ON STRONGLY DEGENERATE PARABOLIC PROBLEMS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. It is shown that both a model of traffic flow with driver reaction and discontinuous road surface and a model of continuous sedimentation of flocculated suspensions give rise to strongly degenerate parabolic problems with discontinuous coefficients. For one of these models, an existence and uniqueness theory is outlined, which includes the convergence proof for a simple difference scheme. This scheme is used to produce numerical simulations of both applications.

Keywords: strongly degenerate parabolic equation, discontinuous flux, traffic model, continuous sedimentation, difference scheme

AMS classification: 35L65, 35R05, 65M06, 76T20

§1. Introduction

In recent years we have seen an increased interest in degenerate parabolic equations with discontinuous coefficients of the type

$$u_t + f\big(\boldsymbol{\gamma}_1(x), u\big)_x = \big(\gamma_2(x)A(u)_x\big)_x, \quad x \in \mathbb{R}, t > 0, \tag{1}$$

where $\boldsymbol{\gamma}_1(x)$ and $\gamma_2(x)$ are discontinuous parameters, and $\boldsymbol{\gamma}_1(x)$ may be vector-valued. We assume $A'(\cdot) \geq 0$, which includes a first-order conservation law. Solutions of (1) are in general discontinuous and need to be defined as entropy weak solutions. For smooth coefficients, equation (1) is in included in the classical well-posedness (existence and uniqueness) theory by Vol'pert and Hudjaev [9], but this calculus breaks down when fluxes and depend discontinuously on the location x. We refer to [5, 6] for a historical account on conservation laws and degenerate parabolic equations with discontinuous coefficients.

We consider here two applications giving rise to equations of type (1). In §2 we first consider Mochon's extension [7] of the Lighthill-Whitham-Richards (LWR) kinematic traffic model to roads with abruptly changing surface conditions, which gives rise to a flux density function that varies discontinuously with respect to x. This model is then combined with a diffusively corrected kinematic wave model [8], which leads to an additional second-order diffusion term accounting for the drivers' anticipation length and reaction time. The final model appears in two variants (Models 1 and 2). Both have discontinuous parameters in the flux but only Model 2 also includes a discontinuous parameter in the diffusion.

In §3 we outline a model of continuous sedimentation of flocculated suspensions in so-called clarifier-thickener units (Model 3), which includes a degenerate diffusion term with a discontinuous parameter.

In [5, 6] a rather general well-posedness (existence and uniqueness) theory is developed for strongly degenerate convection-diffusion equations with discontinuous coefficients. Since in [5, 6] the diffusion function is not allowed to depend on x, we restrict ourselves in §4 to the initial value problem for Model 1. It seems, however, likely that the analysis can be extended to the x-dependent diffusion case, including Models 2 and 3. This is work in progress [4].

The existence proof for Model 1 outlined in §4 is constructively based on proving convergence of a modification of the Engquist-Osher finite difference scheme. This scheme is applied in §5 to produce two traffic flow simulations for Model 1. Furthermore we present a simulation of the operation of a clarifier-thickener (Model 3) obtained from a semi-implicit variant of that scheme.

§2. Traffic flow with driver reaction and abruptly changing road surface

The classical LWR kinematic model for car traffic on a single-lane highway starts from the conservation equation $\rho_t(x,t) + \big(\rho(x,t)v(x,t)\big)_x = 0$, where ρ is the density of cars as a function of distance x and time t and $v(x,t)$ is the velocity of the car located at point x at time t. The main constitutive assumption is that v is a function of ρ only, $v = v(\rho)$, which yields the conservation law $\rho_t + (\rho v(\rho))_x = 0$ for $x \in \mathbb{R}$ and $t > 0$. Thus, each 'driver' instantaneously adjusts his velocity to the local car density. We assume here that $v(\rho) = v_{\max} V(\rho)$, where $v_{\max}$ is the preferred velocity on a free highway, and $V(\rho)$ is a 'hindrance' function modeling the presence of other cars that urge each driver to reduce his speed. We define

$$\rho v(\rho) = v_{\max} f(\rho), \quad f(\rho) := \chi_{[0,\rho_{\max}]}(\rho)\rho V(\rho), \tag{2}$$

where $\rho_{\max}$ is the maximum car density corresponding to the 'bumper-to-bumper' situation. In this note, we restrict ourselves to the Dick-Greenberg model $V(\rho) = V_{\mathrm{DG}}(\rho) = \min\{1, C\ln(\rho_{\max}/\rho)\}$, where C is a parameter. Common examples besides $V_{\mathrm{DG}}(\rho)$ and further references are listed in [2].

Mochon [7] extended the LWR model to abruptly changing road surface conditions by letting $v_{\max}$ depend discontinuously on x, i.e. $v_{\max} = v_{\max}(x)$. For example, the largest part of the highway may admit a velocity $v^0_{\max}$, and there may be one road segment $[a,b]$ experiencing heavy rainfall, fog, or bad pavement, which enforces a reduction of the maximum velocity to a value $v^*_{\max} < v^0_{\max}$. Alternatively, we could assume that the segment $[a,b]$ admits a higher maximum velocity $v^*_{\max} > v^0_{\max}$. Thus, we consider

$$v_{\max}(x) = \begin{cases} v^0_{\max} & \text{for } x < a \text{ and } x > b, \\ v^*_{\max} & \text{for } x \in (a,b), \end{cases} \quad v^*_{\max} < v^0_{\max} \text{ (Case A) or } v^*_{\max} > v^0_{\max} \text{ (Case B).} \tag{3}$$

We now let $\gamma(x) := v_{\max}(x)$ and rewrite the conservation law $\rho_t + (\rho v(\rho))_x = 0$, modified by (3), as $\rho_t + (\gamma(x)f(\rho))_x = 0$ for $x \in \mathbb{R}$ and $t > 0$.

