Variational Inequalities and Fixed Points Theorems in the Euclidean Space for Non-Continuous Operators

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Abstract. The existence of solutions of general variational inequalities is obtained for some maps defined on a nonempty closed convex subset of the Euclidean space. The convex subset is possibly unbounded, the operator is possibly neither continuous nor coercive, the convex function is possibly non-lower semi-continuous. Two generalizations of the fixed points theorem of Brouwer are deduced for some operators which are non-continuous and defined on some unbounded closed convex subsets.

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§1. Introduction

Various phenomena which occur in physical and economical sciences are mathematically formulated as variational inequalities or as optimization problems where some constraints have to be taken into account. In this paper we consider general variational inequalities allowing to include classical variational inequalities (find $x_0 \in K$ such that $\langle Ax_0 + b, z - x_0 \rangle \geq 0$ for all $z \in K$) and convex optimization problems (find $x_0 \in K$ such that $f(x_0) \leq f(z)$ for all $z \in K$). We consider problems which are formulated in finite dimensional spaces (i.e. $\mathbb{R}^n$ via an isomorphism).

We present some new sufficient conditions for the existence of solutions of general variational inequa-lities. We generalize the classical hypotheses based on the continuity of the operator and the function defining the problem. A more specific study of convex minimization problems and classical variational inequalities follows our general theorem of existence of solutions. Sufficient conditions for the minimization of a non-lower semi-continuous convex function are proposed. Various results on the existence of solutions of variational inequalities in finite dimensional spaces generalize the well-known theorems proved in [4]; the usual hypotheses of continuity and coercivity are relaxed.

We also study the question of existence of fixed points for a class of non-continuous operators defined on nonempty closed convex subsets (find $x_0 \in K$ such that $T(x_0) = x_0$). These results on the existence of fixed points are obtained via the existence of solutions of variational inequalities following the approach of Browder [2]. Generalizations of the fixed points theorem of Brouwer are established for some classes of operators which are not necessarily continuous and defined on closed convex subsets (nonempty in $\mathbb{R}^n$) which are not necessarily bounded.

The main notations used in this paper are the following: $\|\cdot\|$ denotes the norm of the Euclidean space $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product. The relative interior of a subset K of $\mathbb{R}^n$ is denoted by $\mathrm{ri}(K)$: it is the interior of K considered as a subset of the affine hull of K. The interior of a subset K of $\mathbb{R}^n$ is denoted by $\mathrm{int}(K)$; the boundary of K is denoted by $\partial K := K \backslash \mathrm{int}(K)$.

§2. General variational inequalities

2.1. Existence

First, we state the main result about the existence of solutions of general variational inequalities.

Theorem 1. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and K' a dense subset of K. Let $A : K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a convex function such that*

$(A1)$ $\quad \forall z \in K',\ \{x \in K : \langle Ax, x - z \rangle \leq f(z) - f(x)\}$ *is closed,*

$(A2)$ $\quad \exists z \in K,\ \{x \in K : \langle Ax, x - z \rangle \leq f(z) - f(x)\}$ *is bounded.*

Then, $\quad (IVMIN) \quad \exists x_0 \in K,\ \forall z \in K,\ \langle Ax_0, z - x_0 \rangle + f(z) - f(x_0) \geq 0.$

The proof of Theorem 1 uses Corollary 4 (of Proposition 2) which corresponds to the particular case where the convex subset K is compact ((A2) is then satisfied). The proof we give in this paper follows the approach of Browder [2] (using the concept of partition of the unity) and uses the fixed points theorem of Brouwer in a crucial way.

