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THEORETICAL ASPECTS OF WAVE PROPAGATION FOR BIOT'S CONSOLIDATION PROBLEM

H. Barucq, M. Madaune-Tort and P. Saint-Macary

Abstract. We consider a coupled system of mixed hyperbolic-parabolic type which describes the Biot consolidation model in poro-elasticity as well as a coupled quasi-static problem in thermoelasticity. In this work, we intend to develop the existence-uniqueness theory for the multi-dimensional systems in the linear case using classical functional arguments in the Sobolev background. For the consolidation model, our approach involves Galerkin approximations to establish the existence of a solution to the problem while we prove that the thermo-elastic and the quasi-static systems are limit cases of the consolidation model. The treatment of the uniqueness is based on an energy inequality even if, in the quasi-static system, it requires some adjustments because of a lack of regularity.

Keywords: Poro-Elastic Media, Hyperbolic/Parabolic, Linear

AMS classification: 35K20, 35L20, 35L70, 35Q35

§1. Introduction

The Biot consolidation model in poro-elasticity in which we are interested is the following:

$$\begin{cases} \rho\dfrac{\partial^2\mathbf{u}}{\partial t^2} - \bigtriangledown(\lambda^*\dfrac{\partial}{\partial t}div\ \mathbf{u}) - \bigtriangledown((\lambda+\mu)div\ \mathbf{u}) - div(\mu\bigtriangledown\mathbf{u}) + \alpha\bigtriangledown p = \mathbf{f}(t,\mathbf{x}) \\ c_0\dfrac{\partial p}{\partial t} + \alpha\ div\dfrac{\partial \mathbf{u}}{\partial t} - div(k\bigtriangledown p) = h(t,\mathbf{x}), \end{cases} \tag{1}$$

where the physical parameters ρ and λ^* may be equal to zero.
This coupled mixed hyperbolic-parabolic system can describe the phenomena arising when a soil is submitted to a load (in particular the consolidation effects) as well as the ultrasonic propagation in fluid-saturated porous media like cancellous bone. The displacement vector field of the system, denoted by $\mathbf{u}$, satisfies the conservation of momentum while the fluid pressure p satisfies a diffusion equation. K. Terzaghi was the first in interesting in the consolidation phenomena arising in porous media under a load [18]. He showed the similarities between this phenomena and the exit of a flow out of a porous media, that contributed to model fluid flows in saturated deformable porous media as a coupled flow-deformation process. Later, M. A. Biot studied these problems assuming that the continuum mechanics laws are applicable. He developed thus the now classical theory of poro-elasticity and proved that a linear theory of consolidation could be established by using the Darcy law for laminar flows combined with the

momentum balance equations with Hooke law for elastic deformations [4, 5, 6, 7]. His results were justified *a posteriori* by homogenization methods [1, 8, 13]. Indeed, a porous medium consists in connected pores that are fluid-filled in a solid medium. We can then consider the number of heterogeneities and, if the ratio between this number and the wavelength is smaller than one, we can involve a homogenization process to study wave propagation phenomena.

Numerous works have followed the pioneering ones of Terzaghi and Biot. The coupled thermoelastic problem which deals with $\lambda^* = 0$ in the first equation of (1) was studied from a theoretical point of view by C. M. Dafermos [9]. In this paper, he constructs strong solutions to the thermoelastic problem which describes the flow of heat trough an elastic structure. Another approach was used by R. E. Showalter and his coworkers in several papers [14, 15, 16] constructing weak solutions using the linear semi-group theory in Hilbert spaces. He began by the study of the degenerate quasi-static case ($\rho = \lambda^* = 0$ in (1)) [14] before consider the case of composite deformable porous media described by a system of the same type as (1) which involves two pressures solutions to diffusion processes coupled by a distributed exchange term [15]. At last, the case of visco-plastic media was tackled in [16]. Herein, for several space dimensions, we have chosen to deal with the full dynamic system (1) describing the most complete phenomenon of consolidation including secondary one which is sometimes neglected. The study of mono-dimensional open sets is already done in [2]. Moreover, the parameters, considered here as constants, can be functions of $\mathbf{x}$: $\rho(\mathbf{x})$, $\lambda^*(\mathbf{x})$, $\lambda(\mathbf{x})$, $\mu(\mathbf{x})$, $c_0(\mathbf{x})$ and $k(\mathbf{x})$. Providing some technical adjustments, the study of (1) is developed in [3] for variable parameters.

As far as the constants of the system are concerned, $\rho > 0$ is the density of the porous and permeable medium Ω; $\lambda^* > 0$ is a physical parameter arising in connection with secondary consolidation effects; the positive coefficients λ and μ respectively denote the dilatation and shear modulus of elasticity, the so-called Lamé constants; $\alpha > 0$ is the Biot-Willis constant that accounts for the pressure-deformation coupling; $c_0 > 0$ is the combined porosity of the medium and compressibility of the fluid; $k > 0$ is the hydraulic conductivity and it contains both the permeability of the medium and the viscosity of the fluid.

We associate to (1) the following initial conditions:

$$(\mathbf{u}(0,\mathbf{x}), p(0,\mathbf{x})) = (\mathbf{u}_0(\mathbf{x}), p_0(\mathbf{x})), \tag{2}$$

and when $\rho > 0$

$$\partial_t \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_1(\mathbf{x}) \tag{3}$$

with boundary conditions we will precise later.

In this paper, we will consider the study of the model when $\rho > 0$ in Section 2, including in this case the vanishing of the constant λ^*, and we develop the case $\rho = 0$ in Section 3.