We now turn to the second ingredient of the new model. The realism of the assumption $v = v(\rho)$ has been seriously questioned. A potentially more realistic model due to Nelson [8] considers a reaction time τ, representing drivers' delay in their response to events, and an anticipation distance L that partially compensates this delay. Consequently, the velocity function $V(\rho(\cdot,\cdot))$ is evaluated not at the point x, but at $x+L$. The reaction time is included by replacing the argument t by $t-\tau$, and reducing the argument $x+L$ by the distance $v_{\max}V\tau$ traveled in time τ, see [8]. These considerations lead to $v(x,t) = v_{\max}V(\rho(x+L-v_{\max}V\tau, t-\tau))$. Expanding this expression around $\rho(x,t)$, we arrive at [2, 8]

$$\rho v = v_{\max}\big[\rho V(\rho) + \rho V'(\rho)\big(L + \tau\rho v_{\max}V'(\rho)\big)\partial_x\rho\big] + \mathcal{O}(\tau^2 + L^2). \tag{4}$$

The reaction length L can also be considered to depend on $v(\rho)$, i.e., $L = L(\rho) = L(v(\rho))$, see [8]. Neglecting the $\mathcal{O}(\tau^2+L^2)$ term, inserting the right-hand side of (4) into $\rho_t + (\rho v)_x = 0$, and assuming that up to a critical car density $0 \leq \rho_{\mathrm{c}} \leq \rho_{\max}$, the diffusion effect is not present, we finally obtain

$$\rho_t + f(\rho)_x = D(\rho)_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{5}$$

$$\text{where} \quad D(\rho) := \int_0^\rho d(s)\,ds, \qquad d(\rho) := -\chi_{[\rho_{\mathrm{c}},\rho_{\max}]}(\rho)\rho v_{\max}V'(\rho)\big(L(\rho) + \tau v_{\max}\rho V'(\rho)\big), \tag{6}$$

where $f(\rho)$ is given by (2). There are at least two motivations for postulating a critical density ρ_{c}. One of them [8] is based on using $V(\rho) = V_{\mathrm{DG}}(\rho)$. Since $V'_{\mathrm{DG}}(\rho) = 0$ for $\rho \leq \rho_{\mathrm{c}} := \rho_{\max}\exp(-1/C)$, we see that setting $V(\rho) = V_{\mathrm{DG}}(\rho)$, (6) is satisfied. A more general viewpoint is that drivers' reaction is instantaneous in relatively free traffic flow, when $\rho \leq \rho_{\mathrm{c}}$, and otherwise is modeled by the diffusion term. Both views give rise to the same behaviour of the diffusion coefficient. We assume that $V(\rho)$ is chosen such that $\tilde{d}(\rho) > 0$ for $\rho_{\mathrm{c}} < \rho < \rho_{\max}$. Thus, the right-hand side of (5) vanishes on $[0,\rho_{\mathrm{c}}]$, and possibly at $\rho_{\max}$. Thus, (5) is a second-order degenerate parabolic quasilinear partial differential equation. Since (5) degenerates on the ρ-interval of positive length $[0,\rho_{\mathrm{c}}]$, we call (5) strongly degenerate parabolic.

Finally, we point out that Nelson [8] suggests a dependence $L = L(v(\rho))$ of the type $L(v(\rho)) = \tilde{L}(v(\rho)) := \max\{(v(\rho))^2/(2a), L_{\min}\}$, where the first argument is the distance required to decelerate to full stop from speed $v(\rho)$ at deceleration a, and the second is a minimal anticipation distance.

The two modifications of the LWR model discussed so far are now combined into an equation for traffic flow with drivers' anticipation *and* changing road surface conditions. We will do so by admitting two variants, referred to as 'Model 1' and 'Model 2'. Model 1 assumes that the diffusion term models 'driver psychology' and should therefore be independent of road surface conditions. Thus, Model 1 is produced by replacing $f(\rho)$ in (5) by the expression $\gamma(x)f(\rho)$ appearing in $\rho_t + (\gamma(x)f(\rho))_x = 0$, but at the same time the driver reaction is determined by the constant value $v_{\max} = v^0_{\max}$ in (3).

The more involved Model 2 is based on replacing *every* occurrence of $v_{\max}$ by $v_{\max}(x)$ in the derivation of (5). Since the expansion (4) remains valid if we replace the constant $v_{\max}$ by $v_{\max}(x)$, even when $v_{\max}$ is discontinuous, it is straightforward to derive the strongly degenerate convection-diffusion equation

$$\rho_t + \big(\gamma(x)f(\rho)\big)_x = D_2\big(\rho,\gamma(x)\big)_{xx}, \quad D_2\big(\rho,\gamma(x)\big) := \int_0^\rho \chi_{[\rho_{\mathrm{c}},\rho_{\max}]}(s)\gamma(x)r\big(s;\tau,L(s,\gamma(x))\big)\,ds \tag{7}$$

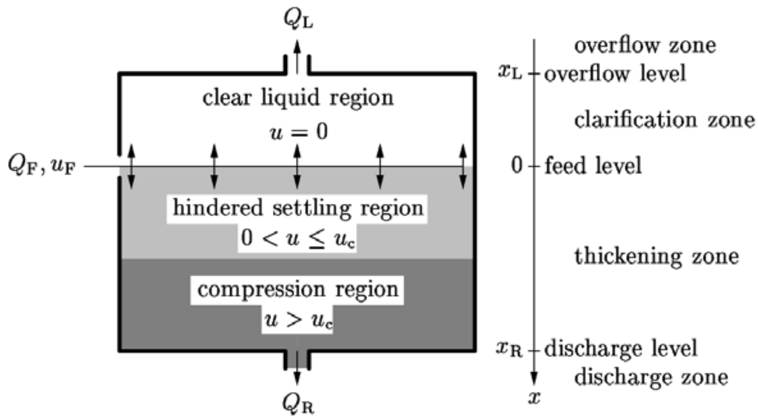

Figure 1: Schematic drawing of a clarifier-thickener unit.

for Model 2. Note that L in general depends on $v(\rho)$. Thus, if $v(\rho)$ depends on $\gamma(x)$, then $L = L(\rho, \gamma(x))$. Observe that the use of $L = \tilde{L}(v(\rho))$, for example, implies that D_2 depends nonlinearly on $\gamma(x)$.

