A POSTERIORI ERROR ESTIMATOR FOR FINITE VOLUME SCHEMES

Mohamed Amara, Lionel Nadau and David Trujillo

Abstract. In this paper, we are interested in the study of *a posteriori* estimators in linear parabolic case for the classical 5 points or 9 points Cells Centered Finite Volume method. Introducing the gradient of the unknown, the FV methods can be interpreted in terms of mixed finite element methods. And we use this interpretation of the scheme in order to construct an appropriate *a posteriori* estimator. The analysis of this estimator is done and we present numerical simulations in order to illustrate the theoritical results.

Keywords: A posteriori estimators, linear elliptic equation, linear parabolic equation, Finite Volume

AMS classification: 65G20, 65M60, 60M50, 35K05, 65M50, 65G20

§1. Introduction

The Finite Volume (FV) methods are often used to solve conservative equations like those encountered in fluid flows. These methods permit to keep locally the conservative aspect of these equations since we integrate the latters on a small domain which is called control volume. Furthermore, the dynamical adaptive mesh technics become an important tool in the developpement of simulators. They permit to improve the quality of the solution and decrease the computational time. *A posteriori* estimator is an efficient tool to perform these techniques. We can distinguish three kinds of *a posteriori* errors: residual estimation, estimation by solving local problems and hierarchical estimation.
In this work, we only study the residual estimator class for a linear parabolic equation with a Cells Centered Finite Volume method.
In the literature, we can find three kinds of results in the case of parabolic equations. First, we have estimators developped for semi-discrete time equations (see [8, 10]). On the other hand, space estimators class are obtained in a fixed time t_n for a linear parabolic equation with a FEM in one, two or three dimensions [5, 3]. At last, two approaches are investigated to establish a time and space estimator. The first one is to find an estimator in time and space simultaneously (see [6, 7, 12]), the secund one shares the study in two part: time in one hand, space on the other hand. All the works presented above used FEM or Galerkin method. Bergam *et al.* [2] established similar results in linear and nonlinear case with a Finite Element Finite Volume method.
This work take as a starting point the techniques developped in Bergam *et al.* [2]. The space discretisation is classical 5 points or 9 points Cells Centered Volume method and an implicit

Euler's scheme is used in the time discretisation. The reader could refer to [1] for the elliptic case details and the proofs encountered here.
First we briefly recall the Finite Volume 5 or 9 points, then we present the *a posteriori* errors results and we finished this paper with numerical results which prove the efficiency of the constructed estimator.

§2. Five or nine points Finite Volume scheme

For simplicity, one presents the method on the Laplace's problem and on a rectangular mesh. (But the method can be extended for a general elliptic operator and on unstructured meshes).
Let be $\Omega \subset \mathbb{R}^2$. In order to simplify the presentation we will suppose here that Ω is a rectangular domain.
We study the following problem, where f is a function of $L^2(\Omega)$:

$$\begin{cases} \text{Find } u \text{ solution of :} \\ -\Delta u = f \quad \text{ in } \Omega, \\ u = 0 \quad \text{ on } \Gamma. \end{cases} \tag{1}$$

2.1. Heuristic derivative of $5, 9$ points cell centered scheme

Let $\mathcal{P}_h$ be a Ω rectangulation regular family (the closed rectangle are called Volume Control) in the sense that:

- for each h, $\bar{\Omega}$ is the union of all elements of $\mathcal{P}_h$,
- for each h, the intersection of two different elements of $\mathcal{P}_h$, if it is not empty, is a corner or a whole edge of both of them.