§2. The case $\rho > 0$

In this section, we will prove the existence and the uniqueness of a solution to the Biot consolidation model (ρ and $\lambda^* > 0$) and to the thermoelastic case ($\rho > 0$ and $\lambda^* = 0$) using a Galerkin approximation method for the first problem and a regularization technique for the second one. As far as the proof of the uniqueness is concerned, it involves Ladyzenskaja's test-functions.

2.1. Existence for The Biot Consolidation Model

In this subsection, we study the most complete model of consolidation. As said before, we denote the porous medium by Ω whose boundary Γ is lipschitzian. We also consider two distinct partitions of the boundary in complementary parts: Γ_1, Γ_1^c and Γ_2, Γ_2^c. We associate to (1) the following boundary conditions:

$$\begin{cases} \mathbf{u} = 0 \text{ sur } \Gamma_1 \times [0,T] \\ \lambda^* \partial_t div \mathbf{u}\, n_i + (\lambda+\mu) div\, \mathbf{u}\, n_i + \mu \nabla u_i \cdot \mathbf{n} - \alpha p n_i + \displaystyle\sum_{j=1}^{n} A_{ij} u_j = 0 \text{ sur } \Gamma_1^c \times [0,T] \\ p = 0 \text{ sur } \Gamma_2 \times [0,T] \\ k \nabla p \cdot \mathbf{n} + Bp = 0 \text{ sur } \Gamma_2^c \times [0,T], \end{cases} \tag{4}$$

where the elasticity modulus A and the pressure modulus B are respectively a positive definite symmetric $n \times n$ matrix and a positive number. It states that the body is rigidly clamped on $\Gamma_1 \times [0,T]$ and there is no pressure due to the fluid on $\Gamma_2 \times [0,T]$ while it is elastically clamped on $\Gamma_1^c \times [0,T]$ and it undergoes a pressure on $\Gamma_2^c \times [0,T]$. We define the spaces $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{v} = 0$ on $\Gamma_1\}$ and $\tilde{V} = \{q \in H^1(\Omega)$ such that $q = 0$ on $\Gamma_2\}$ which are Hilbert spaces once equipped with the usual semi-norm. We associate to (1) the initial conditions (2) and (3) with $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$, $p_0 \in L^2(\Omega)$.
We call solution to the problem described by {(1), (2), (3), (4)} any pair $(\mathbf{u}, p)$ which satisfies the following variational formulation:

$$\begin{cases} \text{Find } (\mathbf{u},p) \in L^\infty(0,T;\mathbf{V}) \times L^2(0,T;\tilde{V}) \text{ such that} \\ \partial_t \mathbf{u} \in L^2(0,T;H(div,\Omega)), \quad \partial_t^2 \mathbf{u} \in L^2(0,T;\mathbf{V}'), \quad \partial_t p \in L^2(0,T;\tilde{V}'), \\ \text{verifying for } a.e.\ t \in]0,T[,\ \forall (\mathbf{v},q) \in \mathbf{V} \times \tilde{V} : \\ \rho < \partial_t^2 \mathbf{u}, \mathbf{v} >_{\mathbf{V}',\mathbf{V}} + \lambda^* \displaystyle\int_\Omega div \partial_t \mathbf{u}\, div \mathbf{v} dx + (\lambda+\mu) \int_\Omega div \mathbf{u}\, div \mathbf{v}\, dx \\ + \mu \displaystyle\int_\Omega \nabla \mathbf{u} \otimes \nabla \mathbf{v} dx - \alpha \int_\Omega p\, div \mathbf{v} dx + \int_{\Gamma_1^c} A\mathbf{u} \otimes \mathbf{v} d\sigma = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx, \\ c_0 < \partial_t p, q >_{\tilde{V}',\tilde{V}} + \alpha \displaystyle\int_\Omega q\, div \partial_t \mathbf{u} dx + k \int_\Omega \nabla p \cdot \nabla q dx + B \int_{\Gamma_2^c} pq d\sigma = \int_\Omega hq dx, \\ (\mathbf{u}(0,\mathbf{x}), p(0,\mathbf{x})) = (\mathbf{u}_0(\mathbf{x}), p_0(\mathbf{x})), \quad \partial_t \mathbf{u}(0,\mathbf{x}) = \mathbf{u}_1(\mathbf{x}), \end{cases} \tag{5}$$

and we will prove the:

Theorem 1. *Under the hypotheses* $\mathbf{u}_0 \in \mathbf{V}$, $p_0 \in L^2(\Omega)$, $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$, $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $h \in L^2(0,T;L^2(\Omega))$, *Problem (5) admits at least a solution* $(\mathbf{u}, p)$.

Proof. To prove the existence of a solution to (5), we proceed to a Faedo-Galerkin approximation. Indeed, we intend to construct a solution to (5) as the limit of a sequence of approximate regular solutions denoted by $(\mathbf{u}_m, p_m)_{m \in \mathbb{N}^*}$. Let $(\mathbf{w}_j)_{j \in \mathbb{N}^*}$ and $(\chi_j)_{j \in \mathbb{N}^*}$ respectively be a basis of $\mathbf{V}$ and $\tilde{V}$. Then, we set $(\mathbf{u}_m, p_m)$ in $\mathbf{V}_m \times \tilde{V}_m$ where $\mathbf{V}_m = Span\{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ and $\tilde{V}_m = Span\{\chi_1, \dots, \chi_m\}$.

