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Determining stability regions in highly perturbed, non-linear dynamical systems using periodic orbits.

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Abstract

We rely on numerically determined periodic orbits to explore stability of motion over three dimensional space around highly perturbed nonlinear dynamical systems. It is known that families of three dimensional periodic orbits appear in the vicinity of planar, resonant, periodic orbits. Thus, by computing several of these "resonant" families and studying their stability properties as they evolve, we find that the stability type of the periodic orbits changes at certain critical inclinations. We use these stability transitions in order to determine regions in three dimensional space where orbital motion is stable.

Keywords: Periodic orbits, stability, celestial mechanics, asteroids

AMS Classification: 37C27, 70F15

1 Introduction

General perturbation theory is often used for the integration or simplification of non integrable dynamical systems. It is generally applicable for systems that are slightly perturbed from an integrable system but will often fail for highly perturbed systems. In these cases one of the remaining portals with which to understand the dynamics of the problem is through the determination of periodic orbits (which Poincaré noted [6]). Once a periodic orbit is computed, the stability of that orbit can also be computed, which sheds light on the character of phase space in the vicinity of the orbit. This situation occurs for orbital dynamics in close proximity to asteroids where, due to their distended shapes and rapid spin rates, motion is far from integrable. In terms of specific perturbations, we find that the ellipticity coefficient of the body often has the same order as the oblateness coefficient, meaning that the classical theories for motion close to spheroidal bodies cannot be applied.

Planar periodic orbits around rotating asteroids are relatively well understood, but the distribution and properties of three dimensional periodic orbits for this problem are not as well understood [7]. It is known that resonances of the mean motion of an orbiter with the rotation rate of the asteroid produce families that bifurcate into three dimensional orbits, periodic in the frame rotating with the asteroid. In general, these families evolve from planar direct orbits through 180 degrees of inclination to planar retrograde orbits [5, 3]. We compute several of these bifurcation families and their stability properties as they evolve and find that these resonant families change their stability type at certain critical inclinations. We use these stability transitions in order to determine regions around the asteroid (in 3-D space) where orbital motion is stable. Specifically, we try to relate the inclination of these critical orbits to the averaged semimajor axis of the periodic orbits.

2 Asteroid (433) Eros: The model

Let us consider the motion of a satellite referred to a synodic reference frame. The origin of the reference frame is at the center of mass of Eros, and the axes coincide with the principal axes of inertia. We consider the satellite as a mass point, and take up to the second order in the potential expansion. We also suppose that Eros rotates around the z-axis with constant velocity ω. Under these assumptions, the Lagrangian defining the motion is

$$\mathcal{L} = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \omega(x\dot{y} - y\dot{x}) + \Omega(x, y, z), \tag{1}$$

where Ω is the effective potential function

$$\Omega = \tfrac{1}{2}\omega^2(x^2 + y^2) - \mathcal{V}(x, y, z), \tag{2}$$

and the potential $\mathcal{V}$ is

$$\mathcal{V} = -\frac{\mu}{r}\left[1 + \left(\frac{\alpha}{r}\right)^2 \left\{3C_{2,2}\frac{x^2 - y^2}{r^2} - \tfrac{1}{2}C_{2,0}\left(1 - 3\frac{z^2}{r^2}\right)\right\}\right], \tag{3}$$

where μ is the gravitational constant, $r = \sqrt{x^2 + y^2 + z^2}$ is the radial distance of the satellite, α the equatorial radius and the harmonic coefficients are $C_{2,0} < 0 < C_{2,2}$ because Eros spins around its axis of greatest inertia. The numerical values we use are

$$\begin{aligned}
\alpha &= 16.5\,\mathrm{km} \\
\mu &= 4.463\,10^{-4}\,\mathrm{km}^3/\mathrm{s}^2 \\
\omega &= 3.31182\,10^{-4}\mathrm{s}^{-1} \\
C_{2,0} &= -0.1102314049586777 \\
C_{2,2} &= 0.05282644628099174
\end{aligned}$$

Using suitable units, both ω and μ can be set equal to 1. The equations of motion corresponding to the Lagrangian (1) are

$$\ddot{x} - 2\omega\,\dot{y} = \frac{\partial \Omega}{\partial x}, \qquad \ddot{y} + 2\omega\,\dot{x} = \frac{\partial \Omega}{\partial y}, \qquad \ddot{z} = \frac{\partial \Omega}{\partial z}. \tag{4}$$

Since the force function Ω does not show explicit dependence on time, System (4) admits the Jacobian integral

$$2\Omega - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = C, \tag{5}$$

where C is the so-called Jacobian constant.

