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On a critical Hamiltonian system on $\mathbb{R}^N$.

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Abstract

In this paper, we elucidate how abstract concentration compactness established in [7], can be used in solving variational systems, by giving an application to a problem treated in [6] by concentration compactness of P. L. Lions.

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1 Introduction and main result

Via an abstract concentration compactness approach, we prove the existence of a weak solution of the problem:

$$(P): \begin{cases} -\Delta u = q|v|^{q-2}v \text{ in } \mathbb{R}^N \\ -\Delta v = p|u|^{p-2}u \text{ in } \mathbb{R}^N \\ \lim\limits_{|x|\longrightarrow\infty} u(x) = 0 \\ \lim\limits_{|x|\longrightarrow\infty} v(x) = 0 \end{cases}$$

where $N \geq 3$, and p, q are two real numbers satisfying $\frac{1}{p} + \frac{1}{q} = \frac{N-2}{N} = \frac{2}{2^*}$.
P.L.Lions has proved in [6], that the corresponding scalar equation $-\Delta((-\Delta u)^{1/q}) = u^p$, has radial ground states for all values of p and q, on the critical hyperbola.
In [3], authors have proved uniqueness of solution and its asymptotic behavior.
In this paper, we prove again, the existence result, by combining linking and abstract concentration compactness method.
Note that if $p = q$, then $p = q = 2^*$, $v = u$, and

$$(p) \Longleftrightarrow \begin{cases} -\Delta u = 2^*|u|^{2^*-1} \text{ in } \mathbb{R}^N \\ \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}$$

A weak solution of the problem (P), is a critical point of the functional J, defined by:

$$J(u,v) := \int_{\mathbb{R}^N} \nabla u.\nabla v dx - \int_{\mathbb{R}^N} [|u|^p + |v|^q]\, dx$$

We note by $\mathcal{H}$, the Banach space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ equipped with the norm

$$\|(u,v)\|_{\mathcal{H}} = \|u\|_{H^1(\mathbb{R}^N)} + \|v\|_{H^1(\mathbb{R}^N)}.$$

Theorem 1.1 *(P) has a weak solution in $\mathcal{H}$, for all positive real numbers p and q, satisfying $\frac{1}{p} + \frac{1}{q} = \frac{N-2}{N} = \frac{2}{2^*}$.*

2 Minimax theorem and Plais-Smale sequence

In this section, we establish the linking geometry of J, to give a Palais-Smale sequence by the minimax principle used in [9] and [1].

Definition 2.1 *Let S be a closed subset of a Banach space X, and Q a submanifold of X with relative boundary ∂Q.*
We say that S and ∂Q link if:

1. *$S \cap \partial Q = \emptyset$.*

2. *$\forall h \in C^0(X,X)$ such that $h_{|\partial Q} = id$, there holds $h(Q) \cap S \neq \emptyset$.*

Theorem 2.2 *Let $J : X \longrightarrow \mathbb{R}$ be a C^1 functional. Consider a closed subset $S \subset X$, and a submanifold $Q \subset X$ with relative boundary ∂Q. Suppose:*

1. *S and ∂Q link.*

2. *$\exists \delta > 0$ such that*

$$J(z) \geq \delta \; \forall z \in S,$$

$$J(z) \leq 0 \; \forall z \in \partial Q.$$

Let

$$\Gamma := \{h \in C^0(X,X) \;/\; h_{|\partial Q} = id\},$$

and

$$c := \inf_{h\in\Gamma} \sup_{z\in Q} J(h(z)) \geq \delta.$$

Then there exists a sequence $(z_k)_{k\in\mathbb{N}} \subset X$*, such that*

$$J(z_k) \xrightarrow[k\to\infty]{} c,$$

$$J'(z_k) \xrightarrow[k\to\infty]{} 0.$$

We choose numbers $\mu > 1,\ \nu > 1$, such that $\dfrac{1}{p} < \dfrac{\mu}{\mu+\nu}$, and $\dfrac{1}{q} < \dfrac{\nu}{\mu+\nu}$.
The following propositions give the linking geometry of J. Its proofs are similar to those in [1], and will be omitted.