Model 1 leads to the simpler strongly degenerate convection-diffusion equation

$$\partial_t \rho + \partial_x\big(\gamma(x) f(\rho)\big) = \partial_x^2 D_1(\rho), \quad x \in \mathbb{R},\ t > 0, \quad \text{with } D_1(\rho) := D_2(\rho, v_{\max}^0). \tag{8}$$

Observe that we can combine the two models into the single equation

$$\rho_t + \big(\gamma^1(x) f(\rho)\big)_x = \big(\gamma^2(x) R(\rho, \gamma^2(x))_x\big)_x, \quad R\big(\rho, \gamma^2(x)\big) := \big(1/\gamma^2(x)\big) \int_0^\rho d\big(s, \gamma^2(x)\big)\, ds, \tag{9}$$

where $\gamma^1(x) := \gamma(x) = v_{\max}(x)$ for both models, $\gamma^2 \equiv v_{\max}^0$ for Model 1 and $\gamma^2(x) := \gamma(x)$ for Model 2, where $v_{\max}^0$ and $v_{\max}(x)$ are defined in (3). For $L = \tau = 0$, (9) reduces to the first-order equation $\rho_t + (\gamma(x) f(\rho))_x = 0$. For simplicity, we set $[a, b] = [0, 1]$, and consider a prescribed initial car density

$$\rho(x, 0) = \rho_0(x), \quad x \in \mathbb{R}. \tag{10}$$

§3. Clarifier-thickener models

The settling of a monodisperse, flocculated suspension of small particles in a fluid [1] can be described by the following second-order PDE for the solids concentration u as a function of depth x and time t:

$$u_t + \big(q(x,t) u + h(u)\big)_x = A(u)_{xx}, \tag{11}$$

where $q(x, t)$ is the local bulk velocity of the mixture, the function $h(u)$ is the hindered settling function satisfying $h(u) = 0$ for $u \leq 0$ and $u \geq 1$, $h(u) > 0$ for $0 < u < 1$, $h'(0) > 0$ and

$h'(1) \leq 0$, and $A(u)$ is an integrated diffusion coefficient accounting for sediment compressibility. Usually it is assumed that

$$A(u) := \int_0^u a(s)\,dx, \quad a(u) := \frac{h(u)\sigma_{\rm e}'(u)}{\Delta\varrho g u}, \quad \sigma_{\rm e}'(u) := \frac{d}{du}\sigma_{\rm e}(u), \quad \sigma_{\rm e}(u), \sigma_{\rm e}'(u) \begin{cases} = 0 & \text{for } u \leq u_{\rm c}, \\ > 0 & \text{for } u > u_{\rm c}, \end{cases} \tag{12}$$

where $\sigma_{\rm e}(u)$ is the effective stress function (the second function besides $h(u)$ characterizing the suspension), $\Delta\varrho > 0$ is the solid-fluid density difference, g is the acceleration of gravity, and $u_{\rm c}$ is a critical concentration at which the solid particles are assumed to touch each other. See [1] for details. From (12) we infer that $A(u) = 0$ (and thus (11) is first-order hyperbolic) for $u \leq u_{\rm c}$ and $u = 1$, and otherwise $A'(u) > 0$ and (11) is second-order parabolic. Consequently, (11) is strongly degenerate parabolic.

We now extend (11) to continuously operated clarifier-thickener units as drawn in Figure 1. At $x = 0$, fresh suspension (with known solids concentration $u_{\rm F}$) is pumped into the unit at a volume rate $Q_{\rm F} \geq 0$. This inflow is divided into an upwards-directed overflow $Q_{\rm L} \leq 0$ and a downwards-directed discharge underflow $Q_{\rm R} \geq 0$. We always have $Q_{\rm F} = Q_{\rm L} - Q_{\rm R}$, so that $Q_{\rm L}$ and $Q_{\rm R}$ are independent control flow variables. At $x = x_{\rm L}$ and $x = x_{\rm R}$, the mixture leaves the unit through a thin pipe in which the solids and the fluid move at the same speed. Thus, the functions $h(u)$ and $a(u)$ are "switched off" for $x \leq x_{\rm L}$ and $x \geq x_{\rm R}$. We are interested in u as a function of x and t. Figure 1 shows a typical steady-state situation, in which no solids pass into the upper (clarification) zone, and the lower (thickening) zone is divided into a hindered settling region (where $u \leq u_{\rm c}$) and a compression region (where $u > u_{\rm c}$).