$(\mathbf{u}_m, p_m)_{m\in\mathbb{N}^*}$ satisfies the discrete formulation: $\forall m \in \mathbb{N}^*$, $\forall j \in \mathbb{N}^*$, $1 \le j \le m$

$$\left\{\begin{array}{l} \rho \displaystyle\int_\Omega \partial_t^2 \mathbf{u}_m \cdot \mathbf{w}_j dx + \lambda^* \int_\Omega div\partial_t \mathbf{u}_m div\mathbf{w}_j dx + (\lambda+\mu)\int_\Omega div\mathbf{u}_m div\mathbf{w}_j dx \\ +\mu \displaystyle\int_\Omega \nabla \mathbf{u}_m \otimes \nabla \mathbf{w}_j dx - \alpha \int_\Omega p_m div \mathbf{w}_j dx + \int_{\Gamma_1^c} A\mathbf{u}_m \otimes \mathbf{w}_j d\sigma = \int_\Omega \mathbf{f}\cdot \mathbf{w}_j dx, \\ c_0 \displaystyle\int_\Omega \partial_t p_m \chi_j dx + \alpha \int_\Omega \chi_j div\partial_t \mathbf{u}_m dx + k\int_\Omega \nabla p_m \cdot \nabla \chi_j dx + B\int_{\Gamma_2^c} p_m \chi_j d\sigma = \int_\Omega h\chi_j dx \end{array}\right. \tag{6}$$

with $(\mathbf{u}_m(0), p_m(0)) = (\mathbf{u}_{0m}, p_{0m}) \to (\mathbf{u}_0, p_0)$ in $\mathbf{V}\times L^2(\Omega)$ and $\partial_t \mathbf{u}_m(0) = \mathbf{u}_{1m} \to \mathbf{u}_1$ in $\mathbf{L}^2(\Omega)$. According to the theory of linear differential equations, there exists a single solution $(\mathbf{u}_m, p_m)_{m\in\mathbb{N}^*}$ to this approximate problem in $H^2(0,T;\mathbf{V})\times H^1(0,T;\tilde{V})$. Next, it is a matter to derive an *a priori* estimate to prove that $(\mathbf{u}_m, p_m)_{m\in\mathbb{N}^*}$ is bounded in a suitable functional framework which allows us to extract a subsequence of $(\mathbf{u}_m, p_m)_{m\in\mathbb{N}^*}$ that converges to $(\mathbf{u}, p)$ solution to (5).
We multiply the first equation of (6) by $u'_{jm}(t)$ and the second one by $p_{jm}(t)$. Then, we add the resulting equalities and we integrate them on $(0,t)$ for any $0 \le t \le T$ and we sum each term from $j=1$ to $j=m$. Using hypotheses on the initial conditions, the data and Cauchy-Schwarz, Young and Poincaré inequalities and Gronwall lemma, we obtain that there exists a constant κ such that: for each $m \in \mathbb{N}^*$,

$$||\mathbf{u}_m||_{L^\infty(0,T;\mathbf{V})}, ||\partial_t \mathbf{u}_m||_{L^\infty(0,T;\mathbf{L}^2(\Omega))}, ||div\partial_t \mathbf{u}_m||_{L^2(0,T;L^2(\Omega))}, ||p_m||_{L^\infty(0,T;L^2(\Omega))}, ||p_m||_{L^2(0,T;\tilde{V})} \le \kappa.$$

Then, only to extract a subsequence later, we get $\mathbf{u}_m \to \mathbf{u}$ in $L^\infty(0,T;\mathbf{V})$ weak *, $\partial_t \mathbf{u}_m \to \partial_t \mathbf{u}$ in $L^\infty(0,T;\mathbf{L}^2(\Omega))$ weak *, $div\partial_t \mathbf{u}_m \to div\partial_t \mathbf{u}$ in $L^2(0,T;L^2(\Omega))$ weakly, $p_m \to p$ in $L^\infty(0,T;L^2(\Omega))$ weak * and in $L^2(0,T;\tilde{V})$ weakly. We can then pass to the limit in (6) when m tends to infinity and using the properties of the basis $(\mathbf{w}_j)_{j\in\mathbb{N}^*}$ and $(\chi_j)_{j\in\mathbb{N}^*}$ and the same kind of techniques as [10] page 623, we show that the pair $(\mathbf{u}, p)$ satisfies the variational equations in (5). The proof of Theorem 1 is achieved checking that $(\mathbf{u}, p)$ fits into the initial conditions at $t=0$. According to the previous *a priori* estimate, $(\mathbf{u}_m)_{m\in\mathbb{N}^*}$ is bounded in $W(0,T;\mathbf{V},\mathbf{L}^2(\Omega))$. Using the continuous imbedding of this space into $C^0([0,T];\mathbf{L}^2(\Omega))$, we show that $\mathbf{u}(0,.) = \mathbf{u}_0(.)$. To show that $\partial_t \mathbf{u}(0,.) = \mathbf{u}_{1,}$ and $p(0,.) = p_0(.)$, we multiply Equations (5) written with $\mathbf{v} = \mathbf{w}_j$ and $q = \chi_j$ and Equations (6) by a function ψ belonging to $C^1([0,T])$ such that $\psi(T) = 0$. $\square$

2.2. Existence for the Thermo-Elastic Case

This subsection deals with the particular case when $\lambda^* = 0$ in System (1). We neglect the secondary consolidation term taking $\lambda^* = 0$ in the first equation. In this case, we are interested in the thermo-elastic model studied by C. M. Dafermos as recalled in Section 1. We consider the system with the initial conditions (2), (3) and the boundary conditions (4) where now $\lambda^* = 0$ which become:

$$\left\{\begin{array}{l} \mathbf{u} = \mathbf{0} \text{ on } \Gamma_1 \times [0,T] \\ (\lambda+\mu) div\, \mathbf{u}\, n_i + \mu \nabla u_i \cdot \mathbf{n} - \alpha p n_i + \displaystyle\sum_{j=1}^n A_{ij} u_j = 0 \text{ on } \Gamma_1^c \times [0,T] \\ p = 0 \text{ on } \Gamma_2 \times [0,T] \\ k\nabla p \cdot \mathbf{n} + Bp = 0 \text{ on } \Gamma_2^c \times [0,T]. \end{array}\right. \tag{7}$$

