GENERALIZED POLARIZATION ALGEBRA

J. J. Gil, J. M. Correas, P. A. Melero, C. Ferreira

Abstract. A new model for the mathematical representation of polarimetric properties of light and optical systems is presented. The polarimetric states are characterized by means of complex correlation matrices "coherency matrices" which contain all the physical measurable information. The physical magnitudes arise as the coefficients of the expansion of the coherency matrix in a set of hermitic trace-orthogonal matrices constituted by the generators of the SU(n) group and the $n \times n$ identity matrix. The cases of light ($n = 2$, $n = 3$) and optical systems ($n = 4$) are analyzed on the basis of a unified model which is presented in a generic formulation that can be also applied to other n-dimensional phenomena. A global index of purity is defined as a non-dimensional measure of the statistical polarimetric mixture of pure physical states.

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§1. 2D Polarization Matrix

As usual in polarization optics, when a uniform light beam maintains fixed its propagation direction, the evolution of its two transversal field components provides the corresponding state of polarization. The polarization matrix $\mathbf{P}$ (or coherency matrix) [8, 1, 3] of a light beam contains all the measurable information about its state of polarization (including intensity). This Hermitian matrix is defined as

$$\mathbf{P} = \langle \varepsilon(t) \otimes \varepsilon^{+}(t) \rangle, \tag{1}$$

where

- ε is the Jones vector whose two components are the analytic signal of the wavefield.
- Brackets indicates time averaging: $\langle X \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t)dt$
- $\otimes$ stands for Kronecker product.
- ε^{+} is the transposed conjugated vector of ε.

A proper invariant and non-dimensional magnitude for describing the stability of the polarization ellipse is the "degree of polarization", which can be expressed as [4]

$$G_{(2)} = \left(\frac{2Tr(\mathbf{P})^2}{(Tr\mathbf{P})^2} - 1 \right)^{1/2} . \tag{2}$$

$\mathbf{P}$ can be expressed as a linear expansion, with real coefficients, on the basis constituted by the set composed of the three Pauli matrices and the identity matrix:

$$\mathbf{P} = \frac{1}{2} \sum_{i=0}^{3} s_i \sigma_i, \quad s_i = Tr(\mathbf{P}\sigma_i),\ i = 0,1,2,3 \tag{3}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{4}$$

The well known set of four real "Stokes parameters" s_i determines completely the state of polarization.

§2. 3D Polarization Matrix

Some authors have previously dealt with the general description of states of polarization where the three components of the wave field should be considered [7, 2] so that generalized polarization matrix (or "coherency matrix") is defined by

$$\mathbf{R} = \langle \varepsilon(t) \otimes \varepsilon^{+}(t) \rangle, \tag{5}$$

where ε represents the generalized Jones vector, constituted by the three analytic signals corresponding to the three wavefield components.

Nevertheless, for an appropriate treatment of this subject it is necessary to consider some recent works [6] where the set of Hermitian, trace-orthogonal matrices constituted by the adequately normalized Gell-Mann matrices and the identity matrix, is used as a basis for the expansion of $\mathbf{R}$.

$$\begin{gathered}
\mathbf{\Omega}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{\Omega}_1 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{\Omega}_2 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\mathbf{\Omega}_3 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{\Omega}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \mathbf{\Omega}_5 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\mathbf{\Omega}_6 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \mathbf{\Omega}_7 = \sqrt{\frac{3}{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \mathbf{\Omega}_8 = \sqrt{\frac{3}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{gathered} \tag{6}$$

Then,

$$\mathbf{R} = \frac{1}{3} \sum_{i,j=0}^{8} q_j \mathbf{\Omega}_i; \quad q_i = Tr(\mathbf{R}\mathbf{\Omega}_i),\ i = 0,1,\ldots,9, \tag{7}$$

where the nine coefficients q_i can be properly called the "generalized Stokes parameters" or the "3D Stokes parameters".

Now, the 3D "degree of polarization" or "degree of purity" [6] can be defined as

$$G_{(3)} = \left[\frac{1}{2}\left(\frac{3Tr(\mathbf{R}^2)}{(Tr\mathbf{R})^2} - 1\right)\right]^{1/2}. \tag{8}$$

This invariant, non-dimensional parameter is restricted to $0 \leq G_{(3)} \leq 1$, so that $G_{(3)} = 1$ corresponds to the case of $\mathbf{R}$ has only one non-null eigenvalue (total polarimetric purity and total correlation between the field variables), whereas for $G_{(3)} = 0$ the three eigenvalues of $\mathbf{R}$ are equal (equiprobable mixture of states of polarization, and null correlation between the field variables).