The splitting of the feed flow into two diverging bulk flows and the reduction of (11) to a linear transport equation (since $h(u)$ and $a(u)$ are "switched off") at $x \leq x_{\rm L}$ and $x \geq x_{\rm R}$ imply the following governing equation for the clarifier-thickener unit, where $S(x)$ is the cross-sectional area:

$$S(x)u_t + G(x,u)_x = \big(\gamma_1(x)A(u)_x\big)_x, \quad x \in \mathbb{R}, \quad t > 0, \gamma_1(x) := \chi_{(x_{\rm L},x_{\rm R})}(x), \tag{13}$$

$$G(x,u) = \begin{cases} Q_{\rm L}(u-u_{\rm F}) & \text{for } x < x_{\rm L}, \quad Q_{\rm R}(u-u_{\rm F}) + S(x)h(u) \quad \text{for } 0 < x < x_{\rm R}, \\ Q_{\rm L}(u-u_{\rm F}) + S(x)h(u) & \text{for } x_{\rm L} < x < 0, \quad Q_{\rm R}(u-u_{\rm F}) \quad \text{for } x > x_{\rm R}. \end{cases}$$

In this note we limit ourselves to a vessel with constant interior cross-sectional area, i.e., we let $S(x) = S_0$ for $x < x_{\rm L}$ and $x > x_{\rm R}$ and $S(x) = S_{\rm int}$ otherwise. In this case, the solution does not depend on the value of S_0, as is shown in [4], and defining the velocities $q_{\rm R} := Q_{\rm R}/S_{\rm int}$, $q_{\rm L} := Q_{\rm L}/S_{\rm int}$, and the diffusion functions $\mathsf{a}(\cdot) := a(\cdot)/S_{\rm int}$, $\mathsf{A}(\cdot) := A(\cdot)/S_{\rm int}$, we finally obtain the clarifier-thickener model

$$u_t + g(x,u)_x = \big(\gamma_1(x)\mathsf{A}(u)_x\big)_x, \quad x \in \mathbb{R}, \quad t > 0, \tag{14}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{15}$$

$$g(x,u) := \begin{cases} q_{\rm L}(u-u_{\rm F}) & \text{for } x < x_{\rm L}, \quad q_{\rm R}(u-u_{\rm F}) + h(u) \quad \text{for } 0 < x < x_{\rm R}, \\ q_{\rm L}(u-u_{\rm F}) + h(u) & \text{for } x_{\rm L} < x < 0, \quad q_{\rm R}\big(v-u_{\rm F}\big) \quad \text{for } x > x_{\rm R}. \end{cases} \tag{16}$$

We refer to (14)–(16) as *Model 3*. Observe that the variation of $S(x)$ at $x = x_{\rm L}$ and $x = x_{\rm R}$ does no longer appear in (14). This significantly facilitates the analysis. Finally, we define the vector of discontinuity parameters $\boldsymbol{\gamma} := (\gamma_1, \gamma_2)$ with $\gamma_2(x) = Q_{\rm L}$ for $x < 0$ and $\gamma_2(x) = Q_{\rm R}$ for $x > 0$, and the flux function $f(\boldsymbol{\gamma}(w), v) := g(x,u) = (\gamma_1(x)/S_{\rm int})h(u) + (\gamma_2(x)/S_{\rm int})(u-u_{\rm F})$.

§4. Mathematical analysis

We here recall the main results of [5, 6], applied to Model 1, under the following assumptions on ρ_0:

$$\rho_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R});\ \ \rho_0(x) \in [0, \rho_{\max}],\ \ A(\rho_0) \text{ is absolutely continuous on } \mathbb{R};\ \ A(\rho_0)_x \in BV(\mathbb{R}). \tag{17}$$

Thus, any jump in ρ_0 must be contained within $[0, \rho_{\mathrm{c}}]$. However, this is a technical condition for the mathematical analysis, but one may use more general initial conditions for numerical simulations (§ 5).

Independently of the smoothness of $\gamma(x) = v_{\max}(x)$, solutions to (8), (10) can be discontinuous since $D_1'(\rho) = 0$ for $\rho \leq \rho_{\mathrm{c}}$. Therefore (8), (10) need to be interpreted in the weak (distributional) sense. Moreover, one needs an additional condition, the entropy condition, to single out a unique solution.

We denote by $\mathcal{M}(\Pi_T)$ the finite Radon (signed) measures on Π_T. The space $BV(\Pi_T)$ of functions of bounded variation is defined as the set of locally integrable functions $W : \Pi_T \to \mathbb{R}$ for which $\partial_x W, \partial_t W \in \mathcal{M}(\Pi_T)$. In this paper we use the space $BV_t(\Pi_T)$ of locally integrable functions $W : \Pi_T \to \mathbb{R}$ for which only $\partial_t W \in \mathcal{M}(\Pi_T)$. Of course, we have $BV(\Pi_T) \subset BV_t(\Pi_T)$. We can also define the space $BV_x(\Pi_T)$ by replacing the condition $\partial_t W \in \mathcal{M}(\Pi_T)$ by $\partial_x W \in \mathcal{M}(\Pi_T)$. Moreover, we denote by $\mathcal{D}(\Pi_T)$ the space of test functions on Π_T that vanish for $t = 0$ and $t = T$, i.e., $\mathcal{D}(\Pi_T) = C_0^\infty(\mathbb{R} \times (0, T))$.

Equipped with these definitions, we can state entropy solution concept as follows.