3 Families of periodic orbits

Far away from the origin, the Keplerian approximation will provide an almost periodic solution of the perturbed problem. The use of differential corrections will improve the periodicity of the orbit until finding a true periodic orbit. Then, by means of tangent predictions followed by isoenergetic corrections we can continue the *natural* family of periodic orbits (see [2]) for either retrograde or direct motion. Both families are presented in Fig. 1, where a representation of the stability indices versus the Jacobian constant is provided.

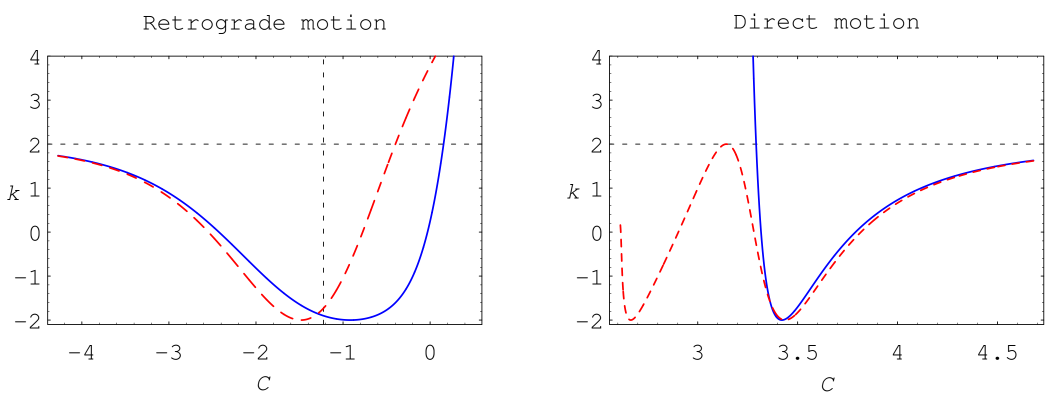

Figure 1: Stability indices $k_{\rm h}$ (full line) and $k_{\rm v}$ (dashed line) of the families of planar periodic orbits. Abscissas present the Jacobian constant. For retrograde motion, values of $C > -1.224$ (on the right of the vertical axis) correspond to collision orbits.

As it is known, the linear stability of a periodic orbit is determined from two stability indices $k_{1,2}$, where the condition k_i real and $|k_i| < 2$ $(i = 1, 2)$ applies for linear stability [1]. For planar solutions, one of the indices ($k_{\rm h}$) measures the in-plane stability, while the other ($k_{\rm v}$) shows the behavior of the orbit when suffering perturbations in the out-of-plane direction [4]. The critical value ± 2 for any of the stability indices means that, probably, a new family of periodic orbits bifurcate from the original one. When the index taking the value ±2 is k_v the bifurcating family will continue in the direction orthogonal to the plane; on the contrary, $|k_h| = 2$ means that the new family appears in the plane.

Planar retrograde orbits are almost circular, and retrograde motion is always stable for non collision orbits of our model (see Fig. 1). We find a critical point at $C \approx -1.484$ where $k_v = -2$ and a period doubling, vertical bifurcation can occur.

Direct motion is a different case. For values $x > 30.12$ km ($C > 3.29$) the orbits are stable. Passed that point the motion becomes highly unstable ($k_h > 2$) but remains planar ($|k_v| < 2$). At $C \approx 3.14$ we find a critical point ($k_v = 2$) where a vertical bifurcation can occur. Direct orbits cannot go above a minimum distance of 27.996 km on the x-axis (for $C \approx 3.07$). For decreasing values of the Jacobian constant the orbits become more and more eccentric, and parts of the motion are seen like retrograde in the rotating frame (see Fig. 2).

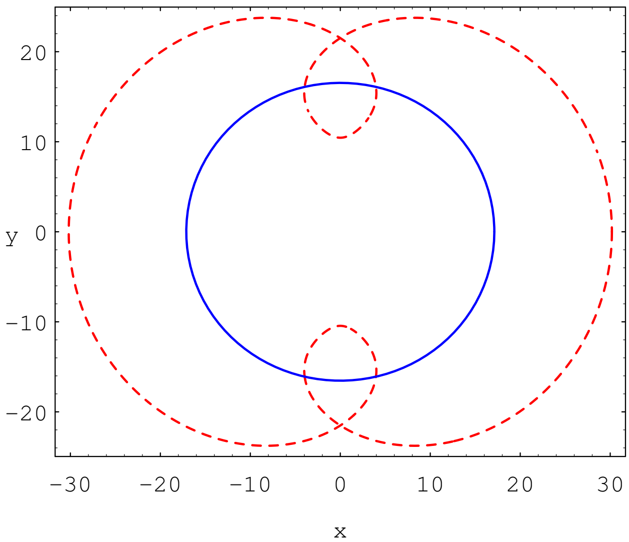

Figure 2: Periodic orbits in the rotating frame. One almost circular orbit with retrograde motion (full line), and one highly unstable, eccentric orbit with mixed direct and retrograde motion (dashed line).

