\widetilde{\mathcal{M}}_u$. We can use the essentially unique generator of this ideal to clasify the semi-classical matrix functional similarly to the scalar case.

Definition 3. Let $u \in \mathbb{P}^{(m)'}$ be semi-classical and let $\alpha \in \mathbb{P}\backslash\{0\}$ be a polynomial with smallest degree such that $D(u\alpha I) = uB$, $B \in \mathbb{P}^{(m)}$. Then, we say the class of u is s =max$\{$dg$\alpha - 2$, dg$B - 1\}$.

The following theorem provides with characterizations of semi-classical matrix functionals in terms of structure relations and differo-differential equations of second order.

Theorem 2. *Let $u \in \mathbb{P}^{(m)'}$ be quasi-definite and let $(P_n)_{n\geq 0}$ be the associated sequence of left orthogonal matrix polynomials. Then, the following statements are equivalent:*

(i) u is semi-classical.

(ii) (Structure relation) There exists a polynomial $\alpha \in \mathbb{P}\backslash\{0\}$ *with dg*$\alpha = p$, $s \geq$*max*$\{0,\ p-2\}$ *and* $\Theta_j^{(n)} \in \mathbb{C}^{(m,m)}$ $(n \geq 0, -s \leq j \leq p)$, *so that,*

$$\alpha(x)P'_{n+1}(x) = \sum_{j=-s}^{p} \Theta_j^{(n)} P_{n+j}(x),$$

where $\Theta_{-s}^{(n)} \neq \theta$ *for some* $n \geq s$ *(we use the convention* $P_k = \theta$ *for* $k < 0$).

(iii) (Differo-differential equation) There exist two polynomials $\alpha, \beta \in \mathbb{P}$ *with dg*$\alpha = p \geq 0$, *dg*$\beta = q$ *and matrices* $\Lambda_k^{(n)} \in \mathbb{C}^{(m,m)}$ $(n \geq 0, -s \leq k \leq s)$, *such that,*

$$\alpha(x)P''_n(x) + \beta(x)P'_n(x) = \sum_{k=-s}^{s} \Lambda_k^{(n)} P_{n+k}(x),$$

where $s \geq$*max*$\{p-2, q-1\}$ *(we use the convention* $P_k = \theta$ *for* $k < 0$*).*

Proof. See [2]. □

§3. Classical Hermite type matrix polynomials

We will define the classical functionals anagously to the scalar case.

Definition 4. We will say that $u \in \mathbb{P}^{(m)'}$ is classical if it is semi-classical and its class is $s = 0$.

Let us suppose a classical functional $u \in \mathbb{P}^{(m)'}$, that is, satisfying a Pearson type equation,

$$D\,(u\alpha I) = uB, \qquad \alpha \in \mathbb{P}\backslash\{0\}, \qquad B \in \mathbb{P}^{(m)}, \tag{2}$$

with dg$\alpha \leq 2$ and dg$B \leq 1$. We will write

$$B(x) = B_1 x + B_0, \qquad B_0, B_1 \in \mathbb{P}^{(m)}.$$

Taking into account the different posibilities for the degree and the roots of α, we get that every classical functional of $\mathbb{P}^{(m)'}$ belongs, up to afine transformations of x, to one of the following “canonical” types:

⋆ Hermite: $\alpha = 1$
⋆ Laguerre: $\alpha = x$
⋆ Bessel: $\alpha = x^2$
⋆ Jacobi: $\alpha = x^2 - 1$

The study of classical matrix orthogonal polynomials implies the analysis of the functionals and polynomials that belong to each canonical type, obtaining the related parameters like the coefficients of the recurrence and the structure relations.

Notice that Theorem 2 does not ensure that the classical matrix orthogonal polynomials satisfy a second order differential equation, since $\Lambda_k^{(n)}$ can be different from zero for $k \neq 0$ in the differo-differential equation. Thus, apart from the previous work, it remains open a question: do the classical matrix orthogonal polynomials satisfy a second order differential equation?

We will solve above problems for the Hermite case.

In what follows $(P_n)_{n\geq 0}$ will be the sequence of monic orthogonal matrix polynomials related to the quasi-definite functional u. We will normalize this functional by means of $< I, u >= C_0 = I$. We will also use the notation

$$E_n = \langle P_n, P_n \rangle_u$$

$$P_n(x) = x^n + \pi_n x^{n-1} + \dots$$

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The corresponding distributional equation is

$$Du = u\left(B_1 x + B_0\right), \qquad B = B_1 x + B_0,$$

which means that

$$< (x^k)', u >= - < x^k, uB >, \qquad k \geq 0.$$

This condition is equivalent to the following relation between the moments

$$C_{k+1}B_1 + C_k B_0 + kC_{k-1} = 0, \qquad k \geq 1 \tag{3}$$

$$C_1 B_1 + B_0 = 0, \qquad k = 0. \tag{4}$$

Hence, $C_1 = -B_0 B_1^{-1}$ because B_1 is a non-singular matrix since

$$E_1 B_1 =< P_1, uB >= - < P_1', u >= -I.$$

Taking into account that $\langle P_1, u \rangle = \langle P_1, P_0 \rangle_u = 0$, we get for $P_1(x) = xI + \pi_1$ that

$$\pi_1 = -C_1 = B_0 B_1^{-1}.$$

As a consequence, π_1 is non-singular too.