Proposition 2. *Let K be a nonempty compact convex subset of $\mathbb{R}^n$ and K' be a nonempty subset of K. Let $A : K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a convex function satisfying $(A1)$. Then there exists $x_0 \in K$ such that $\langle Ax_0, z - x_0 \rangle + f(z) - f(x_0) \geq 0$ for all $z \in K'$.*

Proof. Assume there is no point x_0 in K such that $\langle Ax_0, z - x_0 \rangle \geq f(x_0) - f(z)$ for all $z \in K'$. To each $x \in K$ corresponds at least one point $z \in K'$ such that $\langle Ax, z - x \rangle < f(x) - f(z)$. Thus $K = \bigcup_{z \in K'} O_z$ where $O_z := \{x \in K : \langle Ax, z - x \rangle < f(x) - f(z)\}$ is an open subset of K by (A1). The family $(O_z)_{z \in K'}$ being an open covering of the compact subset K, there exists a finite number of points $z_1, \ldots, z_p \in K'$ such that $K = \bigcup_{i=1,\ldots,p} O_{z_i}$. We consider a partition of unity associated with this open covering, composed of continuous functions $\varphi_i : K \to [0, +\infty[$ whose support is included respectively in O_{z_i} and such that $\sum_{i=1,\ldots,p} \varphi_i(x) = 1$ for all $x \in K$. We define the map F by $F(x) = \sum_{i=1,\ldots,p} \varphi_i(x) z_i$ for all $x \in K$. Since F is continuous on the nonempty compact convex subset K into itself, the theorem of Brouwer gives the existence of a fixed point $y \in K : F(y) = y$. We remark that for at least one index $j \in \{1, \ldots, p\}$ the real number $\varphi_j(y)$ is positive (all are nonnegative with a nonzero sum). If $i \in \{1, \ldots, p\}$ is such that $\varphi_i(y) > 0$ (otherwise $\varphi_i(y) = 0$) then $y \in O_{z_i}$ and thus $\langle Ay, z_i - y \rangle < f(y) - f(z_i)$. Consequently, $\langle Ay, F(y) -$

$y\rangle = \sum_{i=1,\ldots,p} \varphi_i(y)\langle Ay, z_i - y\rangle < \sum_{i=1,\ldots,p} \varphi_i(y)\left(f(y) - f(z_i)\right)$. However, by convexity of f, we have $\sum_{i=1,\ldots,p} \varphi_i(y)\left(f(y) - f(z_i)\right) = f(y) - \sum_{i=1,\ldots,p} \varphi_i(y) f(z_i) \leq f(y) - f(\sum_{i=1,\ldots,p} \varphi_i(y) z_i) = f(y) - f(F(y)) = 0$. We get a contradiction ($\langle Ay, F(y) - y\rangle < 0$ with $F(y) = y$) which allows us to conclude. □

The following lemma proves that in finite dimensional spaces a general variational inequality is sa-tisfied on a convex subset as soon as it is satisfied on a dense subset. We remark that we cannot use an upper semi-continuity argument since there exist convex functions defined on a convex subset of $\mathbb{R}^2$ which are not upper semi-continuous (cf. [5]).

Lemma 3. *Let K be a nonempty convex subset of $\mathbb{R}^n$ and K' a dense subset of K. Let $A : K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a convex function. If $x_0 \in K$ satisfies $\langle Ax_0, z - x_0\rangle + f(z) - f(x_0) \geq 0$ for all $z \in K'$ then x_0 satisfies $\langle Ax_0, z - x_0\rangle + f(z) - f(x_0) \geq 0$ for all $z \in K$.*

Proof. Since f is a convex function on the convex subset K of the Euclidean space $\mathbb{R}^n$, f is continuous on $\mathrm{ri}(K)$, the relative interior of K (cf. [5]). By density of K' in K and by continuity, we conclude that $\langle Ax_0, z - x_0\rangle + f(z) \geq f(x_0)$ for all $z \in \mathrm{ri}(K)$. We fix $c \in \mathrm{ri}(K)$ (nonempty). For any $z \in K$, we have $z_t := (1-t)z + tc \in \mathrm{ri}(K)$ for all $t \in (0, 1]$. We have $f(x_0) \leq \langle Ax_0, z_t - x_0\rangle + f(z_t) \leq \langle Ax_0, z_t - x_0\rangle + (1-t)f(z) + tf(c)$. Letting $t \to 0^+$ we conclude that $f(x_0) \leq \langle Ax_0, z - x_0\rangle + f(z)$. □