We associate to this problem the following weak formulation:

$$\left\{\begin{array}{l}\text{Find } (\mathbf{u},p)\in L^\infty(0,T;\mathbf{V})\times L^2(0,T;\tilde{V}) \text{ such that}\\ \partial_t\mathbf{u}\in L^\infty(0,T;\mathbf{L}^2(\Omega)),\quad \partial_t^2\mathbf{u}\in L^2(0,T;\mathbf{V}'),\quad \partial_t(c_0p+\alpha div\mathbf{u})\in L^2(0,T;\tilde{V}'),\\ \text{verifying for } a.e\ t\in]0,T[,\ \forall(\mathbf{v},q)\in\mathbf{V}\times\tilde{V}:\\ \rho<\partial_t^2\mathbf{u},\mathbf{v}>_{\mathbf{V}',\mathbf{V}}+(\lambda+\mu)\displaystyle\int_\Omega div\mathbf{u}\, div\mathbf{v}dx+\mu\int_\Omega\nabla\mathbf{u}\otimes\nabla\mathbf{v}dx\\ -\alpha\displaystyle\int_\Omega p\, div\mathbf{v}dx+\int_{\Gamma_1^c}A\mathbf{u}\otimes\mathbf{v}d\sigma=\int_\Omega\mathbf{f}\cdot\mathbf{v}dx,\\ <\partial_t(c_0p+\alpha div\mathbf{u}),q>_{\tilde{V}',\tilde{V}}+k\displaystyle\int_\Omega\nabla p\cdot\nabla qdx+B\int_{\Gamma_2^c}pqd\sigma=\int_\Omega hqdx,\\ (\mathbf{u}(0,\mathbf{x}),p(0,\mathbf{x}))=(\mathbf{u}_0(\mathbf{x}),p_0(\mathbf{x})),\quad\partial_t\mathbf{u}(0,\mathbf{x})=\mathbf{u}_1(\mathbf{x})\end{array}\right.\tag{8}$$

and we will prove the:

Theorem 2. *Under the hypotheses* $\mathbf{u}_0\in\mathbf{V}$, $p_0\in L^2(\Omega)$, $\mathbf{u}_1\in\mathbf{L}^2(\Omega)$, $\mathbf{f}\in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $h\in L^2(0,T;L^2(\Omega))$, *Problem (8) admits at least a solution* $(\mathbf{u},p)$.

Proof. We use a regularization of Formulation (8): for all $\varepsilon>0$, we consider the variational formulation

$$\left\{\begin{array}{l}\text{Find } (\mathbf{u}_\varepsilon,p_\varepsilon)\in L^\infty(0,T;\mathbf{V})\times L^2(0,T;\tilde{V}) \text{ such that}\\ \partial_t\mathbf{u}_\varepsilon\in L^2(0,T;H(div,\Omega)),\quad \partial_t^2\mathbf{u}_\varepsilon\in L^2(0,T;\mathbf{V}'),\quad \partial_tp_\varepsilon\in L^2(0,T;\tilde{V}'),\\ \text{verifying for } a.e.\ t\in]0,T[,\ \forall(\mathbf{v},q)\in\mathbf{V}\times\tilde{V}:\\ \rho<\partial_t^2\mathbf{u}_\varepsilon,\mathbf{v}>_{\mathbf{V}',\mathbf{V}}+\varepsilon\displaystyle\int_\Omega div\partial_t\mathbf{u}_\varepsilon\, div\mathbf{v}dx+(\lambda+\mu)\int_\Omega div\mathbf{u}_\varepsilon\, div\mathbf{v}\,dx\\ +\mu\displaystyle\int_\Omega\nabla\mathbf{u}_\varepsilon\otimes\nabla\mathbf{v}dx-\alpha\int_\Omega p_\varepsilon\, div\mathbf{v}dx+\int_{\Gamma_1^c}A\mathbf{u}_\varepsilon\otimes\mathbf{v}d\sigma=\int_\Omega\mathbf{f}\cdot\mathbf{v}dx,\\ c_0<\partial_tp_\varepsilon,q>_{\tilde{V}',\tilde{V}}+\alpha\displaystyle\int_\Omega q\, div\partial_t\mathbf{u}_\varepsilon dx+k\int_\Omega\nabla p_\varepsilon\cdot\nabla qdx+B\int_{\Gamma_2^c}p_\varepsilon qd\sigma=\int_\Omega hqdx,\\ (\mathbf{u}_\varepsilon(0,\mathbf{x}),p_\varepsilon(0,\mathbf{x}))=(\mathbf{u}_0(\mathbf{x}),p_0(\mathbf{x})),\quad\partial_t\mathbf{u}_\varepsilon(0,\mathbf{x})=\mathbf{u}_1(\mathbf{x}).\end{array}\right.\tag{9}$$