3.1 Bifurcations produced by resonances in retrograde motion

Noting n the mean motion of the orbiter in the inertial frame, and $\tilde{n} = n - \omega$ the mean motion in the rotating frame, the resonance $-n/\omega = N/D$ in N revolutions of the orbiter in the inertial frame and D rotations of the asteroid, will occur when $-\tilde{n}/\omega = 1 + N/D$, or for a value

$$P_R = \frac{2\pi}{1 + N/D}, \tag{6}$$

of the period in the rotating frame $P_R = 2\pi/\tilde{n}$. The orbit will close in the inertial frame after a time $T = 2\pi D = (D + N)P_R$ producing the resonance. Therefore, the D:N-

resonant orbit is obtained by locating the orbit with period P_{R} given by Eq. (6) on the retrograde family. The corresponding initial conditions with multiple period $P = (D+N)P_{\mathrm{R}}$ are then used for the computation of the family for variations towards greater values of the Jacobian constant, until reaching the vertical bifurcation. The vertically bifurcated branch family can then be computed.

Depending on the distance to the asteroid, in general we find two different kinds of families that bifurcate out of the plane at different resonances. The first passes from retrograde to direct motion through the 180° of inclination, and the other ends at certain inclination.

Far away from the asteroid the orbital behavior is very similar to the perturbed Keplerian case (see [5]). Starting from a retrograde, planar, almost circular, periodic orbit the inclination decreases continuously as the Jacobian constant increases until ending on a planar orbit of the direct family. Figure 3 illustrates this behavior for the 1:3-resonance.

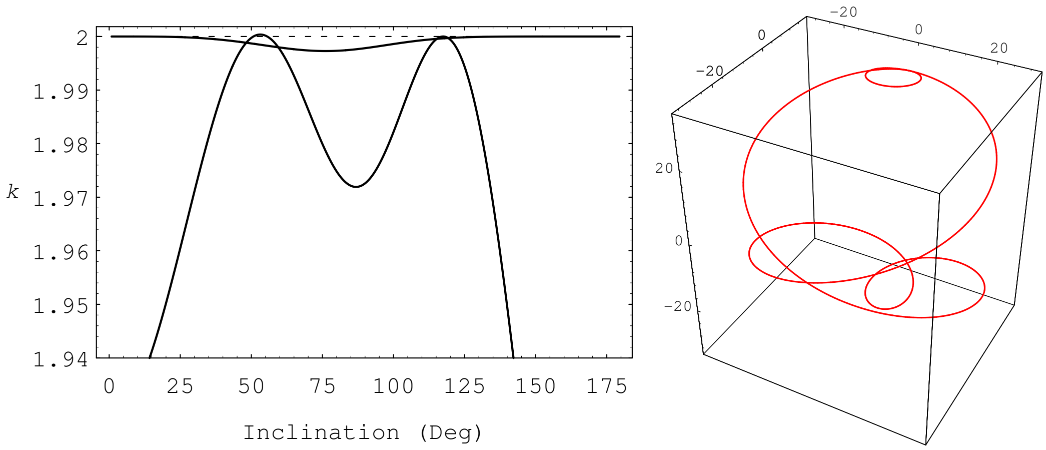

Figure 3: Left: Evolution of the stability of the 3D orbits of the family bifurcated at the 1:3-resonance. The orbits remain with linear stability for all inclinations. Right: A sample orbit of the family for $I \approx 90°$. Distances are km.

Due to the high perturbations of the dynamical system we are dealing with, the orbital behavior dramatically changes when the families of three-dimensional periodic orbits are in a close vicinity of Eros. Again, starting from an almost circular orbit of the retrograde family the inclination decreases continuously as the Jacobian constant increases. But at difference from the previous case, once that a certain value of the inclination has been reached, the almost circular orbits of the family change their stability to instability. Later, the instability grows very high for small decrements of the inclination making very difficult the continuation of the family. We illustrate this behavior in Fig. 4, where the family of three-dimensional periodic orbits that bifurcate from the 5:11-resonance of the retrograde family is plotted.