Proposition 2.3 *There exist* $\rho > 0,\ \delta > 0$*, such that if we define*

$$S := \{(\rho^{\mu-1}u, \rho^{\nu-1}u)/\|(u,u)\| = \rho,\ u \in D(\nabla)\}$$

then $J(z) \geq \delta\ \forall z \in S$*.*

Proposition 2.4 *There exist* $\sigma > 0,\ M > 0$*, such that if we define*
$Q = \{\tau(\sigma^{\mu-1}u, \sigma^{\nu-1}u) + (\sigma^{\mu-1}v, -\sigma^{\nu-1}v)\ /0 \leq \tau \leq \sigma,\ 0 \leq \|(v,-v)\|_{\mathcal{H}} \leq M,$
and $u, v \in D(\nabla)\}$*,*
then $J(z) \leq 0\ \forall z \in \partial Q$*, where* ∂Q *is the boundary of* Q *relative to the subspace*

$$\{\tau(\sigma^{\mu-1}u, \sigma^{\nu-1}u) + (\sigma^{\mu-1}v, -\sigma^{\nu-1}v)/\ \tau \in \mathbb{R},\ v \in D(\nabla)\}$$

3 Abstract concentration compactness

In this section, we recall the abstract concentration compactness due to I.Schindler, and K.Tintarev [7], and we give a version adapted to our problem.
Let $\mathcal{E}$ be a separable reflexive Banach space, and let G be an infinite multiplicative group of bounded linear operator on $\mathcal{E}$.
Let $u,\ u_k \in \mathcal{E}$. We say that u_k converges to u weakly with concentration, and we note $u_k \xrightarrow{cw} u$, if $\forall \phi \in \mathcal{E}^*$

$$\lim_{k\to\infty} \sup_{g\in G} (g(u_k - u), \phi) = 0.$$

If G is a compact group, concentrated weak convergence is equivalent to weak convergence.

Definition 3.1 *Let* $\{\phi_k\}$ *be a normalized basis for* $\mathcal{E}^*$*. Then we define the norm*

$$\|u\|_G := \sup_{g\in G} \left(\sum_{k=1}^{\infty} \frac{|(gu, \phi_k)|^2}{2^k} \right)^{\frac{1}{2}}.$$

We suppose that G satisfies:

P1) $\sup_{g\in G} \|g\| < \infty$, where $\|g\| := \sup_{\|u\|=1} \|gu\|$.

P2) If $(g_k)_{k\in\mathbb{N}} \subset G$, and for all $u \in \mathcal{E}$ $g_k u \underset{k\to\infty}{\longrightarrow} g_0 u$, then $g_0 \in G$.

P3) If $g_k u \rightharpoonup w \neq 0$ for a $u \in \mathcal{E}$, then g_k has a strongly convergent subsequence.

P4) Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{E}$ be a bounded sequence, and let $\{w^{(n)}\} \subset \mathcal{E}$ and $\{g_k^{(n)}\} \subset G$, $k, n \in \mathbb{N}$, be such that the sequence

$$(g_k^{(n)} g_k^{(m)^{-1}})_{k\in\mathbb{N}} \text{ dissipates for } m \neq n$$

and

$$g_k^{(n)} u_k \rightharpoonup w^{(n)},\ n \in \mathbb{N},$$

then $\|w^{(n)}\|_G \underset{n\to\infty}{\longrightarrow} 0$.