According to Subsection 2.1, we are ensured that Formulation (9) admits a single solution $(\mathbf{u}_\varepsilon,p_\varepsilon)$ for each $\varepsilon>0$. We will show that we can construct a solution to (8) as the limit of the sequence $(\mathbf{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$ when ε tends to zero.
First, using the calculations developed in the proof of Theorem 1, we show that $(\mathbf{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$ is bounded in $W(0,T;\mathbf{V},\mathbf{L}^2(\Omega))\times(L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;\tilde{V}))$ and $\sqrt{\varepsilon}(div\partial_t\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\Omega))$. Moreover, using these results and the equations of (9), we also prove that $(\partial_t^2\mathbf{u}_\varepsilon)_{\varepsilon>0}$ and $(\partial_t(c_0p_\varepsilon+\alpha div\partial_t\mathbf{u}_\varepsilon))_{\varepsilon>0}$ are respectively bounded in $L^2(0,T;\mathbf{V}')$ and $L^2(0,T;\tilde{V}')$. Hence, we deduce that there exists a subsequence extracted from $(\mathbf{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$ also denoted $(\mathbf{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$ for the sake of conciseness and which satisfies $\mathbf{u}_\varepsilon\to\mathbf{u}$ in $L^\infty(0,T;\mathbf{V})$ weak *, $\varepsilon div\partial_t\mathbf{u}_\varepsilon\to0$ in $L^2(0,T;L^2(\Omega))$ weak, $\partial_t^2\mathbf{u}_\varepsilon\to\partial_t^2\mathbf{u}$ in $L^2(0,T;\mathbf{V}')$ weak, $p_\varepsilon\to p$ in $L^\infty(0,T;L^2(\Omega))$ weak * and in $L^2(0,T;\tilde{V})$ weak, $\partial_t(c_0p_\varepsilon+\alpha div\mathbf{u}_\varepsilon)\to\partial_t(c_0p+\alpha div\mathbf{u})$ in $L^2(0,T;\tilde{V}')$. This pair $(\mathbf{u},p)$ defined as the limit state to $(\mathbf{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$ when $\varepsilon\to0$ is then solution to the variational equations of (8).
To achieve the proof of the existence of a solution to (8) only consists in checking that the pair $(\mathbf{u},p)$ at time $t=0$ fits into the given initial data $(\mathbf{u}_0,p_0)$ and $\mathbf{u}_1$. Using the results of the

a priori estimate obtained, we know that $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ and $(\partial_t\mathbf{u}_\varepsilon)_{\varepsilon>0}$ are respectively bounded in $W(0,T;\mathbf{V},\mathbf{L}^2(\Omega))$ and $W(0,T;\mathbf{L}^2(\Omega),\mathbf{V}')$. Using the fact that $W(0,T;\mathbf{V},\mathbf{L}^2(\Omega))$, $W(0,T;\mathbf{L}^2(\Omega),\mathbf{V}') \hookrightarrow C^0([0,T];\mathbf{L}^2(\Omega))$, we are able as in Subsection 2.1 to conclude that $\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x})$ and $\partial_t\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_1(\mathbf{x})$ in Ω.
Moreover, we know that $(c_0 p_\varepsilon + \alpha div\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $W(0,T;L^2(\Omega),\tilde{V}') \hookrightarrow C^0([0,T];L^2(\Omega))$ and using the information we have on $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$, we can conclude that $p(0,.) = p_0(.)$ in $\mathcal{D}'(\Omega)$ that achieve the proof of the existence of a solution to Problem (8). $\square$

2.3. Uniqueness

Now we are interested in the question of the uniqueness for Problems (5) and (8). As these problems only differ from the term of secondary consolidation, it is sufficient to develop the proof of the uniqueness of a solution to (5). Moreover as this problem is linear, it amounts to verify that the homogeneous variational problem associated to (5) (*i.e.* $\mathbf{f}\equiv 0$, $h\equiv 0$, $\mathbf{u}_0\equiv 0$, $\mathbf{u}_1\equiv 0$, $p\equiv 0$) only admits the trivial solution $(\mathbf{u},p)=(0,0)$. As the first equation is second order in time hyperbolic type, we should take $\mathbf{v}=\partial_t\mathbf{u}$ as test-function in the first equation to obtain an energy inequality. But, because of a lack of regularity, we cannot use the natural pair $(\partial_t\mathbf{u},p)$ to eliminate the coupling terms. That is why we consider Ladyzenskaja's test-functions [11, 12] to compensate for this difficulty. Let s be in $]0,T[$ and let consider the pair (φ_1,φ_2) defined by:

$$\varphi_1(t,\mathbf{x}) = \begin{cases} -\displaystyle\int_t^s \mathbf{u}(\sigma,\mathbf{x})d\sigma \\ \mathbf{0} \end{cases} \text{and } \varphi_2(t,\mathbf{x}) = \begin{cases} -\displaystyle\int_t^s \left(\int_0^\tau p(\sigma,\mathbf{x})d\sigma\right)d\tau & \text{if} \quad t\le s, \\ 0 & \text{if} \quad t\ge s. \end{cases}$$

We also define $\tilde{\varphi}_1(t,\mathbf{x}) = \displaystyle\int_0^t \mathbf{u}(\sigma,\mathbf{x})d\sigma$ and $\tilde{\varphi}_2(t,\mathbf{x}) = \displaystyle\int_0^t\left(\int_0^\tau p(\sigma,\mathbf{x})d\sigma\right)d\tau$.
The following properties are thus satisfied:

$$\begin{cases} \varphi_1(t,\mathbf{x}) = \tilde{\varphi}_1(t,\mathbf{x}) - \tilde{\varphi}_1(s,\mathbf{x}) & \varphi_1(0,\mathbf{x}) = -\tilde{\varphi}_1(s,\mathbf{x}) & \varphi_1(s,\mathbf{x}) = \mathbf{0} \\ \partial_t\varphi_1(t,\mathbf{x}) = \mathbf{u}(t,\mathbf{x}) & \partial_t\varphi_1(0,\mathbf{x}) = \mathbf{0} & \partial_t\varphi_1(s,\mathbf{x}) = \mathbf{u}(s,\mathbf{x}) \end{cases} \tag{10}$$

and:

$$\begin{cases} \varphi_2(t,\mathbf{x}) = \tilde{\varphi}_2(t,\mathbf{x}) - \tilde{\varphi}_2(s,\mathbf{x}) & \varphi_2(0,\mathbf{x}) = -\tilde{\varphi}_2(s,\mathbf{x}) & \varphi_2(s,\mathbf{x}) = 0 \\ \partial_t\varphi_2(t,\mathbf{x}) = \displaystyle\int_0^t p(\sigma,\mathbf{x})d\sigma & \partial_t\varphi_2(0,\mathbf{x}) = 0 & \partial_t\varphi_2(s,\mathbf{x}) = \displaystyle\int_0^s p(\sigma,\mathbf{x})d\sigma \\ \partial_t^2\varphi_2(t,\mathbf{x}) = p(t,\mathbf{x}) & \partial_t^2\varphi_2(0,\mathbf{x}) = 0 & \partial_t^2\varphi_2(s,\mathbf{x}) = p(s,\mathbf{x}). \end{cases} \tag{11}$$