We also introduce the nodes which are the centers of these volumes.
Now, we explain the Finite Volume method.
Integrating the relation $div\,\nabla u = f$ over the volumes and using Green's formula, we obtain

$$\forall\, V \in \mathcal{P}_h \qquad \int_{\partial V} \nabla u.\mathbf{n}\, d\sigma = -\int_V f\, dx.$$

Different Finite Volume schemes can be obtained by various flux interpolations. Let be $(\nabla u.\mathbf{n})_{\gamma_i}$ $i = 1, 2, 3, 4$ the flux through out the edge γ_i and $\mathbf{n}$ the outward unit normal . If we suppose the flux to be constant on each edge of volume V we obtain the following formula, in regular mesh as considered here:

$$(\nabla u.\mathbf{n})_{i+\frac{1}{2},j} = \frac{u_{i+1,j} - u_{i,j}}{dx_i},$$

where dx_i is the length between two centers of volumes and $\gamma_i = i + \frac{1}{2}, j$ is the interface in the middle of the segment $[u_{i,j}, u_{i+1,j}]$. The same formula is applied to the other three edges. We therefore obtain the well-known scheme named the 5 points Finite Volume scheme.
In the same manner, the 9 points scheme is held by the coming estimated formula of the flux:

$$(\nabla u.\mathbf{n})_{i+\frac{1}{2},j} = \frac{1}{dx_i}\left(\frac{3}{4}u_{i+1,j} + \frac{1}{8}u_{i+1,j+1} + \frac{1}{8}u_{i+1,j-1} - (\frac{3}{4}u_{i,j} + \frac{1}{8}u_{i,j+1} + \frac{1}{8}u_{i,j-1})\right).$$

In the next paragraph, we give a mathematical formulation for these Finite Volume schemes using a Finite Element (MFE).

2.2. Interpretation of the schemes in term of a MFE method

Considering the mesh $\mathcal{P}_h$ introduced in the latter paragraph, we construct a dual mesh named $\mathcal{U}_h$ by taking the medians of each rectangle and the center of each rectangle for the vertices of this dual mesh. The nodes of mesh $\mathcal{U}_h$ are defined as follows:

- The centers of $\mathcal{P}_h$ volumes,
- On the boundary of Ω: the centers of the faces (or edges) of the $\mathcal{P}_h$ volumes and in addition the Ω conners.

We will call V a cell $\mathcal{P}_h$ and $V^\star$ an element of $\mathcal{U}_h$.
We introduce the discretisation spaces:

$$W_h = \{p_h \in H(div, \Omega); \forall\, V \in \mathcal{P}_h, p_h|_V \in P_{1,0}(V) \times P_{0,1}(V)\}.$$

$$M_h = \{u_h \in H_0^1(\Omega); \forall\, V^\star \in \mathcal{U}_h, u_h|_{V^\star} \in Q_1(V^\star)\},$$

We introduce $p_h \in W_h$ an approximation of ∇u and we enforce the next integral relation:

$$\forall\, V \in \mathcal{P}_h,\ u_h \in M_h \int_{\partial V} p_h.\mathbf{n}\, d\sigma = \int_{\partial V} \nabla u_h.\mathbf{n}\, d\sigma.$$

As $p_h \in P_{1,0}(V) \times P_{0,1}(V)$, p_h is entirely determinated by is value on the edges of the volume V. Let be $(p_h.\mathbf{n})_{\gamma_i}$ $i = 1, 2, 3, 4$ the flux through out of the edge γ_i and $\mathbf{n}$ the outward unit normal.
If we used a 5 points Finite Volume, we have the next formula:

$$(p_h.\mathbf{n})_{i+\frac{1}{2},j} = \frac{u_{i+1,j} - u_{i,j}}{dx_i},$$

where dx_i is the length between two centers of volumes and $\gamma_i = i + \frac{1}{2}, j$ is the interface in the middle of the segment $[u_{i,j}, u_{i+1,j}]$. In the same idea, the 9 points scheme can be obtained.

Remark 1. We note p_h is different from ∇u_h.

In the next paragraph, we done *a posteriori* estimators for residual estimation method in the case of the heat equation. This estimator is based on the difference: $p_h - \nabla u_h$.

