REDUCTION, RELATIVE EQUILIBRIA AND STABILITY FOR A GYROSTAT IN THE N-BODY PROBLEM

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Abstract. We consider the non-canonical Hamiltonian dynamics of a gyrostat in the n-body problem. Using the symmetries of the system we carry out a reduction process in two steps, giving explicitly at each step the Poisson structure of the reduced system. Next, we obtain general properties of the relative equilibria of the problem and if we restrict to different approximations of the gravitational potential function, some particular cases are studied and, by means of energy-Casimir and spectral methods, sufficient and necessary conditions of stability can be obtained. We extend some results by Fanny and Badaoui (1998) and by Mondéjar, Vigueras and Ferrer (2001). In [3] the case of a rigid body in the three-body problem, in terms of the global variables in the unreduced problem, was considered, and in [12] the case of a gyrostat in the three-body problem was studied, but working now in the reduced system; in this way natural simplifications in the conditions of the equilibria appear. As a particular case, the problem of three bodies is considered for an arbitrary approximation of the potential function and the conditions for existence of Eulerian and Lagrangian equilibria are given in different cases. Here, as in [12], we use geometric methods, developed in part by Marsden, and others (see [6], [7], [8], [9] and [10]).

Keywords: Lie-Poisson systems, reduction, relative equilibria, stability, gyrostat, n-body problem, Eulerian and Lagrangian equilibria

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§1. Introduction

In the last years many papers about the problem of roto-translational motion of celestial bodies have appeared. They show a new interest in the study of configurations of relative equilibria by differential geometry methods or by more classical ones.

We will mention here the papers of Wang, Krishnaprasad and Madocks (1991) about the problem of a rigid body in a central Newtonian field; Maciejewski (1995) about the problem of two rigid bodies in mutual Newtonian attraction. These papers have been generalized to the case of a gyrostat in a central Newtonian field by Wang, Lian and Chen (1995) and by Mondéjar and Vigueras (1999) to the case of two gyrostats in mutual Newtonian attraction.

In the problem of three rigid bodies we would like to mention that Vidiakin (1977) and Duboshine (1984) proved the existence of Euler and Lagrange configurations of equilibria when the bodies possess symmetries; Zhuralev and Petruskii (1990) made a review of the results up to 1990; Fanny and Badaoui (1998) studied the configuration of the equilibria in terms of the global variables in the unreduced problem, where the two bodies have spherical distribution of mass and the third is rigid, and in Mondéjar, Vigueras and Ferrer (2001), the problem of three bodies where two bodies have spherical distribution of mass and the third one of them is a gyrostat is considered. Working in the reduced problem global considerations about the conditions for relative equilibria are made. Finally, in an approximated model of the dynamics (zero order approximation dynamics), a complete study of the relative equilibria is made.

Here, we will consider the non-canonical Hamiltonian dynamics of $n+1$ bodies in Newtonian attraction, where n of them are rigid bodies with spherical distribution of mass and the other one is a triaxial gyrostat. First, we obtain the equations of the problem. Using the symmetries of the system we carry out two reductions, giving in each step the Poisson structure of the reduced space. Then, we will obtain necessary and sufficient conditions for existence of relative equilibria and its stability, by means of spectral and Energy-Casimir methods. As a particular case, the problem of three bodies is considered for an arbitrary approximation of the potential function and the conditions for Euler and Lagrange equilibria type are given in these cases. The obtained results generalize other previous results but new problems are open. A complete treatment and more details of these topics will appear in [14].

§2. Configuration and phase space

A gyrostat is a mechanical system G composed of a rigid body and other bodies (deformable or rigid) connected to it such that their relative motion do not change the distribution of mass of G.

Let S_0 be a gyrostat of mass m_0; S_1, S_2, ..., S_n, n rigid bodies with spherical symmetry of masses m_1, m_2, ...,m_n respectively; $\mathcal{I} = \{O, u_1, u_2, u_3\}$ an inertial reference frame; $\mathcal{B} = \{C_0, b_1, b_2, b_3\}$ a body frame fixed at the center of mass C_0 of S_0; R_i are the vector positions of the center of masses of S_i in $\mathcal{I}$

Then a particle of S_0 with coordinates Q in $\mathcal{B}$ is represented in the inertial frame $\mathcal{I}$ by the vector

$$q = R_0 + BQ,$$

where $B \in SO(3)$.

The configuration space of the problem is the Lie group

$$\mathbf{Q} = SE(3) \times \mathbb{R}^3 \times \overset{n)}{\dots} \times \mathbb{R}^3$$

where $SE(3)$ is the known semidirect product of $SO(3)$ and $\mathbf{R}^3$, with elements $((B, R_0), R_1, ..., R_n)$. And the Kinetic energy of the system is

$$T = \frac{1}{2}\int_{S_o} |\, \dot{\mathbf{q}} \,|^2 \, dm(\mathbf{Q}) + \frac{1}{2}\sum_{i=1}^{n} m_i \,|\, \dot{\mathbf{R}}_i \,|^2 \tag{1}$$

The previous expression of the Kinetic energy simplifies (Cid & Vigueras 1985) to

$$T = \frac{1}{2}\mathbf{\Omega} \cdot \mathbb{I}\mathbf{\Omega} + \mathbf{l}_r \cdot \mathbf{\Omega} + \frac{1}{2}\sum_{i=0}^{n} m_i \,|\, \dot{\mathbf{R}}_i \,|^2 + T_r \tag{2}$$

where $\mathbf{I}$ is the tensor of inertia of S_0 in the body frame, Ω is the angular velocity of S_0 defined by $\dot{B} = B\widehat{\Omega}$, L_r and $\mathcal{T}_r$ are the momentum and the Kinetic energy of the moving part of the gyrostat respectively and

$$\widehat{X} = \begin{pmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{pmatrix}$$

The gravitational potential energy is the function $\mathcal{V} : \mathbf{Q} \to \mathbb{R}$ given by

$$V = -\sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{Gm_i m_j}{\mid \mathbf{R}_i - \mathbf{R}_j \mid} - \sum_{i=1}^{n} \int_{S_o} \frac{Gm_i dm(\mathbf{Q})}{\mid B\mathbf{Q} + \mathbf{R}_0 - \mathbf{R}_i \mid} \tag{3}$$

In what follows we assume that $\mathcal{T}_r$ is a known function of the time and L_r is constant. Then, the Lagrangian of the problem is $\mathcal{L} : \mathbf{TQ} \to \mathbb{R}$

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \circ \tau \tag{4}$$

where $\tau : \mathbf{TQ} \to \mathbf{Q}$ is the canonical projection.