For the 2D case, $G_{(2)}$ appears as a relative difference between the two eigenvalues of $\mathbf{P}$

$$G_{(2)} = \frac{\lambda_1 - \lambda_2}{Tr\mathbf{P}}. \tag{9}$$

For the 3D case, the statistics that underlies the state of polarization is more complex so that several relative balances can be considered in order to describe "structure of purity" of the corresponding state of polarization. Then, in addition to $G_{(3)}$, which provides a global measurement of the polarimetric purity, two new "indices of purity" can be defined from the eigenvalues of $\mathbf{R}$.

$$P_1 = \frac{\lambda_1 - \lambda_2}{Tr\mathbf{R}}, \qquad P_2 = \frac{\lambda_1 + \lambda_2 - 2\lambda_3}{Tr\mathbf{R}}. \tag{10}$$

These indices are restricted to the following limits

$$0 \leq P_1 \leq P_2 \leq 1. \tag{11}$$

>From the above equations, the following quadratic relation between the global degree of purity $G_{(3)}$ and the two indices of purity P_1 and P_2 can be obtained

$$G_{(3)} = \frac{1}{2}\left(3P_1^2 + P_2^2\right)^{1/2}. \tag{12}$$

Then, the two indices of purity provide complete information about the polarimetric purity of the corresponding polarization state. This enhanced description based on invariant, non-dimensional parameters, has special physical significance.

The physically accessible region in the space P_1, P_2 is the following:

§3. 4D Coherency matrix associated with a Mueller matrix

The Stokes-Mueller algebra allows representing, in a general way, the transformation of the state of polarization of a light beam that interacts with an optical medium. The fundamental equation that relates the incident and emerging Stokes vectors with the Mueller matrix of the medium is the following

$$s' = \mathbf{M}s. \tag{13}$$

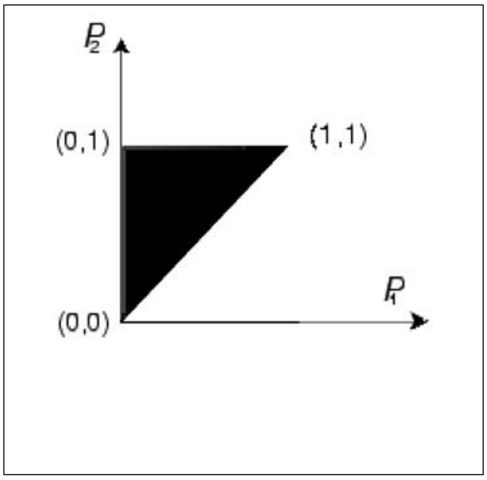

Figure 1: Region for the indices of purity

All the information about the polarimetric behavior of the medium is contained in its corresponding Mueller matrix $\mathbf{M}$, and it is possible to define a positive semi-definite Hermitian matrix $\mathbf{H}$ associated with $\mathbf{M}$ [4, 5], hence containing the same information

$$\mathbf{H} = \frac{1}{4} \sum_{k,l=0}^{3} m_{kl} \mathbf{E}_{kl}, \tag{14}$$

where m_{kl} are the Mueller matrix elements and $\mathbf{E}_{kl}$ are the 16 Dirac matrices defined as

$$\mathbf{E}_{kl} = \sigma_k \otimes \sigma_l, \qquad k, l = 0, 1, 2, 3, \tag{15}$$

so that m_{kl} can be expressed as

$$m_{kl} = Tr(\mathbf{E}_{kl} H). \tag{16}$$

It is clear that the relation between the "coherency matrix" $\mathbf{H}$ and the Mueller matrix $\mathbf{M}$ is analogous to the relation between the coherency matrix of the wave and its corresponding Stokes parameters (1D or 3D). In fact we can see that the relevant polarimetric magnitudes are given by the coherency matrices (positive-semidefinite Hermitian matrices):

- Polarization matrix (2D) $\mathbf{P}$. The coefficients of its expansion in the basis of trace-orthogonal Hermitian matrices σ_i, given by the generators of the group. SU(2) and the identity matrix, are the measurable magnitudes (2D Stokes parameters). The mentioned basis is constituted by the three Pauli matrices and the identity matrix.

- Polarization matrix (3D) $\mathbf{R}$. The coefficients of its expansion in the basis of trace-orthogonal Hermitian matrices $\mathbf{\Omega}_i$, given by the generators of the group. SU(3) and the identity matrix, are the measurable magnitudes (3D Stokes parameters). The mentioned basis is constituted by the eight Gell-Mann matrices and the identity matrix.

- Coherency matrix of the optical medium (4D) $\mathbf{H}$. The coefficients of its expansion in the basis of trace-orthogonal Hermitian matrices $\mathbf{E}_{kl}$, given by the generators of the group. SU(4) and the identity matrix, are the measurable magnitudes (Mueller matrix elements). The mentioned basis is constituted by the sixteen Dirac matrices including the identity matrix.