§3. The linear heat equation

3.1. Formulation of the linear heat equation

Let be $\Omega \in \mathbb{R}^2$ a rectangle domain and T a fixed nonnegative real. We consider the time interval $[0,T]$. The study problem writes

$$\left\{ \begin{array}{l} \text{Find } u \text{ such that :} \\ \dfrac{\partial u}{\partial t} - \Delta u = f \quad \text{in }]0,T[\times\Omega, \\ u = 0 \quad \text{on }]0,T[\times\Gamma, \\ u(0,.) = u_0(.) \quad \text{in } \Omega. \end{array} \right. \tag{2}$$

where $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in L^2(\Omega)$.
This problem admits a unique solution u such that $u \in \mathcal{C}^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ (*cf.* Lions et Magenes ([9],*chapter 3 §4*)).

3.2. Time and space discrete problem

We use an implicit Euler's scheme for the time discretisation and the 5 or 9 points Finite Volume scheme for the space discretisation. We seek two stair functions with respect of time: $u_h \in L^2(0,T;M_h)$ and $p_h \in L^2(0,T;W_h)$ such that u_h^n and p_h^n are the restriction of u_h and p_h to I_n respectively. p_h is an approximation of $p = \nabla u$. The time and space discrete problem writes:
$\forall\, n,\ 1 \leq n \leq N$, $\forall\, V \in \mathcal{P}_h,$

$$\left\{ \begin{array}{l} \dfrac{1}{mes\, V} \displaystyle\int_V (u_h^n - u_h^{n-1})\, dV - \tau_n\, div\, p_h^n = \tau_n\, f_h^n, \\ u_h^0 = u_0 \quad \text{in } V. \end{array} \right. \tag{3}$$

where $f_h^n|_V = \dfrac{1}{mes\, V} \displaystyle\int_V f^n\, dx$ and τ_n the length $t_n - t_{n-1}$ such that $\displaystyle\sum_{n=1}^{N} \tau_n = T$.

3.3. Notations

For all step function $v \in L^2(0,T;H_0^1(\Omega))$, constant in time on each subinterval I_n, we define the linear interpolation function in time v_τ de $\mathcal{C}^0([0,T],H_0^1(\Omega))$ such that:

$$v_\tau(t) = \frac{t - t_{n-1}}{\tau_n}\, v^n + \frac{t_n - t}{\tau_n}\, v^{n-1},$$

where v^n is the restriction of v onto I_n $1 \leq n \leq N$. Similar definitions are applied to $p_\tau,\ v_{h\tau},\ p_{h\tau}$
We define the norms to be used as
$\forall\, v \in \mathcal{C}^0(0,T;H_0^1(\Omega))$, $\forall\, p \in \mathcal{C}^0(0,T;(L^2(\Omega))^2)$, $\forall\, t \in [0,T]$,

$$|||(v,p)(t)|||^2 = ||v(t)||_{0,\Omega}^2 + \int_0^t \{|v(s)|_{1,\Omega}^2 + ||p(s)||_{0,\Omega}^2\}\, ds$$

In order to establish the *a posteriori* estimators, we share in two parts the quantity $|||((u,p)(t_n) -(u_{h\tau},p_{h\tau}))(t_n)|||$ by a triangle inequality technique. We also obtain a time error part, namely, $|||((u,p)-(u_\tau,p_\tau))(t_n)|||$ and a space error part $|||((u_\tau,p_\tau)-(u_{h\tau},p_{h\tau}))(t_n)|||$. We study separetaly this two quantities. Furthermore, the estimators issued from this quatities are said "optimal" estimators in the sense of Bernardi *et al.* [4] if they bound the exact norm error.