The phase space is $\mathbf{T}^*\mathbf{Q}$, and its elements can be written as

$$\Xi = ((B, \mathbf{R}_0), \mathbf{R}_1, \ldots, \mathbf{R}_n; (B\widehat{\mathbf{\Pi}}, \mathbf{P}_0), \mathbf{P}_1, \ldots, \mathbf{P}_n) \tag{5}$$

where $\Pi = \mathbf{I}\Omega + L_r$ is the total angular momentum vector of the gyrostat in the body frame $\mathcal{B}$, $P_i = m_i \dot{R}_i$ $(i = 0, 1, ..., n)$ are the linear momenta of the bodies in the fixed frame $\mathcal{I}$, and $\mathbf{T}^*\mathbf{Q}$ carries a canonical symplectic structure ω defined as

$$w = w^{\mathbf{SE}(3)} + w^{\mathbb{R}^3} + \overset{n)}{\ldots} + \ w^{\mathbb{R}^3} \tag{6}$$

Associated to the symplectic structure on $\mathbf{T}^*\mathbf{Q}$ given by ω we have a Poisson structure where the Poisson bracket takes the form

$$\{f, g\}_{T^*\mathfrak{Q}}(\Xi) =$$
$$(< D_B f, \frac{\partial g}{\partial B\widehat{\mathbf{\Pi}}} > - < D_B g, \frac{\partial f}{\partial B\widehat{\mathbf{\Pi}}} > + \sum_{i=0}^{n} \left(\frac{\partial f}{\partial \mathbf{R}_i}\frac{\partial g}{\partial \mathbf{P}_i} - \frac{\partial g}{\partial \mathbf{R}_i}\frac{\partial f}{\partial \mathbf{P}_i} \right))(\Xi) \tag{7}$$

By the Legendre transformation, we obtain the Hamiltonian of the problem $\mathcal{H} : \mathbf{T}^*\mathbf{Q} \to \mathbb{R}$

$$H = \frac{1}{2}\mathbf{\Pi} \cdot \mathbb{I}^{-1}\mathbf{\Pi} - \mathbf{l}_r \cdot \mathbb{I}^{-1}\mathbf{\Pi} + \sum_{i=0}^{n} \frac{\mid \mathbf{P}_i \mid^2}{2m_i} + V \circ \tau_{T^*\mathfrak{Q}} \tag{8}$$

§3. Symmetries and reduction

The problem can be reduced by the action of the group $SE(3)$. In this case we can proceed by stages (Marsden, 1992):

• In the first stage, in a similar way to [12], using the regular reduction theorem ([10]), a symplectic reduction procedure by the translation group $\mathbb{R}^3$ is made and introducing the barycentric coordinates in the following way: $\mathbf{R}_{i,j} = \mathbf{R}_j - \mathbf{R}_i$ (it is the mutual vector between S_i and S_j),

$\mathbf{G}_k$ are the center of mass of the systems $\{S_n, S_{n-1}, ..., S_{n-k}\}$ $(k = 1, 2, ..., n-1)$, $M_j = \sum_{k=j-1}^{n} m_k$, $(j = 1, \ldots, n)$
Then, we define these coordinates as

$$\begin{array}{ccc} \rho_1 = \mathbf{R}_{n-1,n} & \rho_2 = \mathbf{R}_{n-2} - \mathbf{G}_1 & \ldots \\ \rho_k = \mathbf{R}_{n-k} - \mathbf{G}_{k-1} & \ldots & \rho_n = \mathbf{R}_0 - \mathbf{G}_{n-1} \end{array} \tag{9}$$

and the reduced masses by means of

$$\begin{array}{ccc} g_1 = \dfrac{m_n m_{n-1}}{M_n} & g_2 = \dfrac{m_{n-2} M_n}{M_{n-1}} & \ldots \\ g_i = \dfrac{m_{n-i} M_{n-i+2}}{M_{n-i+1}} & \ldots & g_n = \dfrac{m_0 M_2}{M_1} \end{array} \tag{10}$$

The linear momenta corresponding to these coordinates are

$$\begin{array}{ccc} \widetilde{\rho}_1 = g_1 \rho_1 & \widetilde{\rho}_2 = g_2 \rho_2 & \widetilde{\rho}_3 = g_3 \rho_3 \\ \widetilde{\rho}_i = g_i \rho_i & \ldots & \widetilde{\rho}_n = g_n \rho_n \end{array} \tag{11}$$

Then the reduced space is given by the symplectic manifold (M, ω_m), where $M = SO(3) \times so(3)^* \times \mathbf{T}^*\mathbb{R}^3 \times \overset{n)}{\ldots} \times \mathbf{T}^*\mathbb{R}^3$, and the symplectic form is defined by $\omega_M = \omega^{\mathbf{SO}(3)} + \omega^{\mathbb{R}^3} + \overset{n)}{\ldots} + \omega^{\mathbb{R}^3}$.
The Poisson bracket $\{f, g\}_M(\mathbf{m})$ associated to this symplectic form ω_M is given by

$$(< D_B f, \frac{\partial g}{\partial B\widehat{\mathbf{\Pi}}} > - < D_B g, \frac{\partial f}{\partial B\widehat{\mathbf{\Pi}}} > + \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \rho_i} \frac{\partial g}{\partial \widetilde{\rho}_i} - \frac{\partial g}{\partial \rho_i} \frac{\partial f}{\partial \widetilde{\rho}_i} \right))(\mathbf{m}) \tag{12}$$

for any $f, g \in C^\infty(M)$.

The reduced Hamiltonian on M is the function

$$H_I(\mathbf{m}) = \sum_{i=1}^{n} \frac{|\ \widetilde{\rho}_i\ |^2}{2g_i} + \frac{1}{2} \mathbf{\Pi} \cdot \mathbb{I}^{-1} \mathbf{\Pi} - \mathbf{l}_r \cdot \mathbb{I}^{-1} \mathbf{\Pi} + V(\mathbf{m}) \tag{13}$$

where

$$V(\mathbf{m}) = -(\sum_{\substack{i,j=1 \\ i \neq j}}^{n} \frac{G m_i m_j}{|\ \mathbf{R}_{i,j}\ |} + \sum_{i=1}^{n} \int_{S_o} \frac{G m_i dm(\mathbf{Q})}{|\ B\mathbf{Q} + \mathbf{R}_{i,0}\ |}) \tag{14}$$

and $\mathbf{R}_{i,j}$ can be expressed as function of the ρ_i, $(i = 1, \ldots, n)$. In the case $n = 2$ we have:

$$\mathbf{R}_{1,2} = \rho_1 \quad \mathbf{R}_{1,0} = \rho_2 + \frac{m_2}{M_2} \rho_1 \quad \mathbf{R}_{2,0} = \rho_2 - \frac{m_1}{M_2} \rho_1 \tag{15}$$

being now $M_2 = m_1 + m_2$.