Definition 1 (BV_t entropy solution). Let $\Pi_T = \mathbb{R} \times (0, T)$ with $T > 0$ fixed. A function $u : \Pi_T \to \mathbb{R}$ is a BV_t entropy solution of the initial value problem (8), (10) on Π_T if (a) $\rho \in L^1(\Pi_T) \cap BV_t(\Pi_T)$ and $\rho(x,t) \in [0, \rho_{\max}]$ for a.e. $(x,t) \in \Pi_T$, (b) $D_1(\rho)$ is continuous and $D_1(\rho)_x \in L^\infty(\Pi_T)$,

(c) $$\forall c \in \mathbb{R},\ \forall 0 \leq \phi \in \mathcal{D}(\Pi_T) :\quad \iint_{\Pi_T} \Big(|\rho - c|\phi_t + \operatorname{sgn}(\rho - c)\gamma(x)(f(\rho) - f(c))\phi_x + \big|D_1(\rho) - D_1(c)\big|\phi_{xx} \Big)\, dt\, dx + \int_0^T \big|v_{\max}^0 - v_{\max}^*\big| f(c)\, (\phi(a,t) + \phi(b,t))\; dt \geq 0, \tag{18}$$

(d) condition (10) is satisfied in the following strong L^1 sense: (19)

$$\operatorname*{ess\,lim}_{t \downarrow 0} \int_{\mathbb{R}} \big|\rho(x,t) - \rho_0(x)\big|\, dx \to 0.$$

The entropy condition (18) implies that (8) also holds in the weak sense. It is well known that there exists an entropy solution to (8), (10) on Π_T if $\gamma(x)$ depend smoothly on x, and this solution belongs to $BV(\Pi_T)$ if assumptions (17) on ρ_0 are satisfied, see [9]. However, with $\gamma(x)$ depending discontinuously on x, it is hard (if possible) to prove that $u \in BV_x(\Pi_T)$. It is, however, possible to prove that there exist solutions in $BV_t(\Pi_T)$, and this motivates the BV_t requirement in part (a) of Def. 1.

Theorem 1 (L^1 stability and uniqueness). *Let ρ_1, ρ_2 be two BV_t entropy solutions to* (8), (10) *on Π_T with initial data $\rho_{1,0}$, $\rho_{2,0}$ satisfying* (17)*. Then $\|\rho_1(\cdot,t) - \rho_2(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|\rho_{1,0} - \rho_{2,0}\|_{L^1(\mathbb{R})}$ for a.e. $t \in (0,T)$. In particular, there exists at most one BV_t entropy solution of* (8), (10).

The proof of Theorem 1 can be found in [6]. It relies on jump conditions that relate limits from the right and left of the BV_t entropy solution $\rho(\cdot,t)$ at $x=a,b$ (the two points where $\gamma(x)$ has jump discontinuities). To be more precise, we use a Rankine-Hugoniot condition expressing conservation across the jumps at $x=a,b$, and also an entropy jump inequality which is a consequence of the entropy condition (18). Let $\xi=a$ or b, and introduce the notation $\gamma_\pm := \gamma(\xi\pm)$, $\rho_\pm := \rho_\pm(t) := \rho(\xi_m\pm,t)$, $\mathcal{D}_{1,\pm} := \mathcal{D}_{1,\pm}(t) := \mathcal{D}_1(\xi_m\pm,t)$ and $\mathcal{D}_1(x,t) := D_1(\rho)_x(x,t)$. The existence of the above limits (traces) is not entirely obvious since Definition 1 says nothing about the regularity of $\rho(x,t)$ in the x variable. Nevertheless, in [6] it is proved that the above traces exist due to the entropy condition (18) and the assumption that $\rho\in BV_t(\Pi_T)$, i.e., $u_t\in L^1(\Pi_T)$. Equipped with the existence of these traces, one can easily prove that the following Rankine-Hugoniot and entropy jump conditions hold for a.e. $t\in(0,T)$:

$$\gamma_+ f(\rho_+) - \mathcal{D}_{1,+} = \gamma_- f(\rho_-) - \mathcal{D}_{1,-} \quad \text{and} \tag{20}$$

$$\big(\gamma_+ F(\rho_+,c) - \operatorname{sgn}(\rho_+-c)\mathcal{D}_{1,+}\big) - \big(\gamma_- F(\rho_-,c) - \operatorname{sgn}(\rho_--c)\mathcal{D}_{1,-}\big) \le \big|v^0_{\max} - v^*_{\max}\big| f(c) \tag{21}$$

for all $c\in\mathbb{R}$, where $F(u,v)=\operatorname{sgn}(u-v)(f(u)-f(v))$ denotes the Kružkov entropy flux. The jump conditions (20) and (21) are essential ingredients in the uniqueness proof in [6].

Together with Theorem 1, the next theorem shows that our Model 1 is mathematically well posed.

Theorem 2 (Existence). *Suppose that* (17) *holds. Then there exists a BV_t entropy solution $\rho(x,t)$ to the initial value problem* (8), (10).

The proof of Theorem 2 [5] is constructive and is based on proving convergence (compactness) of an explicit finite difference scheme. Let us state a generalization of this finite difference that applies to Model 1 and 2, not just Model 1, and which is used in § 5 for computational purposes. The difference scheme for (9), (10) is a straightforward extension of the scheme analyzed in [5, 6] to diffusion terms including a discontinuous coefficient. To define the scheme, we choose $\Delta x>0$, set $x_j := j\Delta x$, and discretize the parameter vector $\boldsymbol{\gamma}(x) := (\gamma^1(x),\gamma^2(x))$ and the initial datum $\rho_0(x)$ by

$$\boldsymbol{\gamma}_{j+1/2} := \boldsymbol{\gamma}\left(\hat{x}_{j+1/2}+\right), \quad \rho_j^0 := \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\,dx, \quad j\in\mathbb{Z},$$

where $\hat{x}_{j+1/2}$ is any point lying in the interval $I_{j+1/2}=(x_j,x_{j+1})$.