3.4. Time estimator η_n

For all n, $1 \leq n \leq N$, we put η_n equal to:

$$\eta_n = (\frac{\tau_n}{3})^{\frac{1}{2}}(|u_h^n - u_h^{n-1}|_{1,\Omega} + ||p_h^n - p_h^{n-1}||_{0,\Omega}).$$

We underline the results given below are independent of the space discretisation.
Let be π_τ an interpolation operator of continous functions from $\mathcal{C}^0([0,T])$ to $L^2([0,T])$. More precisely, for all function v belongs to $\mathcal{C}^0([0,T])$, $\pi_\tau v$ is a constant function and is equal to $v(t_n)$ on each interval $[t_{n-1},t_n],\ 1 \leq n \leq N$.
We have the following proposition:

Proposition 1. *Assume the function f belongs to $\mathcal{C}^0(0,T;L^2(\Omega))$ and the function u_0 belongs to $H_0^1(\Omega)$. There exists a positive constant c such that*

$$\begin{aligned}|||((u,p)-(u_\tau,p_\tau))(t_n)||| \leq c\Bigg((1+\sigma_\tau^{\frac{1}{2}})|||((u_\tau,p_\tau)-(u_{h\tau},p_{h\tau}))(t_n)||| \\ +(\sum_{m=1}^{n}\eta_m^2)^{\frac{1}{2}} + ||f-\pi_\tau f||_{L^2(0,t_n;L^2(\Omega))}\Bigg).\end{aligned}$$

where $\sigma_\tau = max_n \sigma_n$. For all n, $1 \leq n \leq N$, we have

$$\begin{aligned}\eta_n \leq\ & c\Big(||(u^n,p^n)-(u_h^n,p_h^n)|| + \sigma_\tau^{\frac{1}{2}}||(u^{n-1},p^{n-1})-(u_h^{n-1},p_h^{n-1})|| \\ & + ||\nabla(u-u_\tau)||_{L^2(t_{n-1},t_n;L^2(\Omega))} \\ & + ||\frac{\partial(u-u_\tau)}{\partial t}||_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \\ & + ||f-\pi_\tau f||_{L^2(t_{n-1},t_n;L^2(\Omega))}\Big).\end{aligned}$$

where $||(v^m,p^m)||^2 = ||v^m||_{0,\Omega}^2 + \tau_m|v^m|_{1,\Omega}^2 + \tau_m||p^m||_{0,\Omega}^2$

Proof. see [1] for details. □

3.5. A space estimator $\eta_{n,V}$

Let be $\eta_{n,V}$ the following estimator:

$$\eta_{n,V} = h_V||\frac{u_h^{n-1}-u_h^n}{\tau_n} + f_h^n + divp_h^n||_{0,V} + ||p_h^n - \nabla u_h^n||_{0,V},$$

We intend to prove that $\eta_{n,V}$ is an estimator for the error $|||((u_\tau,p_\tau)-(u_{h\tau},p_{h\tau})(t_n)|||$.

3.5.1. The upper bound of $\eta_{n,V}$

The upper bound of $\eta_{n,V}$ is given by the following proposition:

Proposition 2. *Let be f belongs to $\mathcal{C}^0(0,T;L^2(\Omega))$ and u_0 belongs to $H^1(\Omega)$ such that $u_h^0 \equiv u_0$, then, for all $t_n,\ 1 \le n \le N$, we have the next assessment:*

$$|||((u_\tau, p_\tau) - (u_{h\tau}, p_{h\tau}))(t_n)||| \le c \left(\sum_{m=1}^{n} \tau_m \sum_{V \in \mathcal{P}_h} (h_V^2 ||f^m - f_h^m||_{0,V}^2 + \eta_{m,V}^2) \right)^{\frac{1}{2}}.$$

Proof. In order to obtain the correct assessments, one uses the Clement's interpolation since we have non-continuous functions. See [1] for further details. □

3.5.2. The lower bound of $\eta_{n,V}$

The lower bound of $\eta_{n,V}$ is given by:

Proposition 3. *Assume f belongs to $\mathcal{C}^0(0,T;L^2(\Omega))$ and u_0 belongs to $H_0^1(\Omega)$. For all error estimator $\eta_{n,V}$, we have the local assessment:*

$$\eta_{n,V} \le c \Big(h_V \, || \frac{(u^n - u_h^n) - (u^{n-1} - u_h^{n-1})}{\tau_n} ||_{0,V} + ||p^n - p_h^n||_{0,V} \\ + |u^n - u_h^n|_{1,V} + h_V \, ||f^n - f_h^n||_{0,V} \Big).$$

Proof. The idea of the proof is due to Verfürth's technique [11] which consists of introducing a bubble function of the volume V. See [1] for more details. □

3.6. Numerical tests

In this paragraph, we present two numerical tests. These tests exemplify the local and global behaviours of the previous *a posteriori* error estimator. The first case is computed from a regular solution and the secund case is issued from a non regular source term. We consider the unit square domain $]0,1[\times]0,1[$ for all our numerical tests.