Notice that, for $0 \leq k \leq n-2$

$$0 = \langle P_n(x)x^k, uB(x) \rangle = -k\langle P_n(x)x^{k-1}, u \rangle - \langle P_n'(x)x^k, u \rangle = -\langle P_n'(x), x^k \rangle_u$$

and thus, the structure relation for these polynomials is

$$P_n' = nP_{n-1}.$$

Therefore, identifiying coefficients we get that $n\pi_{n-1} = (n-1)\pi_n$.

The recurrence relation is

$$xP_n(x) = \gamma_n P_{n-1}(x) + \beta_n P_n(x) + P_{n+1}(x).$$

So, we have the relations

$$\gamma_n = E_n E_{n-1}^{-1}, \qquad \beta_n = \pi_n - \pi_{n+1}.$$

By substituting the expresion for the polynomial P_n' given by the structure relation in the recurrence relation we obtain,

$$xP_n(x) = \frac{\gamma_n}{n} P_n'(x) + \beta_n P_n(x) + P_{n+1}(x)$$

and, derivating,

$$xP_n'(x) + P_n(x) = \frac{\gamma_n}{n} P_n''(x) + \beta_n P_n'(x) + (n+1)P_n(x).$$

Therefore,

$$\frac{\gamma_n}{n} P_n''(x) - (xI - \beta_n)P_n'(x) + nP_n(x) = 0$$

is the second order differential equation for the Hermite type polynomials. We can obtain the coefficientes γ_n, using the structure relation to get that

$$E_n B_1 = \langle P_n(x)x^{n-1}, uB(x)\rangle = -(n-1)\langle P_n(x)x^{n-2}, u\rangle - \langle P_n'(x)x^{n-1}, u\rangle = -nE_{n-1}$$

which implies $E_n = -nE_{n-1}B_1^{-1}$ and, together with $E_0 = I$, gives

$$E_n = (-1)^n n! B_1^{-n}.$$

So,

$$\gamma_n = -nB_1^{-1}.$$

Besides, from the relations, $(n-1)\pi_n = n\pi_{n-1}$ and $\pi_1 = B_0 B_1^{-1}$ we find that

$$\pi_n = n\pi_1 = nB_0 B_1^{-1}$$

and, as a consequence,

$$\beta_n = -B_0 B_1^{-1}.$$

Finally, the differential equation is

$$P_n''(x) + \left(B_1 x + B_1 B_0 B_1^{-1}\right) P_n'(x) - nB_1 P_n(x) = 0.$$

Using the orthogonal polynomials $Q_n := B_1^{-1} P_n$, we can write the differential equation and recurrence relation in the following way

$$Q_n''(x) + (B_1 x + B_0)\, Q_n'(x) - nB_1 Q_n(x) = 0,$$
$$B_1 Q_{n+1}(x) = (B_1 x + B_0)Q_n(x) + nQ_{n-1}(x).$$

Concerning the inverse problem, notice that, for any $B_0, B_1 \in \mathbb{C}^{(m,m)}$ with B_1 non-singular, relations (3) and (4) define the moments of a functional u satisfying a Pearson type equation $Du = u(B_1 x + B_0)$. It can be also proved that the non-singularity of B_1 ensures that the functional u is quasi-definite. Therefore, each choice of $B_0, B_1 \in \mathbb{C}^{(m,m)}$, B_1 non-singular, determines a unique sequence (up to non-singular left factors) of Hermite matrix polynomials. This sequence provides a matrix solution Q_n of the recurrence relation

$$B_1 y_{n+1} = (B_1 x + B_0)y_n + ny_{n-1}, \quad n \geq 0, \tag{5}$$
$$y_{-1} = \theta,$$

being the rest of them $y_n = Q_n K_n$, $K_n \in \mathbb{C}^{(m,m)}$. The polynomial solutions of the differential equations

$$y_n'' + (B_1 x + B_0)y_n' - nB_1 y_n = 0, \quad n \geq 0, \tag{6}$$

have the same form because they have a unique matrix polynomial solution of degree n up to right factors. Hence, the Hermite polynomials Q_n are the essentially unique polynomial solutions of (5) and (6) and, thus, any of them characterizes the Hermite matrix polynomials.

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