For $n = 3$:

$$\begin{array}{ccc} \mathbf{R}_{2,1} = \rho_2 + \dfrac{m_3}{M_3} \rho_1 & \mathbf{R}_{1,0} = \rho_3 - \dfrac{M_3}{M_2} \rho_2 & \mathbf{R}_{3,1} = \rho_2 - \dfrac{m_2}{M_3} \rho_1 \\ \mathbf{R}_{2,0} = \rho_3 + \dfrac{m_1}{M_2} \rho_2 + \dfrac{m_3}{M_3} \rho_1 & \mathbf{R}_{2,3} = \rho_1 & \mathbf{R}_{3,0} = \rho_3 + \dfrac{m_1}{M_2} \rho_2 - \dfrac{m_2}{M_3} \rho_1 \end{array} \tag{16}$$

where $M_3 = m_2 + m_3$ and $M_2 = m_1 + m_2 + m_3$.
• In the second stage, as in [12], we do a Poisson reduction procedure by the rotation group $SO(3)$ and a model for $M/SO(3)$ is obtained by means of the function
$\Psi_2 : (M/\mathbf{SO}(3), \{\cdot,\cdot\}_{M/\mathbf{SO}(3)}) \to (\mathbb{R}^{6n+3}, \{\cdot,\cdot\}_{II})$ defined as follows

$$\Psi_2(\mathbf{m}) = (\boldsymbol{\Pi}, \lambda_1, \mathbf{p}_{\lambda_1}, \lambda_2, \mathbf{p}_{\lambda_2}, \ldots, \lambda_n, \mathbf{p}_{\lambda_n}) \tag{17}$$

where $\lambda_i = B^t\,\rho_i$, $\mathbf{p}_{\lambda_i} = B^t\,\widetilde{\rho}_i$, $(i = 1,\ldots,n)$; being Ψ_2 a Poisson diffeomorphism and the corresponding Poisson bracket $\{\cdot,\cdot\}_{II}$ is given by

$$\{f,g\}_{II}(\mathbf{z}) = (\nabla_z f)^t \mathbf{B}(\mathbf{z})\,\nabla_z g \tag{18}$$

with the Poisson tensor $\mathbf{B}(\mathbf{z})$ given by the following expression

$$\mathbf{B}(\mathbf{z}) = \begin{pmatrix} \widehat{\boldsymbol{\Pi}} & \widehat{\lambda}_1 & \widehat{\mathbf{p}_{\lambda_1}} & \cdots & \cdots & \widehat{\lambda_n} & \widehat{\mathbf{p}_{\lambda_n}} \\ \widehat{\lambda}_1 & \mathbf{0} & I_{\mathbb{R}^3} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \widehat{\mathbf{p}_{\lambda_1}} & -I_{\mathbb{R}^3} & \mathbf{0} & \ddots & \ddots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \widehat{\lambda_n} & \mathbf{0} & \vdots & \vdots & \ddots & \mathbf{0} & I_{\mathbb{R}^3} \\ \widehat{\mathbf{p}_{\lambda_n}} & \mathbf{0} & \vdots & \vdots & \mathbf{0} & -I_{\mathbb{R}^3} & \mathbf{0} \end{pmatrix} \tag{19}$$

If $\mathbf{L}$ is the total angular momentum of the system: $\mathbf{L} = \boldsymbol{\Pi} + \sum_{i=1}^{n} \lambda_i \wedge \mathbf{p}_{\lambda_i}$, then for a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(\mid \mathbf{L} \mid)$ are Casimir functions.

The twice reduced Hamiltonian is the function

$$H_{II}(\mathbf{z}) = H_{M/\mathbf{SO}(3)}(\Psi_2^{-1}(\mathbf{z})) = \sum_{i=1}^{n} \frac{\mid \mathbf{p}_{\lambda_i} \mid^2}{2g_i} + \frac{1}{2}\Pi \cdot \mathbb{I}^{-1}\Pi - \mathbf{l}_r \cdot \mathbb{I}^{-1}\Pi + V(\mathbf{z}) \tag{20}$$

where the potential function V is expressed in terms of the λ_i.

§4. Equations of motion. Relative equilibria

The relative equilibria are the equilibria for the twice reduced problem. Whose equations of motion can be computed, using the Poisson brackets, by the following expression

$$\dot{\mathbf{z}} = \{\mathbf{z}, H_{II}(\mathbf{z})\}_{II}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}} H_{II}(\mathbf{z}) \tag{21}$$

Then, the relative equilibria are given by the system

$$\boldsymbol{\Pi} \wedge \boldsymbol{\Omega} + \sum_{i=1}^{n} \lambda_i \wedge \nabla_{\lambda_i} V = \mathbf{0} \tag{22}$$

$$\lambda_i \wedge \mathbf{\Omega}+\frac{1}{g_i}\mathbf{p}_{\lambda_i}=\mathbf{0} \quad \mathbf{p}_{\lambda_i}\wedge \mathbf{\Omega}-\nabla_{\lambda_i}V=\mathbf{0},(i=1,\ldots,n) \tag{23}$$

where $\mathbf{\Omega}=\mathbb{I}^{-1}(\mathbf{\Pi}-\mathbf{l}_r)$ is the angular velocity of S_0.
If $\mathbf{z}_e=(\mathbf{\Pi}_e,\lambda_1^e,\mathbf{p}^e_{\lambda_1},\lambda_2^e,\mathbf{p}^e_{\lambda_2},...,\lambda_i^e,\mathbf{p}^e_{\lambda i},...,\lambda_n^e,\mathbf{p}^e_{\lambda_n})$ is a relative equilibrium, by vector calculus we obtain the equations

$$\lambda_i^e \mid \mathbf{\Omega}_e \mid^2 -(\lambda_i^e\cdot\mathbf{\Omega}_e)\mathbf{\Omega}_e=\frac{1}{g_i}\nabla_{\lambda_i}V_e,\ (i=1,\ldots,n) \tag{24}$$

where $\nabla_{\lambda_i}V_e$ is $\nabla_{\lambda_i}V$ evaluated in $\mathbf{z}_e$ and $\mathbf{\Omega}_e$ the same thing.