3.6.1. A regular solution

We take for this numerical test the following exact solution:

$$u(t,x,y) = sin(\frac{\pi}{2}\, t)\, sin(\pi x)\, sin(\pi y).$$

We compare here the global estimator $||p_h^n - \nabla u_h^n||_{0,\Omega}$ named estimator 1 with the global estimator $\eta_{n,V}$ named estimator 2. By this test, we want to look at the influence of the equation residual. In the same time, we neglect also the estimator η_n in the both case to measure its influency on the total estimator.
The figures (1(a)), (1(b)), (1(c)) present the time global behaviour of the both estimators and the global exact error norm $||(\cdot,\cdot)||_n$ for different size of meshes.

We remark that the two estimators feature are similar. The estimator 1 is suited better than the estimator 2 for the exact error norm in this numerical test (the same behaviour can be observed with the exact polynomial solution $u(t,x,y) = t^2\, x\,(1-x)\, y\,(1-y)$). From this first result, we can also deduce that the time estimator and the term $h^2||f^n - f_h^n||_{0,\Omega}$ could be neglected. On the other hand, an important aspect of the *a posteriori* estimator is its local behaviour. Therefore, we present the local error maps on the figures (1(d)), (1(e)), (1(f)), (1(g)), (1(h)), (1(i)). We note that the local behaviour of these estimators are quite similar at the end of the simulation but they are diffirent at the beginning.
Finally, this numerical test permits to show that the two estimators are very closed but it can not conclude about the efficiency of one of them. However, the estimator $||p_h^n - \nabla u_h^n||_{0,\Omega}$ is easier to compute than the estimator $\eta_{n,V}$ so, for the next test, we just compute the estimator 1 and analyse its behaviour in the case of a non regular solution.

3.6.2. A non regular solution

We compute a function f which support is an ellipse turning around a circle in the unit square. Let be $(x_c(t), y_c(t))$ the ellipse center, (a,b) the two diameters and t a parameter which varies in the interval $[0,1]$. The center $(x_c(t), y_c(t))$ moves around a circle $\mathcal{C}$ which the center is the point $(\frac{1}{2}, \frac{1}{2})$ and the radius is r. We denote $\mathcal{F}_t$ the following set:

$$\mathcal{F}_t \quad \forall (x,y) \in [0,1]\times[0,1],\ \left(\frac{x - x_c(t)}{a}\right)^2 + \left(\frac{y - y_c(t)}{b}\right)^2 \leq 1.$$

The function f is written

$$\forall (x,y) \in [0,1]\times[0,1], \quad f(t,x,y) = \begin{cases} 2 \text{ if } (x,y) \in \mathcal{F}_t, \\ 0 \text{ otherwise.} \end{cases}$$

On Figures (2), we show the local error map on each time step for the *a posteriori* estimator $||p_h^n - \nabla u_h^n||_{0,\Omega}$. We verify that the most important values of the estimator are located on the $\mathcal{F}_t$ set. This simulation shows that this estimator is well suited for an adaptive mesh refinement strategy .