Corollary 4. *Let K be a nonempty compact convex subset of $\mathbb{R}^n$ and K' a dense subset of K. Let $A : K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a convex function satisfying $(A1)$. Then there exists $x_0 \in K$ such that $\langle Ax_0, z - x_0\rangle + f(z) - f(x_0) \geq 0$ for all $z \in K$.*

Proof. Immediate consequence of Proposition 2 and Lemma 3. □

Proof of Theorem 1. From (A2), there exist $z_0 \in K$ and $R > 0$ such that $\{x \in K : \langle Ax, x - z_0\rangle \leq f(z_0) - f(x)\} \subset B(0, R/2)$ (where $B(0, \rho)$ denotes the closed ball with center 0 and radius $\rho > 0$). The subset $K_R := K \cap B(0, R)$ is a nonempty compact convex subset (contains z_0). From (A1), for all $z \in K'_R := K' \cap B(0, R) \subset K_R$, $\{x \in K_R : \langle Ax, x - z\rangle \leq f(z) - f(x)\}$ is closed (intersection of closed subsets). Obviously K'_R is dense in the compact convex subset K_R; we deduce from Corollary 4 there exists $x_R \in K_R$ such that $\langle Ax_R, z - x_R\rangle + f(z) - f(x_R) \geq 0$ for all $z \in K_R$. Let $z \in K$. For $\theta > 0$ small enough, $z_R := x_R + \theta(z - x_R) \in K_R$; indeed $\|z_R\| \leq R/2 + \theta\,\|z - x_R\|$: $z_0 \in K_R$ and thus $\langle Ax_R, z_0 - x_R\rangle + f(z_0) - f(x_R) \geq 0$ so that (using $x_R \in K$) $x_R \in B(0, R/2)$. We have $\langle Ax_R, z - x_R\rangle = \theta^{-1}\langle Ax_R, z_R - x_R\rangle$ and, by convexity, $f(z) - f(x_R) \geq \theta^{-1}(f(z_R) - f(x_R))$. By summing we get $\langle Ax_R, z - x_R\rangle + f(z) - f(x_R) \geq \theta^{-1}(\langle Ax_R, z_R - x_R\rangle + f(z_R) - f(x_R)) \geq 0$ (for $z_R \in K_R$), which allows us to conclude that x_R is a solution of (IVMIN). □

We remark that if (A1) holds and if K' is a dense subset of K, the following property

$(A1^+)$ $\quad \forall z \in K,\ \{x \in K : \langle Ax, x - z\rangle \leq f(z) - f(x)\}$ *is closed,*

is not necessarily satisfied. Consider for instance the case where $A = 0$ and where the convex function $f : K := [-1, 1] \to \mathbb{R}$ is defined by $f(x) = x^2$ for $x \in [-1, 1)$ and $f(1) = 2$ when $K' := (-1, 1]$.

2.2. Minimization

We deduce from Theorem 1 a result on the existence of a minimizer of a convex function (non-lower semi-continuous and non-coercive in general cases) which is defined on a closed convex subset (non-compact in general cases) of the Euclidean space. Conditions (B1) and (B2) used in the next theorem of minimization are relative to subsets of the form $\{x \in K : f(x) \leq f(z)\}$ depending on $z \in K$.

Theorem 5. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ a convex function such that*

$(B1) \quad \forall z \in \mathrm{ri}(K),\ f^{-1}((-\infty, f(z)])$ *is closed,*

$(B2) \quad \exists z \in K,\ f^{-1}((-\infty, f(z)])$ *is bounded.*

Then $\quad (MIN) \quad \exists x_0 \in K,\ f(x_0) = \inf_{z \in K} f(z).$

Proof. Consequence of Theorem 1 with $A = 0$ and $K' = \mathrm{ri}(K)$; the fact that $\mathrm{ri}(K)$ is dense in K is a classical result. □

Obviously, when K is a compact subset, condition (B2) is satisfied and in fact we have

$(B2^+) \quad \forall z \in K,\ f^{-1}((-\infty, f(z)])$ *is bounded.*

It is also obvious that condition (B2$^+$) is satisfied when K is unbounded provided the function f is coercive: $f(u) \to +\infty$ if $\|u\| \to +\infty$ ($u \in K$).