Theorem 3.2 *Let $(u_k)_{k\in\mathbb{N}} \subset \mathcal{E}$ be a bounded sequence. Then there exist $(w^{(n)})_{n\in\mathbb{N}} \subset \mathcal{E}$, $(g_k^{(n)})_{n\in\mathbb{N}} \subset G,\ k \in \mathbb{N}$, such that for a renamed subsequence,*

$$g_k^{(n)^{-1}} g_k^{(m)} \rightharpoonup 0 \textit{ for } n \neq m,$$
$$u_k - \sum_{n\in\mathbb{N}} g_k^{(n)} w^{(n)} \overset{cw}{\longrightarrow} 0.$$

3.1 Concretisation of the abstract concentration compactness on $\mathcal{H} := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$

Let G be the infinite multiplicative group of bounded linear operators
defined on $\mathcal{H}$ by

$$\begin{aligned} g_{t,\alpha}(u,v) &= \left(t^{-\frac{N}{p}} u\left(\frac{.+\alpha}{t}\right), t^{-\frac{N}{q}} v\left(\frac{.+\alpha}{t}\right)\right) \\ &= \left(g^1_{t,\alpha} u, g^2_{t,\alpha} v\right) \end{aligned}$$

where $t \in \mathbb{R}$ and $\alpha \in \mathbb{R}^N$.
G satisfies the properties **P1)-P4)**. See [7] for a proof.
We have the invariance $J(g_{t,\alpha}(u,v)) = J(u,v)$.
The following theorem is a corollary of the Theorem 1.7 of [7].

Theorem 3.3 *Let $(z_k = (u_k, v_k))_k$ be a bounded sequence in $\mathcal{H}$. Then there exist $w^{(1)}, w^{(2)}, \ldots \in \mathcal{H}$, and $(\alpha_k^{(1)}, t_k^{(1)}), (\alpha_k^{(2)}, t_k^{(2)}), \ldots \in \mathbb{R}^N \times \mathbb{R}^+_*$, such that for $r \neq m$, either $t_k^{(r)}/t_k^{(m)} \to \infty$, or $t_k^{(r)}/t_k^{(m)} \to 0$, or $|\alpha_k^{(r)} - \alpha_k^{(m)}| \to \infty$; and $\forall r :\ t_k^{(r)} \to \infty$, or $t_k^{(r)} \to 0$; where*

$$w^{(n)} = w - \lim_{k\to\infty} g_{\frac{1}{t_k^{(n)}}, -\alpha_k^{(n)}} z_k$$

The series $\sum_n g_{t_k^{(n)}, \alpha_k^{(n)}} w^{(n)}$ converges absolutely in $\mathcal{H}$, and on a renamed subsequence:

$$z_k - \sum_n g_{t_k^{(n)}, \alpha_k^{(n)}} w^{(n)} \overset{cw}{\longrightarrow} 0.$$

Lemma 3.4 *Let $(z_k)_k$ be a bounded sequence in $\mathcal{H}$ such that $z_k \overset{cw}{\longrightarrow} 0$.*
Then modulo a subsequence, $\lim\limits_{k\to+\infty} \|z_k\|_{L^p(\mathbb{R}^N)\times L^q(\mathbb{R}^N)} = 0$, for all positive real numbers p and q, satisfying $\frac{1}{p}+\frac{1}{q}=\frac{N-2}{N}=\frac{2}{2^*}$

Proof of Lemma 3.4: Let $z_k := (u_k, v_k) \in \mathcal{H}$.
$\frac{1}{p}+\frac{1}{q}=\frac{2}{2^*} \Longrightarrow (\frac{N}{N-2} < p \leq 2^*$ and $q \geq 2^*)$ or $(p \geq 2^*$ and $\frac{N}{N-2} < q \leq 2^*)$.
Suppose that $\frac{N}{N-2} < p < 2^*$ and $q \geq 2^*$.
Note that $z_k \xrightarrow[k\to+\infty]{cw} 0 \Longrightarrow \forall g \in G: \; gz_k \rightharpoonup 0$.
Let $g = g_{t_k,0}$, where $t_k \xrightarrow[k\to+\infty]{} +\infty$ is chosen such that

$$\int_{|v_k|>t_k^{\frac{N}{q}}} |v_k|^q \xrightarrow[k\to+\infty]{} 0;$$

and let $w_k = g^2_{t_k,0} v_k = t_k^{-\frac{N}{q}} v_k(\frac{\cdot}{t_k})$, i.e $v_k(x) = t_k^{\frac{N}{q}} w_k(t_k x)$.