Observe that the spatial discretization of $\boldsymbol{\gamma}$ is staggered against that of ρ. We choose Δt, set $\lambda := \Delta t/\Delta x$, $\mu=\Delta t/(\Delta x)^2$, and for $n>0$ define the approximations $\{\rho_j^n\}_{j,n}$, $\rho_j^n\approx\rho(x_j,t_n)$ according to the explicit marching formula involving the well-known numerical (Engquist-Osher) flux:

$$\rho_j^{n+1} = \rho_j^n - \Delta_-\big[\lambda\gamma^1_{j+1/2} f^{\mathrm{EO}}\big(\rho^n_{j+1},\rho^n_j\big) - \mu\gamma^2_{j+1/2}\Delta_+ R\big(\rho^n_j,\gamma^2_{j-1/2}\big)\big], \tag{22}$$

$$f^{\mathrm{EO}}(v,u) := \frac{1}{2}\left[f(u)+f(v) - \int_u^v \big|f'(s)\big|\,ds\right]. \tag{23}$$

We use Δ_+ and Δ_- to denote the forward and backward difference operators in the x direction, for example, $\Delta_+\rho_j^n = \rho^n_{j+1}-\rho^n_j = \Delta_-\rho^n_{j+1}$. For Model 1, (22) simplifies to the scheme discussed below:

$$\rho_j^{n+1} = \rho_j^n - \Delta_- \big[\lambda \gamma_{j+1/2} f^{\mathrm{EO}} \big(\rho_{j+1}^n, \rho_j^n \big) - \mu \Delta_+ D_1 \big(\rho_j^n \big) \big]. \tag{24}$$

Finally, we let $t_n := n\Delta t$, $I^n := [t_n, t_{n+1})$ and $I_j := [x_{j-1/2}, x_{j+1/2})$, and define the approximations ρ^Δ and $\boldsymbol{\gamma}^\Delta$ of ρ and $\boldsymbol{\gamma}$ by setting $\rho^\Delta = \rho_j^n$ and $\gamma^\Delta = \gamma_{j+1/2}$ on $I_j \times I^n$ and I_j, respectively.

According to [5, 6], the proof of Theorem 2 consists in establishing two main parts: (i) compactness of the sequence $\{\rho^\Delta\}_{\Delta>0}$, i.e., that there exists at least a subsequence that converges in $L^1_{\mathrm{loc}}(\Pi_T)$ to limit function ρ and (ii) satisfaction of the entropy condition (18) by the limit ρ. Then Theorem 1 implies that the whole sequence $\{\rho^\Delta\}_{\Delta>0}$ converges in $L^1_{\mathrm{loc}}(\Pi_T)$ to the BV_t entropy solution of (8), (10).

We sketch the main steps of (i) and (ii). First of all, the difference scheme (24) is monotone, which means that if the scheme is written in the form $\rho_j^{n+1} = G_j(\rho_{j+1}^n, \rho_j^n, \rho_{j-1}^n, \boldsymbol{\gamma}_{j-1/2}, \boldsymbol{\gamma}_{j+1/2})$, then $\partial \rho_j^{n+1}/\partial \rho_{j+k}^n = \partial G_j / \partial \rho_{j+k}^n \geq 0$ for $k = -1, 0, 1$. This can easily be verified. Moreover, we have that $\rho_j^n \in [0, \rho_{\max}]$ for all j, n if the following CFL condition holds:

$$\lambda \max \big\{ v_{\max}^0, v_{\max}^* \big\} \max_{[0,\rho_{\max}]} |f'| + \mu \max_{[0,\rho_{\max}]} |D_1'| \leq 1/2. \tag{25}$$

An important consequence of monotonicity is L^1 time continuity of the numerical approximations, i.e.

$$\exists C > 0 \text{ independent of } \Delta: \ \forall n \in \mathbb{N} \cup \{0\}: \quad \Delta x \sum_{j \in \mathbb{Z}} \big| \rho_j^{n+1} - \rho_j^n \big| \leq \Delta x \sum_{j \in \mathbb{Z}} \big| \rho_j^1 - \rho_j^0 \big| \leq C \Delta t. \tag{26}$$

It is not possible to derive a corresponding estimate in space since γ is discontinuous (see [5]), but (26) plays a key role in deriving a bound on the space translates of a certain transformed variable (the so-called singular mapping $\Psi(\gamma, \rho)$). In passing, we note that (26) will ensure that any limit of $\{\rho^\Delta\}_{\Delta>0}$ belongs to $BV_t(\Pi_T)$. The singular mapping $\Psi(\gamma, \rho)$ in [5] is designed to be Lipschitz continuous in both variables and strictly increasing as a function of ρ, and it reads

$$\Psi(\gamma, \rho) = \gamma \int_0^\rho \chi_{[0,\rho_{\mathrm{c}}]}(\xi) |f'(\xi)| \, d\xi + D_1(\rho) =: \mathcal{F}(\gamma, \rho) + D_1(\rho), \tag{27}$$

The singular mapping zeroes out the contribution of the convective flux wherever $D_1(\rho)$ is nondegenerate. It is easy to check that $\partial \Psi(\gamma, \rho)/\partial \rho > 0$ for a.e. $\rho \in [0, \rho_{\max}]$, hence $\rho \mapsto \Psi(\gamma, \rho)$ is strictly increasing.

Let $z^\Delta(x,t) := \Psi(\gamma, \rho^\Delta(x,t))$. The strategy is to prove L^1 convergence along a subsequence of $\{z^\Delta\}_{\Delta>0}$. This also implies convergence of $\{\rho^\Delta\}_{\Delta>0}$ since $\rho \mapsto \Psi(\gamma, \rho)$ is invertible. To prove convergence of $\{z^\Delta\}_{\Delta>0}$, [5] proceeds in two separate steps: (a) Convergence of the diffusion part of $\Psi(\gamma, \rho)$, namely $D_1^\Delta(x,t) := D_1(\rho^\Delta)$. (b) Convergence of the hyperbolic part of Ψ, namely $\mathcal{F}^\Delta(x,t) := \mathcal{F}(\gamma^\Delta, \rho^\Delta)$.