For the convex function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = x^2$ for $x \in (-1, 1)$, $f(-1) = \alpha \in [1, +\infty)$ and $f(1) = \beta \in (1, +\infty)$, condition (B1) is satisfied whereas f is not lower semi-continuous. We remark that for a lower semi-continuous function f on K (nonempty and closed) the following property holds:

$(B1^+) \quad \forall z \in K,\ f^{-1}((-\infty, f(z)])$ *is closed.*

Nevertheless, condition (B1$^+$) can be satisfied by a convex function which is not lower semi-continuous (consider $\alpha = 2$, $\beta = 2$ in the previous example). We remark that property (B1$^+$) is not always satisfied in the previous example (consider $\alpha = 1$, $\beta = 2$ and $z = -1$).

The following corollary generalizes the classical theorem of minimization of a lower semi-continuous convex function defined on a nonempty closed convex subset of $\mathbb{R}^n$ when the subset is bounded or when the function is coercive.

Corollary 6. *If $f : K \to \mathbb{R}$ is a convex function defined on a nonempty closed convex subset K of $\mathbb{R}^n$ such that $f^{-1}((-\infty, f(z)])$ is a compact subset for all $z \in \mathrm{ri}(K)$ then (MIN).*

More general results about minimization of some particular non-lower semi-continuous functions are established in [1].

2.3. Variational inequalities

The results of this section are various corollaries of Theorem 1 when $K' = K$ and $f = 0$.

An important class of maps A satisfying property

$(C1) \quad \forall z \in K,\ \{x \in K : \langle Ax + b, x - z\rangle \leq 0\}$ *is closed,* is composed of the continuous maps from K into $\mathbb{R}^n$ ($b \in \mathbb{R}^n$ is given for the sequel).

The next result generalizes [4, Cor. 4.3] in which the following coercivity condition is used (when K is unbounded):

$$(C0) \quad \exists z \in K, \lim_{x\in K, \|x\|\to+\infty} \frac{\langle Ax - Az, x - z\rangle}{\|x - z\|} = +\infty.$$

In fact, $(C0)$ implies

$(C2) \quad \exists z \in K,\ \{x \in K : \langle Ax + b, x - z\rangle \le 0\}$ *is bounded.*

Indeed, there exists $R > 0$ such that $\langle Ax + b, x - z\rangle > 0$ for $\|x\| > R$ (where $x \in K$) and thus $\{x \in K : \langle Ax + b, x - z\rangle \le 0\} \subseteq B(0, R)$ (choose $L > \|Az + b\|$ and $R > \|z\|$ such that $\frac{\langle Ax-Az,x-z\rangle}{\|x-z\|} \ge L$ for $\|x\| > R$).

Corollary 7. *If $A : K \to \mathbb{R}^n$ is a continuous map on a nonempty closed convex subset K of $\mathbb{R}^n$ and if $(C2)$ holds then $(IV) \quad \exists x_0 \in K,\ \forall z \in K,\ \langle Ax_0 + b, z - x_0\rangle \ge 0$.*

The following example shows that Corollary 7 can be applied if K is unbounded and if the operator A is non-coercive: $K = [0, +\infty)$, $b = 0$, $Ax = 1 + x$ if $x \in [0, 1)$, $Ax = 2$ if $x \in [1, +\infty)$. It is immediate that $\{x \in [0, +\infty) : Ax(x - z) \le 0\} = [0, z]$ for all $z \in [0, +\infty)$.

Obviously, property (C2) holds when K is bounded. The next result (as the previous one) generalizes a result of Hartman and Stampacchia (cf. [4, Th. 3.1]).