The dot product of each equation with the corresponding λ_i^e, yields

$$\begin{gathered}\mathbf{\Pi}_e\wedge\mathbf{\Omega}_e+\sum_{i=1}^n\lambda_i^e\wedge\nabla_{\lambda_i}V_e=0\\ \mid\mathbf{\Omega}_e\mid^2\mid\lambda_i^e\mid^2-(\lambda_i^e\cdot\mathbf{\Omega}_e)^2=\frac{1}{g_i}(\lambda_i^e\cdot\nabla_{\lambda_i}V_e),\ (i=1,\ldots,n)\end{gathered} \tag{25}$$

Then, it is easy to prove the following result: **In the equilibria z_e the angular velocity $\mathbf{\Omega}_e$ of S_0 is parallel to the total angular momentum of the system.** Similar results were obtained in Maciejewski (1995) and Mondéjar and Vigueras (1999).
The general problem is open as in the case of the n-body problem. But the last equations suggest the idea of considering two types of equilibria according to the vector Ω be orthogonal or not to the vectors λ_i and these be coplanar. Then, the equations (25) for equilibria simplify to

$$\begin{gathered}\mathbf{\Pi}_e\wedge\mathbf{\Omega}_e+\sum_{i=1}^n\lambda_i^e\wedge\nabla_{\lambda_i}V_e=0\\ \mid\mathbf{\Omega}_e\mid^2\mid\lambda_i^e\mid^2=\frac{1}{g_i}(\lambda_i^e\cdot\nabla_{\lambda_i}V_e),\ (i=1,\ldots,n)\end{gathered} \tag{26}$$

Other way in order to study the equations of equilibria is to impose the existence of equilibria of Euler or Lagrange type and then, to obtain sufficient conditions about the angular velocity and the gyrostatic momentum of the gyrostat. Other simplification of the problem consist in assuming that the potential function is approximated by truncation of the corresponding Taylor series expansion.

§5. Necessary and sufficient conditions for stability

5.1. Necessary conditions

The necessary conditions for stability of relative equilibria, $\mathbf{z}_e$, shall be obtained by spectral analysis of the linearized equations at $\mathbf{z}_e$

$$\dot{\delta\mathbf{z}}=\mathfrak{A}(\mathbf{z}_e)\delta\mathbf{z} \tag{27}$$

being $\delta\mathbf{z}=\mathbf{z}-\mathbf{z}_e$ and $\mathfrak{A}(\mathbf{z}_e)$ is the Jacobian matrix of the system at $\mathbf{z}_e$.

Spectral stability is necessary for stability, but to study spectral stability we must calculate the eigenvalues of $\mathfrak{A}(\mathbf{z}_e)$, whose characteristic polynomial has degree $6n+3$, as zero is an eigenvalue of this polynomial, then the problem is reduced to give necessary and sufficient conditions for the roots of

$$x^{6n+2} + a_1 x^{6n} + \ldots + a_i x^{6n-2i} + \ldots + a_{3n} \tag{28}$$

belong to the imaginary axis.

The matrix $\mathfrak{A}(\mathbf{z}_e)$ can be obtained in the following way

$$\mathfrak{A}(\mathbf{z}_e) = \mathbf{B}(\mathbf{z}_e)\mathbf{d}^2 F(\mathbf{z}_e) \tag{29}$$

being $F = H_{II} + \lambda(\frac{1}{2} \mid \mathbf{L} \mid^2)$ and λ a multiplier to be determinate with the condition

$$\mathbf{d}F(\mathbf{z}_e) = \mathbf{0} \tag{30}$$

5.2. Sufficient conditions

We will use the energy-Casimir method as a tool to study the stability of these solutions, granting sufficient conditions of Lyapunov's stability for equilibrium solutions of mechanical systems with symmetry. This theorem can be seen in [13]

Theorem 1. Let $(M, \{\cdot,\cdot\}, h)$ be a system of Poisson, $m \in M$ an equilibrium solution of the Hamiltonian vector field X_h and $C_1, C_2, ..., C_n \in C^\infty(M)$ integrals of system, verifying

$$\mathbf{d}(h + C_1 + C_2 + ... + C_n)(m) = 0$$

and that

$$\mathbf{d}^2(h + C_1 + C_2 + ... + C_n)(m) \mid_{W \times W}$$

is a positive or negative definite quadratic form on $W \times W$, where W is defined by

$$W = ker\mathbf{d}C_1(m) \cap ker\mathbf{d}C_2(m) \cap ... \cap ker\mathbf{d}C_n(m)$$

Then, $m \in M$ is stable. If $W = \{0\}$, $m \in M$ is also stable.

Also, we prove the following previous results:

Lemma 1. Let be $\mathbf{u} \in \mathbb{R}^3$, and $\widehat{\mathbf{u}} \in \mathbf{so}(3)$ then we have the identities

$$(\widehat{\mathbf{u}}A)^t = -\mathbf{A}^t\widehat{\mathbf{u}}; \quad (\mathbf{A}\widehat{\mathbf{u}})^t = -\widehat{\mathbf{u}}\mathbf{A}^t \tag{31}$$

with $\mathbf{A} \in \mathbf{Mat}_{3\times 3}(\mathbb{R})$.

Lemma 2. Let be $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$, then we have the equalities

$$\begin{gathered} \nabla_{a_1}\cdot(\mathbf{a}_1\wedge\mathbf{a}_2)^t = \widehat{\mathbf{a}}_2 \qquad\qquad \nabla_{a_2}\cdot(\mathbf{a}_1\wedge\mathbf{a}_2)^t = -\widehat{\mathbf{a}}_1 \\ \nabla_{\mathbf{a}_1}\cdot(\mathbf{a}_3\wedge(\mathbf{a}_1\wedge\mathbf{a}_2))^t = -\widehat{\mathbf{a}}_2\widehat{\mathbf{a}}_3; \quad \nabla_{\mathbf{a}_2}\cdot(\mathbf{a}_3\wedge(\mathbf{a}_1\wedge\mathbf{a}_2))^t = \widehat{\mathbf{a}}_1\widehat{\mathbf{a}}_3 \end{gathered} \tag{32}$$

The proofs are a routine calculation.

The following result is important, because of the obtained equations help us to calculate the matrix to consider in the energy-Casimir method.

Lemma 3. Let be $\phi(\frac{1}{2} \mid \mathbf{L} \mid^2)$, where ϕ is a smooth real function and $\mathbf{L}$ is the total angular momentum of the system, then this function verify the following relations

a)

$$\nabla_{\mathbf{\Pi}}(\phi) = \phi'\mathbf{L} \quad \nabla_{\lambda_i}(\phi) = \phi'\widehat{\mathbf{p}_{\lambda_i}}\mathbf{L} \quad \nabla_{\mathbf{p}_{\lambda_i}}(\phi) = -\phi'\widehat{\lambda_i}\mathbf{L} \quad (i = 1, \ldots, n) \tag{33}$$

b)