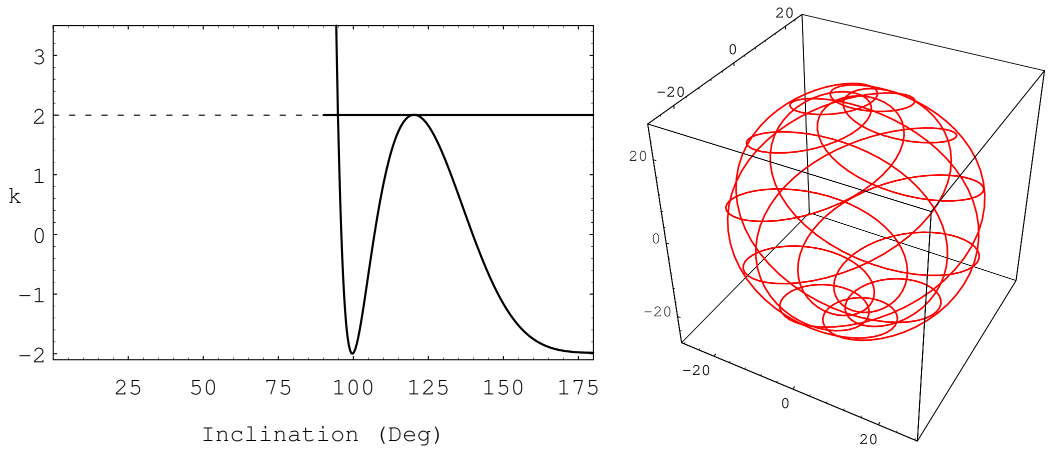

Figure 4: Left: Evolution of the stability of the 3D orbits of the family bifurcated at the 5:11-resonance. The orbits remain with linear stability for inclinations $I > 94.5°$. Right: A sample orbit. Distances are km.

4 Stability regions in three-dimensional motion

As we have seen above, we have available a systematic procedure for the computation of families of three-dimensional periodic orbits that bifurcate from the retrograde family as consequence of resonances in the inertial frame. The determination of the stability of the orbits of these families makes possible the determination of a collection of points, each of them representing the orbit (of a distinct family) at which the stability changes. We can project this points on a plane, relating orbital inclination to the corresponding resonance where the vertical bifurcation happens. By fitting a curve to this collection of points we determine a line separating stable regions in 3D space from unstable ones. In this way we constructed the Fig. 5.

Due to the highly perturbed system we are dealing with, it is difficult to find a simple formula representing the transition line. Therefore we decided not to consider resonances $n/\omega > 2/3$, that correspond to orbits very close to the asteroid. The fit we found is

$$I^\circ = 7235.03 - 11415.5\,\frac{\omega}{n} + 6065.23\left(\frac{\omega}{n}\right)^2 - 1068.39\left(\frac{\omega}{n}\right)^3$$

Note also in Fig. 5 that the 2:3-resonance —corresponding to the abscissa 1.5— apparently divides the plane in two regions, and for distances closer to the asteroid the behavior seems to be much less smooth and the region of stability is reduced to a small region of high retrograde inclinations.

5 Conclusions

Highly perturbed dynamical systems clearly apart from perturbed integrable systems. Therefore, usual perturbation theory does not apply. By computing families of three

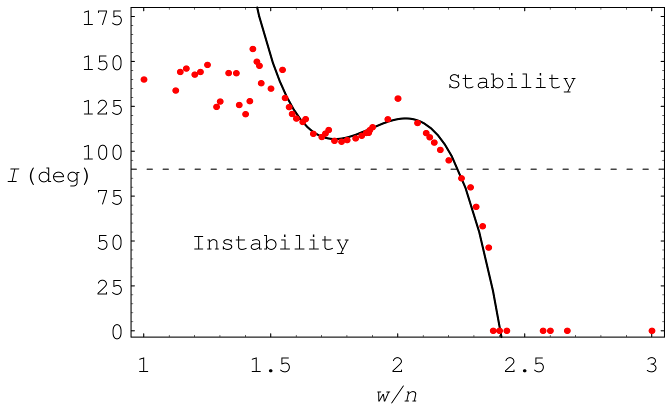

Figure 5: Regions of stability for 3D motion around Eros. Dots mark the inclination I at which the stability character of the periodic orbits change. Abscissas are the resonances ω/n where branch families bifurcate out of the plane. The curve was computed by a Least-squares fit.

dimensional periodic orbits and determining their stability, we can determine regions where three-dimensional motion is stable, thus providing some insight in the character of the phase space. We applied this procedure to the case of asteroid (433) Eros, and draw a line relating the inclination where the stability of the orbits change to the resonance that produces the vertical bifurcation, thus determining regions of stability for three dimensional motion. Additional work is in progress and will be soon reported.

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