Corollary 8. *If K is a nonempty compact convex subset of $\mathbb{R}^n$ and if $A : K \to \mathbb{R}^n$ satisfies $(C1)$ then (IV).*

We give an elementary example of a discontinuous operator for which Corollary 8 can be applied: $K = [0, 1]$, $b = 0$, $A0 = 1$, $Ax = x/2 + \alpha$ if $x \in (0, 1]$, with $\alpha \in (0, 1/2)$ be fixed. The subset $\{x \in [0, 1] : Ax(x - z) \le 0\}$ is the closed line segment with extremities 2α and $z \in [0, 1]$**.**

We can also state (since (C0)⇒(C2)):

Corollary 9. *If K is a nonempty closed convex subset of $\mathbb{R}^n$ and if $A : K \to \mathbb{R}^n$ satisfies $(C0)$ and $(C1)$ then (IV).*

This result is another generalization of [4, Cor. 4.3]. Finally we formulate a particular case of Theorem 1 (which generalizes Corollary 8 but not Corollary 7).

Corollary 10. *If K is a nonempty closed convex subset of $\mathbb{R}^n$ and if $A : K \to \mathbb{R}^n$ satisfies*

$(C3) \quad \forall z \in K,\ \{x \in K : \langle Ax + b, x - z\rangle \le 0\}$ *is compact,*

then (IV).

This result can be applied when the subset K is unbounded and when the operator A is not continuous nor coercive: $K = [0, +\infty)$, $b = 0$, $Ax = 1 + x$ if $x \in [0, 1)$, $Ax = \alpha$ if $x \in [1, +\infty)$ where $\alpha > 2$.

§3. Fixed points

Following the approach of Browder in [2], we deduce some fixed points theorems from results about the existence of solutions of variational inequalities, particularly from Theorem 1.

Theorem 11. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $T : K \to \mathbb{R}^n$ such that*

$(D1)$ $\forall z \in K,\ \{x \in K : \langle x - Tx, x - z\rangle \leq 0\}$ *is closed,*

$(D2)$ $\exists z \in K,\ \{x \in K : \langle x - Tx, x - z\rangle \leq 0\}$ *is bounded,*

$(D3)$ $\forall z \in \partial K, \exists y \in K, [\,\exists r > 0, Tz = z + r(y - z)\,]$ *or* $[\,\langle z - Tz, z - y\rangle > 0\,]$.

Then $\exists x_0 \in K, \quad Tx_0 = x_0.$

Proof. From Theorem 1 (by (D1)-(D2)) there exists $x_0 \in K$ such that $\langle x_0 - Tx_0, x_0 - z\rangle \leq 0$, for all $z \in K$. If $x_0 \in \operatorname{int}(K)$ there exists $r > 0$ such that $x_0 + u \in K$ as soon as $\|u\| \leq r$. Choose any nonzero z in $\mathbb{R}^n$; $x_0 + \theta z \in K$ for $|\theta| \leq r\,\|z\|^{-1}$ and then $\langle x_0 - Tx_0, z\rangle = 0$. We conclude that $x_0 = Tx_0$. If $x_0 \in \partial K$ there exists (by (D3)) $y \in K$ such that $Tx_0 = x_0 + r(y - x_0)$ for some $r > 0$ or else $\langle x_0 - Tx_0, x_0 - y\rangle > 0$. In the first case, $r^{-1}\langle x_0 - Tx_0, Tx_0 - x_0\rangle = \langle x_0 - Tx_0, y - x_0\rangle \geq 0$ and then $\langle x_0 - Tx_0, Tx_0 - x_0\rangle \geq 0$, thus $Tx_0 = x_0$. The second case is impossible: $y \in K$ and then $\langle x_0 - Tx_0, x_0 - y\rangle \leq 0 < \langle x_0 - Tx_0, x_0 - y\rangle$ which leads to a contradiction. □

Remark 1. (i) A particular version of $(D3)$ used in [2] is:

$(D3^+)$ $\forall z \in \partial K, \exists y \in K, \exists r > 0, Tz = z + r(y - z).$

Initially, Halpern introduced the notion of ”inward map”: $Tz \in z + \mathbb{R}^+(K - z)$ for all $z \in K$ (cf. [3]).

(ii) Recall that property $(D3^+)$ holds in particular if $T(\partial K) \subseteq K$ (take $r = 1$ and $y = Tz$ for $z \in \partial K$).