$$\begin{gathered}
Hess_{\mathbf{\Pi},\mathbf{\Pi}}(\phi) = \mathbf{A}_{\mathbf{\Pi},\mathbf{\Pi}} = \phi' I_{\mathbb{R}^3} + \phi''\mathbf{L}\mathbf{L}^t \\
Hess_{\mathbf{\Pi},\lambda_i}(\phi) = \mathbf{A}_{\mathbf{\Pi},\lambda_i} = \phi'\widehat{\mathbf{p}_{\lambda_i}} + \phi''\widehat{\mathbf{p}_{\lambda_i}}\mathbf{L}\mathbf{L}^t \\
Hess_{\lambda_i,\mathbf{\Pi}}(\phi) = \mathbf{A}_{\lambda_i,\mathbf{\Pi}} = -(\phi'\widehat{\mathbf{p}_{\lambda_i}} + \phi''\mathbf{L}\mathbf{L}^t\widehat{\mathbf{p}_{\lambda_i}}) \\
Hess_{\lambda_i,\lambda_j}(\phi) = \mathbf{A}_{\lambda_i,\lambda_j} = -(\phi'\widehat{\mathbf{p}_{\lambda_j}}\widehat{\mathbf{p}_{\lambda_i}} + \phi''\widehat{\mathbf{p}_{\lambda_j}}\mathbf{L}\mathbf{L}^t\widehat{\mathbf{p}_{\lambda_i}}) \\
Hess_{\lambda_i,\mathbf{p}_{\lambda_j}}(\phi) = \mathbf{A}_{\lambda_i,\mathbf{p}_{\lambda_j}} = \begin{cases} \phi'\widehat{\lambda_j}\widehat{\mathbf{p}_{\lambda_i}} + \phi''\widehat{\lambda_j}\mathbf{L}\mathbf{L}^t\widehat{\mathbf{p}_{\lambda_i}}, i \neq j \\ \phi'\widehat{\mathbf{L}} + \phi'\widehat{\lambda_i}\widehat{\mathbf{p}_{\lambda_i}} + \phi''\widehat{\lambda_i}\mathbf{L}\mathbf{L}^t\widehat{\mathbf{p}_{\lambda_i}}, i = j \end{cases} \\
(i, j = 1, \ldots, n)
\end{gathered} \tag{34}$$

$$\begin{gathered}
Hess_{\mathbf{\Pi},\mathbf{p}_{\lambda_i}}(\phi) = \mathbf{A}_{\mathbf{\Pi},\mathbf{p}_{\lambda_i}} = -(\phi'\widehat{\lambda_i} + \phi''\widehat{\lambda_i}\mathbf{L}\mathbf{L}^t) \\
Hess_{\mathbf{p}_{\lambda_i},\mathbf{\Pi}}(\phi) = \mathbf{A}_{\mathbf{p}_{\lambda_i},\mathbf{\Pi}} = A_{\mathbf{p}_{\lambda_i},\mathbf{\Pi}} = \phi'\widehat{\lambda_i} + \phi''\mathbf{L}\mathbf{L}^t\widehat{\lambda_i} \\
Hess_{\mathbf{p}_{\lambda_i},\mathbf{p}_{\lambda_j}}(\phi) = \mathbf{A}_{\mathbf{p}_{\lambda_i},\mathbf{p}_{\lambda_j}} = -(\phi'\widehat{\lambda_j}\widehat{\lambda_i} + \phi''\mathbf{L}\mathbf{L}^t\widehat{\lambda_j}\widehat{\lambda_i}) \\
Hess_{\mathbf{p}_{\lambda_i},\lambda_j}(\phi) = \mathbf{A}_{\mathbf{p}_{\lambda_i},\lambda_j} = \begin{cases} \phi'\widehat{\mathbf{p}_{\lambda_j}}\widehat{\lambda_i} + \phi''\widehat{\mathbf{p}_{\lambda_j}}\mathbf{L}\mathbf{L}^t\widehat{\lambda_i},\ i \neq j \\ -\phi'\widehat{\mathbf{L}} + \phi'\widehat{\mathbf{p}_{\lambda_i}}\widehat{\lambda_i} + \phi''\widehat{\mathbf{p}_{\lambda_i}}\mathbf{L}\mathbf{L}^t\widehat{\lambda_i},\ i = j \end{cases} \\
(i, j = 1, \ldots, n)
\end{gathered} \tag{35}$$

besides, we have:

$$\begin{gathered}
\mathbf{A}_{\lambda_i,\mathbf{\Pi}} = (\mathbf{A}_{\mathbf{\Pi},\lambda_i})^t \quad \mathbf{A}_{\mathbf{p}_{\lambda_i},\mathbf{\Pi}} = (\mathbf{A}_{\mathbf{\Pi},\mathbf{p}_{\lambda_i}})^t \\
\mathbf{A}_{\lambda_i,\lambda_j} = (\mathbf{A}_{\lambda_j,\lambda_i})^t \quad \mathbf{A}_{\mathbf{p}_{\lambda_i},\lambda_j} = (\mathbf{A}_{\lambda_j,\mathbf{p}_{\lambda_i}})^t
\end{gathered} \tag{36}$$

$$\mathbf{A}_{\mathbf{p}_{\lambda_i},\mathbf{p}_{\lambda_j}} = (\mathbf{A}_{\mathbf{p}_{\lambda_j},\mathbf{p}_{\lambda_i}})^t \tag{37}$$

The proof use the results of the previous lemmas.

Now, we consider the following function

$$H_\phi = H_{II} + \phi(\tfrac{1}{2} \mid \mathbf{L} \mid^2) \tag{38}$$

if $\mathbf{z}_e$ verify $\mathbf{d}H_\phi(\mathbf{z}_e) = \mathbf{0}$ and $\mathbf{d}^2 H_\phi(\mathbf{z}_e)$ is a (positive or negative) definite matrix, then $\mathbf{z}_e$ is a Lyapunov stable relative equilibrium of the system.

First, from $\mathbf{d}H_\phi(\mathbf{z}_e) = \mathbf{0}$, we deduce

$$\nabla_\Pi H_\phi(\mathbf{z}_e) = \mathbf{0}; \quad \nabla_{\lambda_i} H_\phi(\mathbf{z}_e) = \mathbf{0}; \quad \nabla_{\mathbf{p}_{\lambda_i}} H_\phi(\mathbf{z}_e) = \mathbf{0}, \ (i = 1, \dots, n) \tag{39}$$

and explicitly we have

$$\mathbf{\Omega}_e + a\mathbf{L}_e = \mathbf{0} \quad \nabla_{\lambda_i} V_e + a(\mathbf{p}^e_{\lambda_i} \wedge \mathbf{L}_e) = \mathbf{0}, \quad \frac{\mathbf{p}^e_{\lambda_i}}{g_i} - a(\lambda^e_i \wedge \mathbf{L}_e) = \mathbf{0}, \ (i = 1, \dots, n) \tag{40}$$

where $a = \phi'(\frac{1}{2} \mid \mathbf{L}_e|^2)$ and $\mathbf{L}_e$ is the total angular momentum of the system evaluated in the equilibrium. Hence $\mathbf{L}_e = -a\mathbf{\Omega}_e$, with $a = -\dfrac{\mid \mathbf{\Omega}_e \mid}{\mid \mathbf{L}_e \mid}$. Then the previous equations are equivalent to the (22)-(23).

In general, it is very difficult to give conditions in order to the matrix $\mathbf{d}^2 H_\phi(\mathbf{z}_e)$ be definite and the algebraic manipulations of these conditions will be possible only by means of a symbolic manipulator as Maple.