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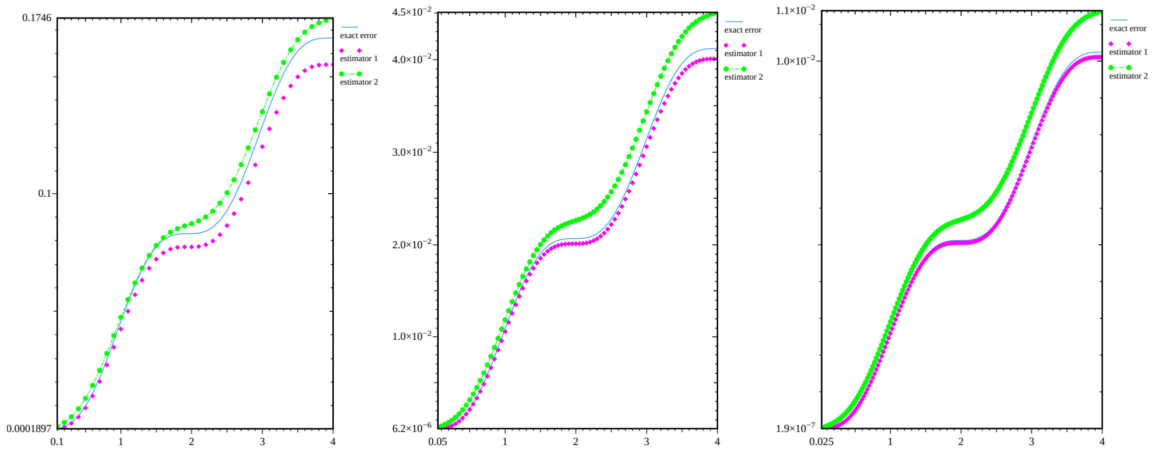
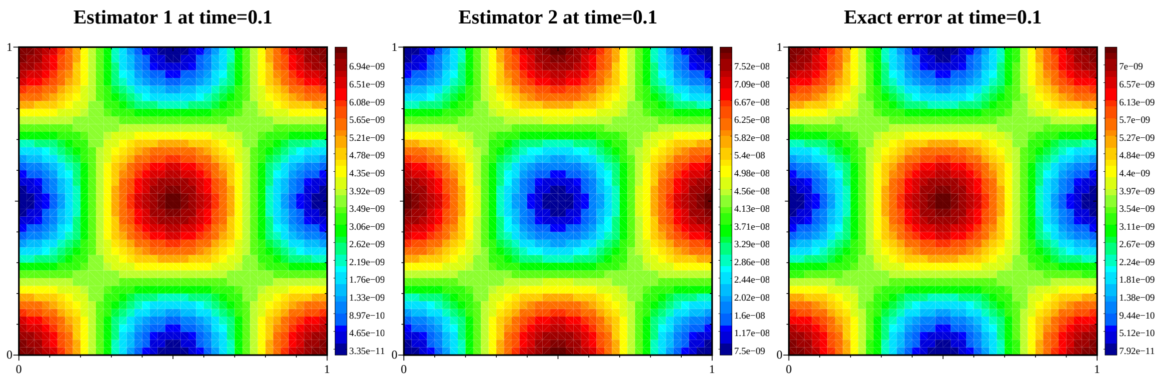
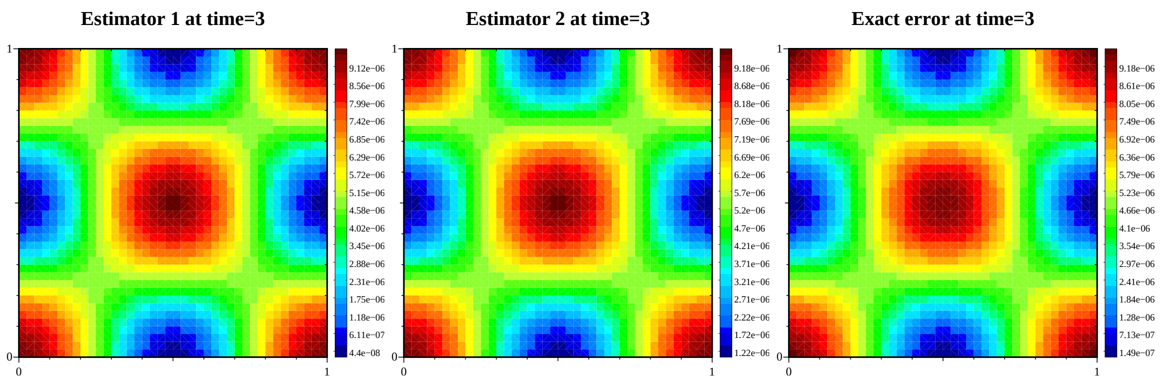