Step (a) is achieved by an energy argument, which implies that $\{D_1^\Delta(x,t)\}_{\Delta>0}$ converges along a subsequence a.e. and in $L^2_{\mathrm{loc}}(\Pi_T)$ to a limit $\overline{D_1} \in L^2(0,T; H^1(\mathbb{R}))$. Moreover, $\overline{D_1} = D_1(\rho)$, where ρ is the $L^\infty(\Pi_T)$ weak-$\star$ limit of ρ^Δ. It is possible to improve the regularity of $D_1(\rho)$. From (26) the space estimate $|D_1(\rho_j^n) - D_1(\rho_i^n)| \leq C|j-i|\Delta x$ for all $i, j \in \mathbb{Z}$ and for some constant C that is independent of Δ, followes easily. One can also prove the time estimate $|D_1(\rho_j^n) - D_1(\rho_j^m)| \leq C(|n-m|\Delta t)^{1/2}$ for all $m, n \in \mathbb{N} \cup \{0\}$, for some constant C

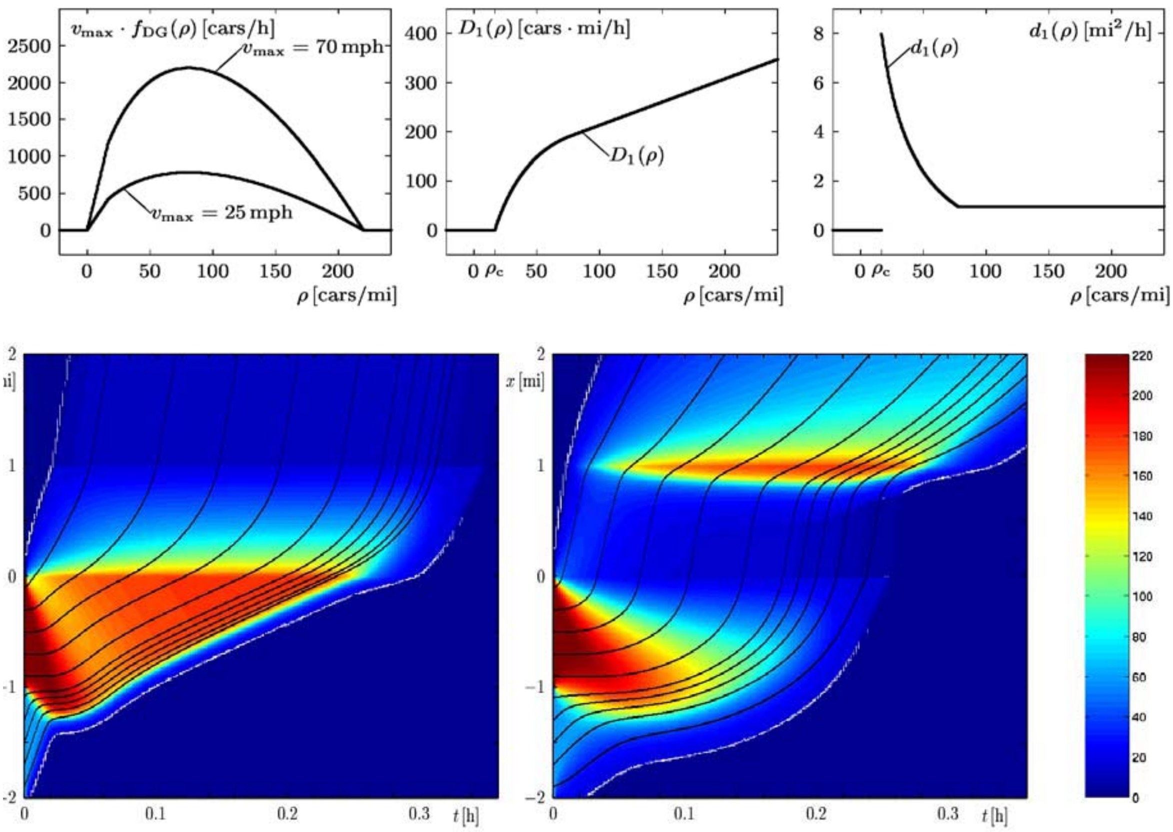

Figure 2: Top: the flux function $v_{\max} f_{\mathrm{DG}}$ (left), the integrated diffusion coefficient $D_1(\rho)$ (middle) and its derivative $d_1(\rho)$ (right) used for the (traffic) Model 1, bottom: numerical simulation of Model 1: Case A (left) and Case B (right) showing plots of the car density ρ combined with selected car trajectories.

that is independent of Δ. These estimates imply eventually that $D_1(\rho)$ is (Hölder) continuous and $D_1(\rho)_x \in L^\infty(\Pi_T)$, i.e., that condition (b) of Definition 1 holds.

Step (b) is achieved by uniformly bounding the total variation of $\mathcal{F}^\Delta(\cdot, t)$ for all $t \in (0, T)$. The proof of this bound relies on a particular cell entropy inequality satisfied by the difference scheme.

In view of (26) and the total variation bound on $\mathcal{F}^\Delta$, $\{\mathcal{F}^\Delta\}_{\Delta>0}$ converges along a subsequence to a limit function $\overline{\mathcal{F}}$ a.e. and in $L^1_{\mathrm{loc}}(\Pi_T)$. Compactness of $\{D_1^\Delta(x,t)\}_{\Delta>0}$ and $\{\mathcal{F}^\Delta(x,t)\}_{\Delta>0}$ separately implies the desired compactness of $\{z^\Delta\}_{\Delta>0}$. Let $z(x,t)$ be a limit point of $\{z^\Delta\}_{\Delta>0}$ and define $\rho(x,t) := \Psi^{-1}(\gamma(x), z(x,t))$. One can prove that $\rho(x,t)$ is a BV_t entropy solution to the initial value problem (8), (10). In particular, one proves that ρ satisfies the entropy condition (18) as a consequence of the cell entropy inequality (not written out here). See [5, 6] for the complete proofs of Theorems 1 and 2.