§6. Relative equilibria and stability for the three-body problem ($n = 2$)

Here we study the problem of three bodies when one of them is a triaxial gyrostat, with gyrostatic momentum constant, in Newtonian attraction with two spherical rigid bodies. Then, the potential function can be written as

$$V(\lambda, \mu) = -(\frac{Gm_1 m_2}{\mid \lambda \mid} + Gm_1 \int_{S_0} \frac{dm(\mathbf{Q})}{\mid \mathbf{Q} + \mu + \frac{m_2}{M_2}\lambda \mid} + Gm_2 \int_{S_0} \frac{dm(\mathbf{Q})}{\mid \mathbf{Q} + \mu - \frac{m_1}{M_2}\lambda \mid}) \tag{41}$$

being $M_2 = m_1 + m_2$ and where $\lambda = \lambda_1, \mu = \lambda_2$.

We suppose that the dimensions of the gyrostat are smaller than the mutual distances between the bodies. We also assume that S_0 is symmetric with respect to the third axis and to the x-y plane of the body. With both assumptions, the gravitational potential function adopts the following Taylor series expansion

$$V(\lambda, \mu) = -(\frac{Gm_1 m_2}{\mid \lambda \mid} + Gm_1 \sum_{i=0}^{\infty} \frac{A_{2i}}{\mid \mu + \frac{m_2}{M_2}\lambda \mid^{2i+1}} + Gm_2 \sum_{i=0}^{\infty} \frac{A_{2i}}{\mid \mu - \frac{m_1}{M_2}\lambda \mid^{2i+1}}) \tag{42}$$

where A_{2i} are coefficients given in Leimanis (1965).

The k^{th} order approximation of the potential is given by the expression

$$V_k(\lambda, \mu) = -(\frac{Gm_1 m_2}{\mid \lambda \mid} + Gm_1 \sum_{i=0}^{k} \frac{A_{2i}}{\mid \mu + \frac{m_2}{M_2}\lambda \mid^{2i+1}} + Gm_2 \sum_{i=0}^{k} \frac{A_{2i}}{\mid \mu - \frac{m_1}{M_2}\lambda \mid^{2i+1}}) \tag{43}$$

Note that

$$\nabla_\lambda V_k = \frac{Gm_1m_2\lambda}{\mid\lambda\mid^3} + \frac{Gm_1m_2}{M_2}\sum_{i=0}^{k}\frac{(\mu+\frac{m_2}{M_2}\lambda)(2i+1)A_{2i}}{\mid\mu+\frac{m_2}{M_2}\lambda\mid^{2i+3}} - \frac{Gm_1m_2}{M_2}\sum_{i=0}^{k}\frac{(\mu-\frac{m_1}{M_2}\lambda)(2i+1)A_{2i}}{\mid\mu-\frac{m_1}{M_2}\lambda\mid^{2i+3}}$$

$$\nabla_\mu V_k = Gm_1\sum_{i=0}^{k}\frac{(\mu+\frac{m_2}{M_2}\lambda)(2i+1)A_{2i}}{\mid\mu+\frac{m_2}{M_2}\lambda\mid^{2i+3}} + Gm_2\sum_{i=0}^{k}\frac{(\mu-\frac{m_1}{M_2}\lambda)(2i+1)A_{2i}}{\mid\mu-\frac{m_1}{M_2}\lambda\mid^{2i+3}} \tag{44}$$

so

$$\nabla_\lambda V_k = \widetilde{A}_{11}\lambda+\widetilde{A}_{12}\mu \qquad \nabla_\mu V_k = \widetilde{A}_{21}\lambda+\widetilde{A}_{22}\mu \tag{45}$$

being

$$\widetilde{A}_{11}(\lambda,\mu) = \frac{Gm_1m_2}{\mid\lambda\mid^3} + \frac{Gm_1m_2^2}{M_2^2}\left(\sum_{i=0}^{k}\frac{\alpha_i}{\mid\mu+\frac{m_2}{M_2}\lambda\mid^{2i+3}}\right) + \frac{Gm_1^2m_2}{M_2^2}\left(\sum_{i=0}^{k}\frac{\alpha_i}{\mid\mu-\frac{m_1}{M_2}\lambda\mid^{2i+3}}\right) \tag{46}$$

$$\widetilde{A}_{12}(\lambda,\mu) = \frac{Gm_1m_2}{M_2}(\sum_{i=0}^{k}\frac{\alpha_i}{\mid\mu+\frac{m_2}{M_2}\lambda\mid^{2i+3}} - \sum_{i=0}^{k}\frac{\alpha_i}{\mid\mu-\frac{m_1}{M_2}\lambda\mid^{2i+3}}) = \widetilde{A}_{21}(\lambda,\mu)$$

$$\widetilde{A}_{22}(\lambda,\mu) = Gm_1\left(\sum_{i=0}^{k}\frac{\alpha_i}{\mid\mu+\frac{m_2}{M_2}\lambda\mid^{2i+3}}\right) + Gm_2\left(\sum_{i=0}^{k}\frac{\alpha_i}{\mid\mu-\frac{m_1}{M_2}\lambda\mid^{2i+3}}\right) \tag{47}$$

with the coefficients $\alpha_i = (2i+1)A_{2i}$.
The k^{th} order approximation dynamics is given by the following differential equations

$$\dot{\mathbf{z}} = \{\mathbf{z}, H_{II}^k(\mathbf{z})\}_{II}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}H_{II}^k(\mathbf{z})$$

where the Hamiltonian function is

$$H_{II}^k(\mathbf{z}) = \frac{\mid\mathbf{p}_\lambda\mid^2}{2g_1} + \frac{\mid\mathbf{p}_\mu\mid^2}{2g_2} + \frac{1}{2}\Pi\cdot\mathbb{I}^{-1}\Pi - \mathbf{l}_r\cdot\mathbb{I}^{-1}\Pi + V_k(\lambda,\mu)$$