(a) size mesh 10×10 (b) size mesh 20×20 (c) size mesh 40×40



(d) size mesh 40×40 (e) size mesh 40×40 (f) size mesh 40×40



(g) size mesh 40×40 (h) size mesh 40×40 (i) size mesh 40×40

Figure 1: The global errors time evolution and local errors map for the both estimators and exact norm in the sinusoïd case.

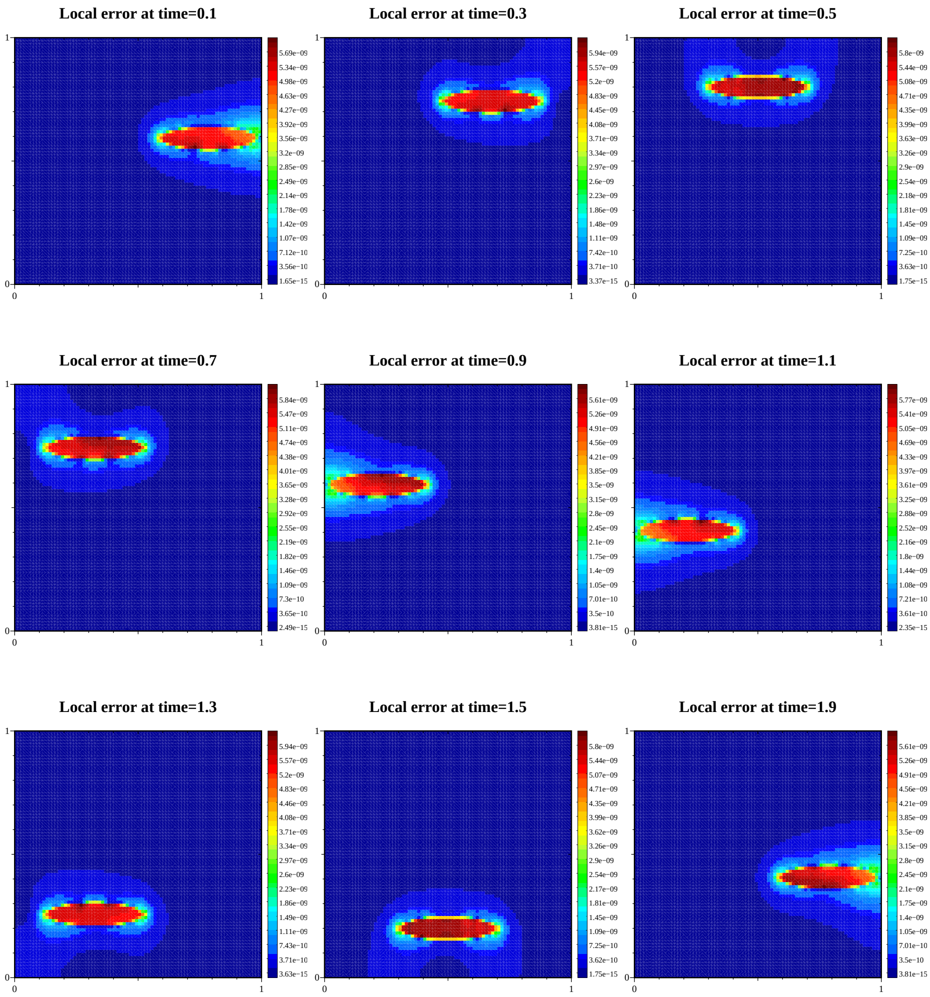


Figure 2: Local error map in the non regular solution case.

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