Remark 2. Condition $(D2)$ is satisfied when $T(K)$ is bounded. In fact, in this case we have

$(D2^+)$ $\forall z \in K,\ \{x \in K : \langle x - Tx, x - z\rangle \leq 0\}$ *is bounded*.

Indeed, if $x \in K$ satisfies $\langle x - Tx, x - z\rangle \leq 0$ then $\|x\|^2 \leq \langle x, z\rangle + \langle Tx, x - z\rangle \leq \alpha\,\|x\| + \beta$ for some positive constants α and β which allows us to conclude immediately.

Formally replacing the operator T by $x \mapsto 2x - T(x)$ we obtain from Theorem 11 its ”dual” version:

Theorem 12. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $T : K \to \mathbb{R}^n$ such that*

$(E1)$ $\forall z \in K,\ \{x \in K : \langle x - Tx, x - z\rangle \geq 0\}$ *is closed,*

$(E2)$ $\exists z \in K,\ \{x \in K : \langle x - Tx, x - z\rangle \geq 0\}$ *is bounded,*

$(E3)$ $\forall z \in \partial K, \exists y \in K, [\,\exists r > 0, Tz = z - r(y - z)\,]$ *or* $[\,\langle z - Tz, z - y\rangle < 0\,]$.

Then $\exists x_0 \in K, \quad Tx_0 = x_0.$

Remark 3. (i) A particular version of $(E3)$ used in [2] is:

$(E3^+)$ $\forall z \in \partial K, \exists y \in K, \exists r > 0, Tz = z - r(y - z).$

The notion of ”outward map” is defined by: $Tz \in z + \mathbb{R}^-(K - z)$ for all $z \in K$ (cf. [3]).

Concerning the case of continuous operators:

Corollary 13. *If $T : K \to \mathbb{R}^n$ is continuous on the nonempty closed convex subset K of $\mathbb{R}^n$ and if $(D2)$ and $(D3)$ hold then T admits a fixed point in K.*

And its dual version:

Corollary 14. *If $T : K \to \mathbb{R}^n$ is continuous on the nonempty closed convex subset K of $\mathbb{R}^n$ and if $(E2)$ and $(E3)$ hold then T admits a fixed point in K.*

Properties (D2) and (E2) are satisfied in particular when K is bounded. This allows us to state:

Corollary 15. *If K is a nonempty compact convex subset of $\mathbb{R}^n$ and if $T : K \to \mathbb{R}^n$ satisfies $(D1)$ and $(D3)$ then T admits a fixed point in K.*

And also its dual version:

Corollary 16. *If K is a nonempty compact convex subset of $\mathbb{R}^n$ and if $T : K \to \mathbb{R}^n$ satisfies $(E1)$ and $(E3)$ then T admits a fixed point in K.*

We achieve this section by two particular cases where compactness is used.

Corollary 17. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $T : K \to \mathbb{R}^n$ satisfying $(D3)$ and*

$(D4)\quad \forall z \in K,\ \{x \in K : \langle x - Tx, x - z\rangle \leq 0\}$ *is compact.*

Then T admits a fixed point in K.

It is obvious that $(D4) \Rightarrow (D1)$-$(D2)$ and $[\,K$ compact and $(D1)\,] \Rightarrow (D4)$. Moreover $[\,T(K)$ compact and $(D1)\,] \Rightarrow (D4)$ since $(D4) \Leftrightarrow (D1)$-$(D2+)$. For the dual version below: $(E4) \Rightarrow (E1)$-$(E2)$ and $[\,K$ compact and $(E1)\,] \Rightarrow (E4)$.

Corollary 18. *Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and $T : K \to \mathbb{R}^n$ satisfying $(E3)$ and*

$(E4)\quad \forall z \in K,\ \{x \in K : \langle x - Tx, x - z\rangle \geq 0\}$ *is compact.*

Then T admits a fixed point in K.

We now present some various applications of our results on variational inequalities and fixed points. The main object is to analyze and compare the different assumptions in concrete cases and thus we study simple operators (with discontinuities except for the first one).

Example 1. We consider the operator T defined by

$Tx = \sqrt{x}$ if $x \in K := [1/2, +\infty)$.