6.1. Relative equilibria in the k^{th} order approximation dynamics

If $\mathbf{z}_e = (\mathbf{\Pi}_e, \lambda^e, \mathbf{p}_\lambda^e, \mu^e, \mathbf{p}_\mu^e)$ is a relative equilibrium in the k^{th} order approximation dynamics then we have

$$\mathbf{\Pi}_e\wedge\mathbf{\Omega}_e+\lambda^e\wedge(\nabla_\lambda V_k)_e + \mu^e\wedge(\nabla_\mu V_k)_e = \mathbf{0} \tag{48}$$

$$\frac{\mathbf{p}_\lambda^e}{g_1} + \lambda^e\wedge\mathbf{\Omega}_e = \mathbf{0} \qquad \frac{\mathbf{p}_\mu^e}{g_2} + \mu^e\wedge\mathbf{\Omega}_e = \mathbf{0}$$

$$\mathbf{p}_\lambda^e\wedge\mathbf{\Omega}_e = (\nabla_\lambda V_k)_e \quad \mathbf{p}_\mu^e\wedge\mathbf{\Omega}_e = (\nabla_\mu V_k)_e \tag{49}$$

and by (46)-(47) we obtain

$$\mathbf{\Pi}_e \wedge \mathbf{\Omega}_e = \mathbf{0}$$

$$|\ \mathbf{\Omega}_e\ |^2|\ \lambda^e\ |^2 - (\lambda^e \cdot \mathbf{\Omega}_e)^2 = \frac{1}{g_1}(\lambda^e \cdot (\nabla_\lambda V_k)_e) \tag{50}$$

$$|\ \mathbf{\Omega}_e\ |^2|\ \mu^e\ |^2 - (\mu^e \cdot \mathbf{\Omega}_e)^2 = \frac{1}{g_2}(\mu^e \cdot (\nabla_\mu V_k)_e)$$

Then we can deduce the following property:

In the relative equilibria for any approximation dynamics there are no moments about the gyrostat, that is the gyrostat move under inertia.

Next, we are going to study the relative equilibria for any approximation dynamics, by assuming that the vectors $\mathbf{\Omega}_e$, λ^e, μ^e verify geometrical properties. So $\mathbf{z}_e$ is a equilibrium **Euler type** when, λ^e, μ^e are collinear, and $\mathbf{\Omega}_e$ is orthogonal to the line defined by the three bodies. And $\mathbf{z}_e$ is a equilibrium **Lagrange type** when λ^e, μ^e are coplanar but no collinear, and $\mathbf{\Omega}_e$ is orthogonal to the plane spanned by λ^e, and μ^e. Other types will be considered in next papers ($\mathbf{\Omega}_e$ is in the plane spanned by λ^e, and μ^e).

In what follows we only give necessary conditions for existence of Eulerian and Lagrangian equilibria in different cases; it is easy to prove that these conditions are sufficient and a complete study about these questions will be made in [14].

6.1.1. Existence of Eulerian equilibria

If $\mathbf{z}_e$ is a Eulerian equilibrium for an approximated dynamics of order k, the following identities hold

$$g_1\ |\ \mathbf{\Omega}_e\ |^2|\ \lambda^e\ |^2 = \lambda^e \cdot (\nabla_\lambda V_k)_e; \quad g_2\ |\ \mathbf{\Omega}_e\ |^2|\ \mu^e\ |^2 = \mu^e \cdot (\nabla_\mu V_k)_e \tag{51}$$

as λ^e and μ^e are collinear, to simplify the notation we shall set

$$\rho = \frac{|\ \mu^e - \frac{m_1}{M_2}\lambda^e\ |}{|\ \lambda^e\ |} \tag{52}$$

and three cases are possible, but we are going to study only the case in which the following relations are verified

$$\mu^e - \tfrac{m_1}{M_2}\lambda^e = \rho\lambda^e; \quad \mu^e + \tfrac{m_2}{M_2}\lambda^e = (1+\rho)\lambda^e \tag{53}$$

hence

$$\mu^e = \frac{((1+\rho)m_1 + \rho m_2)}{M_2}\lambda^e \tag{54}$$

And doing the suitable calculations we have

$$(\nabla_\lambda V_k)_e = \widetilde{A}(\rho)\lambda^e; \quad (\nabla_\mu V_k)_e = \widetilde{B}(\rho)\lambda^e \tag{55}$$

being

$$\widetilde{A}(\rho) = \frac{Gm_1m_2}{|\ \lambda^e\ |^3} + \frac{Gm_1m_2}{M_2}(\sum_{i=0}^{k}\frac{\alpha_i}{|\ \lambda^e\ |^{2i+3}} \cdot \left(\frac{1}{(1+\rho)^{2i+2}} - \frac{1}{\rho^{2i+2}}\right))$$

$$\widetilde{B}(\rho) = \sum_{i=0}^{k}\frac{G\alpha_i}{|\ \lambda^e\ |^{2i+3}}\left(\frac{m_1}{(1+\rho)^{2i+2}} + \frac{m_2}{\rho^{2i+2}}\right) \tag{56}$$

the following identities

$$\begin{aligned} \lambda^e \cdot (\nabla_\lambda V_k)_e &= \mid \lambda^e \mid^2 \widetilde{A}(\rho) \\ \mu^e \cdot (\nabla_\mu V_k)_e &= \frac{((1+\rho)m_1 + \rho m_2)}{M_2} \mid \lambda^e \mid^2 \widetilde{B}(\rho) \end{aligned} \tag{57}$$

help us to deduce the relations

$$\mid \mathbf{\Omega}_e \mid^2 = \frac{\widetilde{A}(\rho)}{g_1}; \quad \mid \mathbf{\Omega}_e \mid^2 = \frac{M_2 \widetilde{B}(\rho)}{g_2\left((1+\rho)m_1 + \rho m_2\right)} \tag{58}$$

and it is possible if and only if the equation

$$g_2\left((1+\rho)m_1 + \rho m_2\right) \widetilde{A}(\rho) = g_1 M_2 \widetilde{B}(\rho) \tag{59}$$

has positive real roots. In this case $\mathbf{\Omega}_e$ will be orthogonal to the line spanned by λ^e, and it will verify

$$\mid \mathbf{\Omega}_e \mid^2 = \frac{\widetilde{A}(\rho)}{g_1} = \frac{M_2 \widetilde{B}(\rho)}{g_2\left((1+\rho)m_1 + \rho m_2\right)} \tag{60}$$

Hence, in this particular case, the problem is reduced to study the positive real roots of the previous equation (59). Next, we are going to study the existence and number of equilibrium solutions in two particular cases corresponding to $k = 0$ and $k = 1$.

Eulerian equilibria in the zero order approximation dynamics In this case the equation (59) is reduced to the classical quintic equation of the three body problem

$$\begin{aligned} &(m_1 + m_2)\rho^5 + (3m_1 + 2m_2)\rho^4 + (3m_1 + m_2)\rho^3 + \\ &-(3m_0 + m_2)\rho^2 - (3m_0 + 2m_2)\rho - (m_0 + m_2) = 0 \end{aligned} \tag{61}$$

This equation whose coefficients are functions of the masses has a unique positive real root, then in this case the existence of Eulerian equilibria is proved being

$$\mid \mathbf{\Omega}_e \mid^2 = \frac{1}{g_1}\left(\frac{Gm_1 m_2}{\mid \lambda^e \mid^3} + \frac{Gm_1 m_2 \alpha_0}{M_2 \mid \lambda^e \mid^3}\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right)\right)$$

where ρ is a positive real root of (61).