Now, we consider the pair (φ_1,φ_2) as test-function in the homogeneous variational formulation associated to (5). Next we integrate on $(0,t)$ with respect to time. Using the properties (10) and (11) and after some calculations, we obtain:

$$\begin{aligned} &\frac{\rho}{2}||\mathbf{u}(s)||^2_{\mathbf{L}^2(\Omega)} + \lambda^*\int_0^s\int_\Omega |div\mathbf{u}|^2dxdt + \frac{\lambda+\mu}{2}||div\tilde{\varphi}_1(s)||^2_{L^2(\Omega)} + \frac{\mu}{2}||\nabla\tilde{\varphi}_1(s)||^2_{\mathbf{L}^2(\Omega)} + \\ &+\tfrac{c_0}{2}||\partial_t\varphi_2(s)||^2_{L^2(\Omega)} + \frac{1}{2}\int_{\Gamma_1^c} A\tilde{\varphi}_1(s)\otimes\tilde{\varphi}_1(s)d\sigma + k\int_0^s\int_\Omega |\nabla\partial_t\varphi_2|^2dxdt + \\ &+B\textstyle\int_0^s\int_{\Gamma_2^c}|\partial_t\varphi_2|^2d\sigma dt \le 0 \end{aligned}$$

which allows us to conclude that $\mathbf{u}(t,\mathbf{x}) = p(t,\mathbf{x}) = 0$ and to expound the:

Theorem 3. *Under the hypotheses* $\mathbf{u}_0 \in \mathbf{V}$*,* $p_0 \in L^2(\Omega)$*,* $\mathbf{u}_1 \in \mathbf{L}^2(\Omega)$*,* $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $h \in L^2(0,T;L^2(\Omega))$*, Problem (5) admits a single solution* $(\mathbf{u}, p)$*.*

§3. The case $\rho = 0$: the Quasi-Static Model

In this part, we deal with the case when $\rho = 0$ and $\lambda^* > 0$ in Formulation (1). This corresponds to the quasi-static case which results from negligible inertia effects and describes the slow deformations associated with consolidation and the associated seepage of the fluid. We consider the homogeneous Dirichlet boundary conditions $(\mathbf{u}, p) = (0,0)$ on Γ and we associate to this problem the following variational formulation :

$$\begin{cases} \text{Find } (\mathbf{u}, p) \in L^\infty(0,T;\mathbf{H}_0^1(\Omega)) \times L^2(0,T;H_0^1(\Omega)) \text{ such that} \\ div\partial_t \mathbf{u} \in L^2(0,T;L^2(\Omega)),\ \partial_t p \in L^2(0,T;H^{-1}(\Omega)), \\ \text{verifying for } \textit{a.e. } t \in]0,T[,\ \forall(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) : \\ \lambda^* \displaystyle\int_\Omega div\partial_t \mathbf{u}\, div\mathbf{v} dx + (\lambda+\mu)\int_\Omega div\mathbf{u}\, div\mathbf{v}\, dx + \mu \int_\Omega \nabla\mathbf{u} \otimes \nabla\mathbf{v} dx \\ -\alpha \displaystyle\int_\Omega p\, div\mathbf{v} dx = \int_\Omega \mathbf{f}\cdot\mathbf{v} dx, \\ c_0 < \partial_t p, q >_{H^{-1}(\Omega),H_0^1(\Omega)} + \alpha \displaystyle\int_\Omega q\, div\partial_t \mathbf{u} dx + k\int_\Omega \nabla p \cdot \nabla q dx = \int_\Omega hq dx, \\ (\mathbf{u}(0,x), p(0,x)) = (\mathbf{u}_0(x), p_0(x)). \end{cases} \tag{12}$$

We define the space $\overline{\mathbf{V}} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \text{ such that } div\mathbf{v} = 0\}^\perp$ and we establish the:

Theorem 4. *Under the hypotheses* $(\mathbf{u}_0, p_0) \in \overline{\mathbf{V}} \times L^2(\Omega)$*,* $\mathbf{f} \in L^2(0,T;Rg(\nabla))$ *and* $h \in L^2(0,T;L^2(\Omega))$*, Problem (12) admits a single solution.*

Proof. We use a regularization technique as in the proof of Theorem 2.
Let $\varepsilon > 0$ be a small parameter. We consider the variational problem as follows:

$$\begin{cases} \text{Find } (\mathbf{u}_\varepsilon, p_\varepsilon) \in L^\infty(0,T;\mathbf{H}_0^1(\Omega)) \times L^2(0,T;H_0^1(\Omega)) \text{ such that} \\ \partial_t \mathbf{u}_\varepsilon \in L^2(0,T;H(div,\Omega)),\ \partial_t^2 \mathbf{u}_\varepsilon \in L^2(0,T;\mathbf{H}^{-1}(\Omega)),\ \partial_t p_\varepsilon \in L^2(0,T;H^{-1}(\Omega)), \\ \text{verifying for } \textit{a.e. } t \in]0,T[,\ \forall(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega) : \\ \varepsilon < \partial_t^2 \mathbf{u}_\varepsilon, \mathbf{v} >_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)} + \lambda^* \displaystyle\int_\Omega div\partial_t \mathbf{u}_\varepsilon\, div\mathbf{v} dx + (\lambda+\mu)\int_\Omega div\mathbf{u}_\varepsilon\, div\mathbf{v}\, dx \\ +\mu \displaystyle\int_\Omega \nabla\mathbf{u}_\varepsilon \otimes \nabla\mathbf{v} dx - \alpha\int_\Omega p_\varepsilon\, div\mathbf{v} dx = \int_\Omega \mathbf{f}\cdot\mathbf{v} dx, \\ c_0 < \partial_t p_\varepsilon, q >_{H^{-1}(\Omega),H_0^1(\Omega)} + \alpha \displaystyle\int_\Omega q\, div\partial_t \mathbf{u}_\varepsilon dx + k\int_\Omega \nabla p_\varepsilon \cdot \nabla q dx = \int_\Omega hq dx, \\ (\mathbf{u}_\varepsilon(0,x), p_\varepsilon(0,x)) = (\mathbf{u}_0(x), p_0(x)),\ \partial_t \mathbf{u}(0,x) = \mathbf{u}_1(x) \end{cases} \tag{13}$$

where $\mathbf{u}_1$ is a given function in $\mathbf{L}^2(\Omega)$. We show as in Subsection 2.1 that the pair $(\mathbf{u}_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ exists and is unique, for any value of $\varepsilon > 0$. We begin as usual deriving some *a priori* estimate. We would like to take the pair $(\mathbf{v}, q) = (\partial_t \mathbf{u}_\varepsilon, p_\varepsilon)$ as test-function in Formulation (13) and to integrate on $(0,t)$ with respect to time. But, because of a lack of regularity, it is not the well-adapted test-functions pair. We consider $\tau > 0$ and we take $\mathbf{v} = \frac{1}{\tau}(\mathbf{u}_\varepsilon(t+\tau) - \mathbf{u}_\varepsilon(t))$.

We integrate on $(0,t)$ and we use an integration by parts formula on the first term of the first equation. We get when τ tends to zero:

$$\left\{\begin{array}{l} \frac{1}{2}\left[\varepsilon\int_\Omega|\partial_t\mathbf{u}_\varepsilon|^2(t)dx+(\lambda+\mu)\int_\Omega|div\mathbf{u}_\varepsilon|^2(t)dx+\mu\int_\Omega|\nabla\mathbf{u}_\varepsilon|^2(t)dx\right.\\ \left.+\,c_0\int_\Omega|p_\varepsilon|^2(t)dx\right]+\lambda^*\int_0^t\int_\Omega|div\partial_t\mathbf{u}_\varepsilon|^2dxds+k\int_0^t\int_\Omega|\nabla p_\varepsilon|^2dxds\\ =\int_0^t\int_\Omega\mathbf{f}\cdot\partial_t\mathbf{u}_\varepsilon dxds+\int_0^t\int_\Omega hp_\varepsilon dxds+\frac{\varepsilon}{2}||\mathbf{u}_1||^2_{\mathbf{L}^2(\Omega)}+\frac{\lambda+\mu}{2}||div\mathbf{u}_0||^2_{L^2(\Omega)}\\ +\frac{\mu}{2}||\nabla\mathbf{u}_0||^2_{\mathbf{L}^2(\Omega)}+\frac{c_0}{2}||p_0||^2_{L^2(\Omega)}. \end{array}\right. \tag{14}$$

Taking hypotheses on initial data into account and using an integration by parts formula, the right member of (14) varies up to and including:

$$\kappa-\int_0^t\int_\Omega\partial_t\mathbf{f}\cdot\mathbf{u}_\varepsilon dxds+\int_\Omega\mathbf{f}(t)\cdot\mathbf{u}_\varepsilon(t)dx-\int_\Omega\mathbf{f}(0)\cdot\mathbf{u}_\varepsilon(0)dx+\int_0^t\int_\Omega hp_\varepsilon dxds.$$

Using Cauchy-Schwarz, Young and Poincaré inequalities, it modifies to:

$$\kappa+\frac{\mu}{4}\int_0^t\int_\Omega|\nabla\mathbf{u}_\varepsilon|^2dxds+\frac{\mu}{4}\int_\Omega|\nabla\mathbf{u}_\varepsilon|^2(t)dx+\frac{c_0}{2}\int_0^t\int_\Omega|p_\varepsilon|^2dxds.$$

We finally apply Gronwall lemma to get the estimate:

$$\left\{\begin{array}{l} \frac{1}{2}\left[\varepsilon\int_\Omega|\partial_t\mathbf{u}_\varepsilon|^2(t)dx+(\lambda+\mu)\int_\Omega|div\mathbf{u}_\varepsilon|^2(t)dx+\frac{\mu}{2}\int_\Omega|\nabla\mathbf{u}_\varepsilon|^2(t)dx\right.\\ \left.+\,c_0\int_\Omega|p_\varepsilon|^2(t)dx\right]+\lambda^*\int_0^t\int_\Omega|div\partial_t\mathbf{u}_\varepsilon|^2dxds+k\int_0^t\int_\Omega|\nabla p_\varepsilon|^2dxds\le\kappa. \end{array}\right.$$

Moreover, using this estimate and the second equation of (13), we prove that $\partial_t p_\varepsilon$ is bounded in $L^2(0,T;H^{-1}(\Omega))$. According to these results, we deduce that there exists a subsequence extracted from $(\mathbf{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$ also denoted by $(\mathbf{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$ which satisfies $\mathbf{u}_\varepsilon\to\mathbf{u}$ in $L^\infty(0,T;\mathbf{H}^1_0(\Omega))$ weak *, $div\partial_t\mathbf{u}_\varepsilon\to div\partial_t\mathbf{u}$ in $L^2(0,T;L^2(\Omega))$ weakly, $p_\varepsilon\to p$ in $L^\infty(0,T;L^2(\Omega))$ weak * and in $L^2(0,T;H^1_0(\Omega))$ weakly, $\partial_t p_\varepsilon\to\partial_t p$ in $L^2(0,T;H^{-1}(\Omega))$ weak. Hence, making ε converge to zero, the pair $(\mathbf{u},p)$ defined as the limit state to $(\mathbf{u}_\varepsilon,p_\varepsilon)_{\varepsilon>0}$ is solution to the variational equations of (12).