$$\begin{aligned}\int_{|v_k|<t_k^{\frac{N}{q}}} |v_k|^q dx &= \int_{|w_k|<1} |w_k|^q dx \\ &\leq \int_{\mathbb{R}^N} |w_k(x)|^2 dx \\ &= t_k^{-\frac{N(2+q)}{q}} \int_{\mathbb{R}^N} |v_k(x)|^2 dx \xrightarrow[k\to\infty]{} 0.\end{aligned}$$

Hence $\|v_k\|_{L^q(\mathbb{R}^N)} \xrightarrow[k\to\infty]{} 0$.
Let now $\{B(y,1),\; y \in Z \subset \mathbb{R}^N\}$ be a cover of $\mathbb{R}^N$, and $g = g_{1,-y}$.
By Rellich-Kondrachov inequality, we obtain:

$$\|u_k\|^p_{L^p(B(y,1))} \leq C\|u_k\|^2_{H^1_0(B(y,1))}\|u_k\|^{p-2}_{L^p(B(y,1))} \tag{3.1}$$

By summing inequalities (3.1) over $y \in Z$, we obtain:

$$\|u_k\|^p_{L^p(\mathbb{R}^N)} \leq C\|u_k\|^2_{H^1(\mathbb{R}^N)} \sup_{y\in Z} \|g^1_{1,-y}u_k\|^{p-2}_{L^p(B(0,1))}.$$

By the compactness of the imbedding of $H^1_0(B(0,1))$ into $L^p(B(0,1))$, it follows that modulo a subsequence, $g^1_{1,-y}u_k \xrightarrow[k\to\infty]{} 0$ in $L^p(B(0,1))$.
Hence, $\|u_k\|_{L^P(\mathbb{R}^N)} \xrightarrow[k\to\infty]{} 0$.

□

4 Proof of the main result

Lemma 4.1 *Let $(z_k = (u_k, v_k))_{k\in\mathbb{N}}$ be a sequence of $\mathcal{H}$ such that*

$$J(z_k) \xrightarrow[k\to\infty]{} c \text{ and } J'(z_k) \xrightarrow[k\to\infty]{} 0. \tag{4.1}$$

Then $(z_k)_{k\in\mathbb{N}}$ is bounded.

The proof is similar to that of Proposition 2.1 in [1].

Proof of Theorem 1.1. Let (z_k) be a sequence satisfying (4.1).
According to Theorem 3.3, there exist $w^{(1)}, w^{(2)}, \cdots \in \mathcal{H}$, and $(\alpha_k^{(1)}, t_k^{(1)})$, $(\alpha_k^{(2)}, t_k^{(2)}), \cdots \in \mathbb{R}^N \times \mathbb{R}_*^+$, such that

$$z_k - \sum_n g_k^{(n)} w^{(n)} \xrightarrow{cw} 0,$$

where $g_k^{(n)} = g_{t_k^{(n)}, \alpha_k^{(n)}}$.
(z_k) does not converge weakly with concentration to 0. In fact, if we suppose that $z_k \xrightarrow{cw} 0$ we will have by Lemma 3.4 $\lim\limits_{k\to\infty} \|z_k\|_{L^p(\mathbb{R}^N)\times L^q(\mathbb{R}^N)} = 0$ (modulo a subsequence), which shows that $J(z_k) \longrightarrow 0$. Contradiction. Then there exists a $w^{(n_0)} \neq 0$.
On the other hand, for some $g_k \in G$, we have

$$g_k z_k \rightharpoonup w^{(n_0)}.$$

Then

$$J'(g_k z_k) \rightharpoonup J'(w^{(n_0)})$$

However, $J'(z_k) \xrightarrow[k\to\infty]{} 0 \Longrightarrow J'(g_k z_k) \xrightarrow[k\to\infty]{} 0$. Then, $J'(w^{(n_0)}) = 0$.

□

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