Eulerian equilibria in the first order approximation Now we have to study the algebraic equation

$$\begin{gathered} m_0 a^2(m_1 + m_2)\rho^9 + m_0 a^2(5m_1 + 4m_2)\rho^8 + m_0 a^2(10m_1 + 6m_2)\rho^7 + \\ +3m_0 a^2(3m_1 + m_2 - m_0)\rho^6 + 3m_0 a^2(m_1 - m_2 - 3m_0)\rho^5 - \\ (6m_0 m_2 a^2 + 10m_0^2 a^2 + \alpha_1(m_1 + m_2 + 5m_0))\rho^4 - \\ (4m_0 m_2 a^2 + 5m_0^2 a^2 + \alpha_1(10m_0 + 4m_2))\rho^3 - \\ (m_0 m_2 a^2 + m_0^2 a^2 + \alpha_1(6m_2 + 10m_0)re)\rho^2 - \alpha_1(5m_0 + 4m_2)\rho - \alpha_1(m_0 + m_2) = 0 \end{gathered} \tag{62}$$

where $a =| \lambda^e |$, and $\alpha_1 = \frac{3}{2}m_0(C-A)$, being C y A the principal moments of inertia of the gyrostat.

If $C > A$ the equation (62) has a unique positive real root. In particular, we can consider that $m_1 = m_2 = m_0$, then this equation can be written as:

$$\begin{array}{c} 2ma^2\rho^9 + 9ma^2\rho^8 + 16ma^2\rho^7 + 9ma^2\rho^6 - 9ma^2\rho^5 - (7\alpha_1 + 16ma^2)\rho^4 \\ -(14\alpha_1 + 9ma^2)\rho^3 - (16\alpha_1 + 2ma^2)\rho^2 - 9\alpha_1\rho - 2\alpha_1 = 0 \end{array} \tag{63}$$

6.1.2. Existence of Lagrangian equilibria

IF $\mathbf{z}_e$ is a Lagrangian equilibrium for an approximated dynamics of order k, we deduce the following identities

$$\begin{array}{cc} \lambda^e \wedge (\nabla_\lambda V_k)_e = \mathbf{0} & \mu^e \wedge (\nabla_\mu V_k)_e = \mathbf{0} \\ g_1 \mid \mathbf{\Omega}_e \mid^2 (\lambda^e \wedge \mu^e) = (\nabla_\lambda V_k)_e \wedge \mu^e; & g_2 \mid \mathbf{\Omega}_e \mid^2 (\lambda^e \wedge \mu^e) = \lambda^e \wedge (\nabla_\mu V_k)_e \end{array} \tag{64}$$

By the formulas

$$\nabla_\lambda V_k = \widetilde{A}_{11}\lambda + \widetilde{A}_{12}\mu \quad \nabla_\mu V_k = \widetilde{A}_{21}\lambda + \widetilde{A}_{22}\mu \tag{65}$$

in an equilibrium solution we have

$$\begin{array}{cc} (\widetilde{A}_{12})_e(\lambda^e \wedge \mu^e) = \mathbf{0}; & (\widetilde{A}_{21})_e(\lambda^e \wedge \mu^e) = \mathbf{0} \\ g_1 \mid \mathbf{\Omega}_e \mid^2 (\lambda^e \wedge \mu^e) = (\widetilde{A}_{11})_e(\lambda^e \wedge \mu^e); & g_2 \mid \mathbf{\Omega}_e \mid^2 (\lambda^e \wedge \mu^e) = (\widetilde{A}_{22})_e(\lambda^e \wedge \mu^e) \end{array} \tag{66}$$

Thus

$$(\widetilde{A}_{12})_e = (\widetilde{A}_{21})_e = 0 \qquad \mid \mathbf{\Omega}_e \mid^2 = \frac{(\widetilde{A}_{11})_e}{g_1} = \frac{(\widetilde{A}_{22})_e}{g_2} \tag{67}$$

If we put $\mid \lambda^e \mid = Z$, $\mid \mu^{\mathbf{e}} + \frac{m_2}{M_2}\lambda^e \mid = X$, $\mid \mu^{\mathbf{e}} - \frac{m_1}{M_2}\lambda^e \mid = Y$, then a Lagrangian equilibrium solution exists if the following algebraic system has positive real roots

$$X^{2k+3} = \sum_{i=0}^{k} \beta_i Z^3 X^{2(k-i)}; \quad Y^{2k+3} = \sum_{i=0}^{k} \beta_i Z^3 Y^{2(k-i)} \;\; k = 0, \ldots, n \tag{68}$$

where Z and $\beta_i = \alpha_i/m_0$ are known parameters of the problem

Lagrangian equilibria in the zero order approximation dynamics For $k = 0$, the equations can be written as follows

$$X^3 = Z^3; \quad Y^3 = Z^3 \tag{69}$$

then $X = Y = Z$ is a equilibrium solution and the three bodies S_0, S_1, S_2 form an equilateral triangle in a plane orthogonal to the angular velocity that verify

$$\mid \mathbf{\Omega}_e \mid^2 = \frac{GM_1}{\mid \lambda^e \mid^3} \tag{70}$$

Lagrangian equilibria in the first order approximation dynamics For $k = 1$, the equations to be considered are

$$X^5 - Z^3X^2 - \beta_1 Z^3 = 0; \quad Y^5 - Z^3Y^2 - \beta_1 Z^3 = 0 \tag{71}$$

being Z and α_1 parameters of the problem.

Lagrangian equilibria in the second order approximation dynamics For $k = 2$, the equations (68) reduce to

$$\begin{aligned} X^7 - Z^3X^4 - \beta_1 Z^3 X^2 - \beta_2 Z^3 = 0 \\ Y^7 - Z^3Y^4 - \beta_1 Z^3 Y^2 - \beta_2 Z^3 = 0 \end{aligned} \tag{72}$$

§7. Conclusions and open problems

This paper represents a first step to understand the geometry and dynamics of the general motion of a gyrostat in Newtonian attraction with n rigid bodies with spherical distribution of mass.

We perform the reduction of the phase space by the semidirect product $SE(3)$ in two step, and we give the equations and some general remarks on the relative equilibria in the same line noted by Mondéjar, Vigueras and Ferrer (2001).

In the second part of the paper we give general criterions to study the stability of the equilibrium solutions of the problem in the general case.

After, we restrict to the three-body problem and different approximation dynamics are considered, giving necessary conditions for existence of equilibrium solutions of Euler and Lagrange type.

Finally, we note that some results of previous papers are included in our configurations equilibria when the gyrostatic momentum is null, and/or the body is spherical. In particular, we note that some results of Fanny and Badaoui (1998) are included in our relative equilibrium configurations when the gyrostatic momentum is null, and the same for our paper above mentioned when $k = 0$.

Open problems: 1) the existence and number of roots of the different equations or systems that appear in our problems, 2) the study of sufficient conditions for existence and stability of Eulerian and Lagrangian equilibria in these models, and 3) more general problems will be considered in next papers.

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