To achieve the proof of the existence of a solution to (12), we have to check that the pair $(\mathbf{u},p)$ fits into the initial conditions. According to the previous *a priori* estimates, we know that $(p_\varepsilon)_{\varepsilon>0}$ is bounded in $W(0,T;H^1_0(\Omega),H^{-1}(\Omega))\hookrightarrow C^0([0,T];L^2(\Omega))$, property which allows us to conclude that $p(0,.)=p_0(.)$ as in the previous subsections. Using the same reasoning with $(div\mathbf{u}_\varepsilon)_{\varepsilon>0}$ which is bounded in $W(0,T;L^2(\Omega),L^2(\Omega))$, we get that $div\mathbf{u}(0,.)=div\mathbf{u}_0(.)$.

To prove that, in fact, we have $\mathbf{u}(0,.)=\mathbf{u}_0(.)$, we use the isomorphism T which associates to each $\mathbf{v}\in\overline{\mathbf{V}}$ its divergence into $L^2_0(\Omega)$ which is the space of functions v in $L^2(\Omega)$ such that $\int_\Omega vdx=0$. Indeed, if $\mathbf{w}\in\overline{\mathbf{V}}$ is such that $div\ \mathbf{w}=div\mathbf{u}_0$ then $\mathbf{w}=\mathbf{u}_0$. That's why we will show that we can define $\mathbf{u}(0)$ in $\overline{\mathbf{V}}$ *i.e.* that $\mathbf{u}(0)\in\mathbf{H}^1_0(\Omega)$ and $\exists q\in L^2(\Omega),-\Delta\mathbf{u}(0)=\nabla q$ according to [17].

For $\mathbf{v}$ in $\mathbf{H}_0^1(\Omega)$ such that $div\mathbf{v} = 0$, the first equation of (12) becomes: for *a.e.* $t \in]0,T[$, $\mu \int_\Omega \nabla \mathbf{u}(t) \otimes \nabla \mathbf{v} dx = \int_\Omega \mathbf{f}(t) \cdot \mathbf{v} dx$. But $\mathbf{f} \in L^2(0,T;Rg(\nabla))$ implies that there exists $q \in L^2(Q)$ such that $\mathbf{f} = \nabla q$. Therefore, for *a.e.* $t \in (0,T)$, $\int_\Omega \mathbf{f}(t) \cdot \mathbf{v} dx = 0$ and $\mathbf{u}(t) \in \overline{\mathbf{V}}$. As T is an isomorphism, there exists $\alpha > 0$ such that for *a.e.* $t \in]0,T[$, $||\mathbf{u}(t)||^2_{\mathbf{H}_0^1(\Omega)} \leq \alpha \int_\Omega |div\mathbf{u}(t)|^2 dx$. For *a.e.* $s,t \in]0,T[$, we get: $||\mathbf{u}(t) - \mathbf{u}(s)||^2_{\mathbf{H}_0^1(\Omega)} \leq \alpha \displaystyle\int_\Omega (div(\mathbf{u}(t) - \mathbf{u}(s)))^2 dx \leq \alpha \int_\Omega (div\mathbf{u}(t) - div\mathbf{u}(s))^2 dx$.
But, according to the previous *a priori* estimates, we know that $div\mathbf{u} \in C^0([0,T];L^2(\Omega))$ and using the previous inequality, we can conclude that $\mathbf{u} \in C^0([0,T];\mathbf{H}_0^1(\Omega))$ and then that $\mathbf{u}(0) \in \mathbf{H}_0^1(\Omega)$. We then get $\mathbf{u}(0) \in \overline{\mathbf{V}}$ and as $div\mathbf{u}(0) = div\mathbf{u}_0$ and T is an isomorphism, we obtain that $\mathbf{u}(0,.) = \mathbf{u}_0(.)$ that achieves the proof of the existence of a solution to (12).
The proof of Theorem 4 is achieved once proved the uniqueness of the solution. To do this, we also use Ladyzenskaja's test-functions like in Subsection 2.3. Indeed, as the first equation is nothing more second order hyperbolic type, we can take $\mathbf{v} = \mathbf{u}$ as test-function in the first equation of the homogeneous formulation associated to (12). But, to be able to eliminate the coupling terms, we cannot choose $q = p$ in the second equation. That is why we take $q = \varphi$ defined by

$$\varphi(t,x) = \begin{cases} \displaystyle\int_t^s p(\sigma,x)d\sigma & \text{if} \quad t \leq s, \\ 0 & \text{if} \quad t \geq s \end{cases}$$

which satisfies the same kind of properties as (10). With this pair of test-function, proceeding as in Subsection 2.3, we obtain an equality which allows us to conclude that the single solution of the homogeneous problem associated to (12) is $(0,0)$ that proves the uniqueness of the solution to (12). $\square$

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Hélène Barucq, Monique Madaune-Tort and Patrick Saint-Macary
Laboratoire de Mathématiques Appliquées, FRE 2570
Université de Pau et des Pays de l'Adour
IPRA, BP1155, 64013 Pau Cedex, France
patrick.saint-macary@univ-pau.fr