The inspection of the expressions of the eigenvalues of $\mathbf{H}$ obtained by algebraic computation shows that it is possible to write them in terms of three non-negative non-dimensional parameters

$$\lambda_0 = \frac{1}{4}Tr\mathbf{H}(1+2P_1+P_2),\ \lambda_1 = \frac{1}{4}Tr\mathbf{H}(1-2P_1+P_2),\ \lambda_2 = \frac{1}{4}Tr\mathbf{H}(1+2P_3-P_2),\ \lambda_3 = \frac{1}{4}Tr\mathbf{H}(1-2P_3-P_2), \tag{17}$$

where the new parameters $P_1,\ P_2,\ P_3$ are defined as

$$P_1 = \frac{\lambda_0-\lambda_1}{Tr\mathbf{H}}, \quad P_2 = \frac{(\lambda_0+\lambda_1)-(\lambda_2+\lambda_3)}{Tr\mathbf{H}}, \quad P_3 = \frac{\lambda_2-\lambda_3}{Tr\mathbf{H}}. \tag{18}$$

We see that these "Purity Indices" are expressed in a similar way to the degree of polarization of a light beam. In fact:

- P_1 is a non-dimensional measure of the differential weight between the two more significant pure components of the system,
- P_2 is a non-dimensional measure of the combined weight of the two more significant pure components relative to the combined weight of the two less significant pure components of the system, and
- P_3 is a non-dimensional measure of the differential weight between the two less significant pure components.

Now we can consider the restrictive relations between the different indices of purity and the limits of their values.

By applying the starting conditions for the eigenvalues, i. e. $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$, we find that the indices of purity are restricted by the following conditions

$$0 \leq P_1,\ 0 \leq P_3,\ P_1+P_3 \leq P_2,\ P_2+2P_3 \leq 1. \tag{19}$$

Figure 2 shows a geometrical synthesized view of the restrictions on the values of the purity indices.

Let us now consider the "Degree of Purity" $G_{(4)}$ defined as

$$G_{(4)} = \left[\frac{1}{3}\left(\frac{4Tr(\mathbf{H}^2)}{(Tr\mathbf{H})^2}\right)-1\right]^{1/2}. \tag{20}$$

This non-dimensional parameter is restricted to the interval $0 \leq G_{(4)} \leq 1$, where the minimum corresponds to an ideal total depolarizer and the maximum to a pure system (i. e. to a deterministic system which preserves the degree of polarization when the incident light is fully polarized).

The Degree of Purity is related with the three indices of purity by the following quadratic relation

$$G_{(4)} = \frac{1}{\sqrt{3}}\left(2P_1^2+P_2^2+2P_3^2\right)^{1/2}. \tag{21}$$

Pure systems are characterized by $G_{(4)} = P_1 = P_2 = 1,\ P_3 = 0$. On the other hand, equiprobable mixtures of four (or more) incoherent elements (i. e., total depolarizers whose Mueller matrix elements are zero except m_{00}) correspond to $G_{(4)} = P_1 = P_2 = P_3 = 0$.

The degree of purity provides a global measure of the purity of the system, whereas a detailed analysis requires consideration of the three indices of purity.

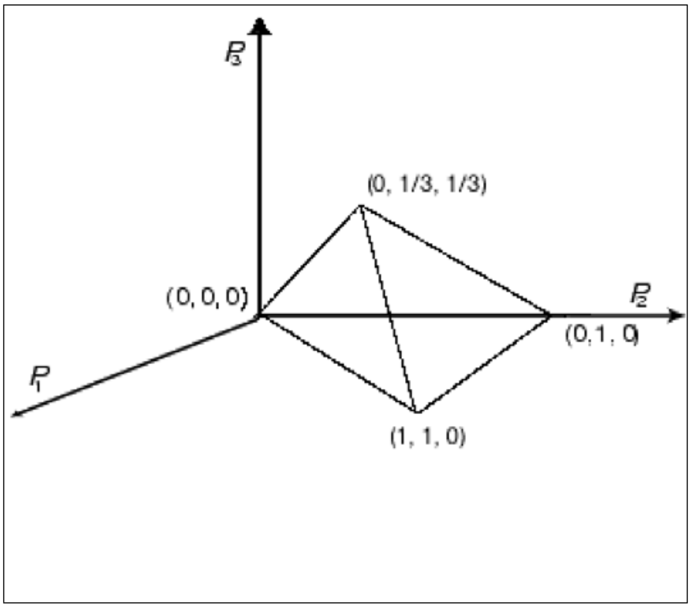

Figure 2: Region for the indices of purity.

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