§5. Numerical examples

We first consider Model 1 for traffic flow with the parameters [2, 8] $\rho_{\max} = 220\,\mathrm{cars/mi}$

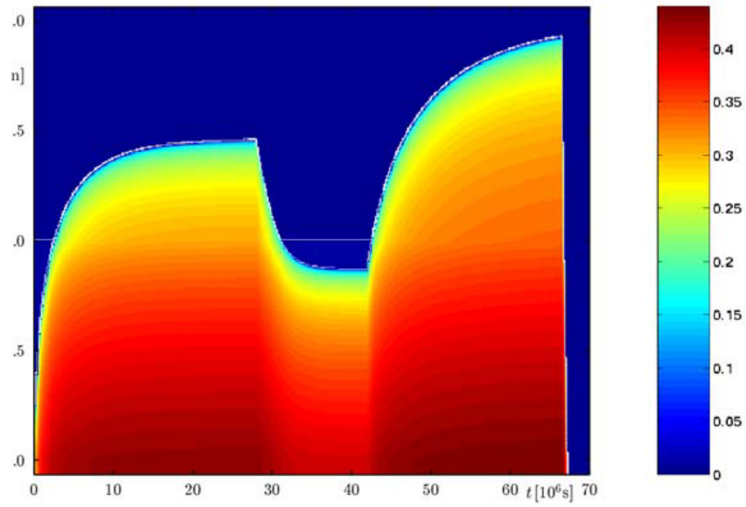

Figure 3: Numerical simulation of a clarifier-thickener treating a flocculated suspension (Model 3).

and $C = \mathrm{e}/7$ = 0.3883. Using these parameters in $V_{\mathrm{DG}}(\rho)$ we obtain the convex flux function plotted in Figure 2 (top left), and $\rho_{\mathrm{c}} = 0.07614\rho_{\max} = 16.751\,\mathrm{cars/mi}$. Furthermore, using $L = \tilde{L}(v(\rho))$ along with $L_{\min} = 0.05\,\mathrm{mi}$, $a = 0.1\,g$, $\tau = 2\,\mathrm{s}$ and $v^0_{\max} = 70\,\mathrm{mph}$, we obtain the diffusion functions $D_1(\rho)$ and $d_1(\rho) = D_1'(\rho)$ shown in the top middle and top right diagrams of Figure 2. Explicit algebraic expressions for $f_{\mathrm{DG}}(\rho)$, $D_1(\rho)$ and $d(\rho)$ are provided in [2]. We present here two numerical simulations produced by $\Delta x = 0.01\,\mathrm{mi}$, $\lambda = 0.0003\,\mathrm{h/mi}$, $v_{\max}(x) = v^{\mathrm{A}}_{\max}(x)$ for Case A and $v_{\max}(x) = 95\,\mathrm{mph} - v^{\mathrm{A}}_{\max}(x)$ for Case B, where

$$\rho_0(x) = \begin{cases} 220\,\mathrm{cars/mi} & \text{for } x \in [-1\,\mathrm{mi}, 0), \\ 60\,\mathrm{cars/mi} & \text{for } x \in [-2\,\mathrm{mi}, -1\,\mathrm{mi}), \\ 0 & \text{otherwise,} \end{cases}$$

$$v^{\mathrm{A}}_{\max}(x) := \begin{cases} 70\,\mathrm{mph} & \text{for } x < 0 \text{ and } x > 1\,\mathrm{mi}, \\ 25\,\mathrm{mph} & \text{for } x \in [0, 1\,\mathrm{mi}]. \end{cases}$$

In both cases the diffusion functions are based on $v^0_{\max} = 70\,\mathrm{mph}$. Figure 2 shows the numerical results.

In Figure 3 we consider a clarifier-thickener unit with $x_{\mathrm{R}} = -x_{\mathrm{L}} = 1\,\mathrm{m}$ treating a suspension with $h(u) = 0.0001\chi_{[0,1]}(u)u(1-u)^5\,\mathrm{m/s}$, $u_{\mathrm{c}} = 0.1$, $\Delta\varrho = 1500\,\mathrm{kg/m^3}$, and $\sigma_{\mathrm{e}}(u) = \chi_{[u_{\mathrm{c}},1]}(u)((u/u_{\mathrm{c}})^6 - 1)\,\mathrm{Pa}$. We choose $q_{\mathrm{L}} = -1.0\times10^{-5}\,\mathrm{m/s}$, $q_{\mathrm{R}} = -2.5\times10^{-6}\,\mathrm{m/s}$, start from a vessel initially full of water ($u_0 \equiv 0$) and attain different steady states before eventually emptying the unit by setting

$$u_{\mathrm{F}}(t) = \begin{cases} 0.086 & \text{for } t \leq 0.4T, \qquad 0.088 \quad \text{for } 0.6T < t \leq 0.95T, \\ 0.08 & \text{for } 0.4T < t \leq 0.6T, \quad 0 \quad \text{for } t > 0.95T, \end{cases} \qquad T := 7.0\times10^7\,\mathrm{s}.$$

The parameters $\Delta x = 1/150\,\mathrm{m}$ and $\lambda = 4000\,\mathrm{s/m}$ refer to a semi-implicit variant of the scheme [4].

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