Property (D2) (and in fact (D2+)) holds; nevertheless K and $T(K)$ are unbounded. Indeed, for all $z \in [1/2, +\infty)$, the subset $D_z := \{x \in [1/2, +\infty) : (x - Tx)(x - z) \leq 0\}$ is the closed line segment with extremities 1 and z. Property (D1) also holds (by continuity of T or by the closeness of each D_z). Property (D3) (and in fact (D3+)) holds for $T(\partial K) = T(\{1/2\}) \subset K$. Theorem 11 proves the existence of a fixed point which cannot be directly obtained by the classical Brouwer's theorem nor by its extension given in [2] without more information (for K is not a compact subset).

Property (E1) holds by continuity of T. Property (E2) does not hold: the subset $E_z := \{x \in [1/2,+\infty) : (x-\sqrt{x})(x-z) \geq 0\}$ contains all real number greater than $\max(1,z)$ for any $z \in [1/2,+\infty)$. We also remark that (E3) does not hold. For this example, Theorem 12 cannot be used.

Example 2. We define the operator T by

$Tx = x + x^{-1}$ if $x \in (0,1]$ and $T0 = -\alpha$ where α is a real parameter.

Property (E2) holds since $K := [0,1]$ is bounded. If $\alpha \leq 0$ then $E_z := \{x \in [0,1] : (x-Tx)(x-z) \geq 0\}$ is closed ($E_z = [0,z]$) for all $z \in [0,1]$; if $\alpha > 0$ then $E_z = (0,z]$ is not closed for $z \in (0,1]$. Thus (E1) holds if and only if $\alpha \leq 0$. When $\alpha \leq 0$ we get from Theorem 1 the existence of a solution for the variational inequality associated with the operator $A : x \mapsto x - Tx$ on $[0,1]$. Moreover (E3+) holds if and only if $\alpha \geq 0$. Consequently, Theorem 12 can be applied when $\alpha = 0$.

Property (D2) always holds (K is bounded). If $\alpha \geq 0$ then $D_z := \{x \in [0,1] : (x - Tx)(x-z) \leq 0\}$ is closed ($D_z = \{0\} \cup [z,1]$) for all $z \in [0,1]$; if $\alpha < 0$ then $D_z = [z,1]$ is closed for all $z \in (0,1]$ but $D_0 = (0,1]$ is not closed. Thus (D1) holds if and only if $\alpha \geq 0$. When $\alpha \geq 0$ we get from Theorem 1 the existence of a solution for the variational inequality associated with the operator $A : x \mapsto Tx - x$ on $[0,1]$. Since (D3) never holds, Theorem 11 cannot be used even when $\alpha = 0$.

Example 3. We define on $K := \mathbb{R} \times \mathbb{R}$ the operator T by

$T(0,0) = (0,0)$ and $T(x,y) = (xy/(x^2+y^2), xy/(x^2+y^2))$ if $(x,y) \neq (0,0)$.

Property (D2) (and in fact (D2+)) holds since $T(\mathbb{R} \times \mathbb{R})$ is contained in the closed unit ball. Property (D1) also holds (but T is not continuous at the origin). Property (D3) is obvious. The existence of a fixed point can be obtained by Theorem 11 (and Corollary 17) but not by the Brouwer's theorem as if T is defined on the closed unit ball because T is discontinuous at the origin.

Example 4. We define on $K := \mathbb{R} \times \mathbb{R}$ the operator T by

$T(0,0) = (0,0)$ and $T(x,y) = (x/(x^2+y^2), x/(x^2+y^2))$ if $(x,y) \neq (0,0)$.

We easily verify that properties (D1) and (D2) hold (and in fact (D2+) since $D_z := \{u \in \mathbb{R} \times \mathbb{R} : \langle u - Tu, u - z\rangle \leq 0\}$ is contained in the closed ball with center the origin and radius $\max(1, \|z\|)$). Property (D3) is obvious. The Brouwer's theorem cannot be applied: T is discontinuous at the origin. We remark that property (D2) can be satisfied for some operators which are not locally bounded at a point. Theorem 11 (and Corollary 17) can be applied to get the existence of a fixed point.

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