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Stability of equilibria for 2-D resonant Hamiltonian systems: a geometrical approach

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Abstract

The stability of an equilibrium point of a 2-D Hamiltonian system, in the presence of resonances, is decided by means of a geometrical criterium, when the corresponding quadratic part is not sign defined. It is proven that this method is the geometrical counterpart of a theorem of Cabral and Meyer which constitutes an extension of the Arnold's theorem.

Keywords: Non linear stability, resonances.

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1 Introduction

Let us consider the problem of the stability of an equilibrium for a Hamiltonian system with two degrees of freedom. Let us assume that the corresponding linearized system is stable and that the eigenvalues associated to the linear part are of the form $\pm\omega_1 i$, $\pm\omega_2 i$ with ω_1, ω_2 positive real numbers, we suppose to be distinct. After a suitable linear transformation the Hamiltonian can be expressed as

$$\mathcal{H} = H_2 + F(q_1, q_2, p_1, p_2), \tag{1}$$

where

$$H_2 = \frac{1}{2}\omega_1(q_1^2 + p_1^2) \pm \frac{1}{2}\omega_2(q_2^2 + p_2^2).$$

There are two different cases to be considered. In the first one, H_2 is definite (the plus sign) and the stability follows from the theorem of Dirichlet [9]. In the second one, H_2 is indefinite (the minus sign) and, now, it is not possible to determine the stability from the linear part and higher orders must be considered.

Let us suppose that $m\omega_1 - n\omega_2 \neq 0$ for m and n integers satisfying $m + n \leq 2l$ and write the Hamiltonian $\mathcal{H}$ in Poincaré action-angle variables $(\Psi_1, \Psi_2, \psi_1, \psi_2)$ defined by

$$q_j = \sqrt{2\Psi_j}\cos\psi_j, \qquad p_j = \sqrt{2\Psi_j}\sin\psi_j, \quad j = 1, 2.$$

Assuming that $\mathcal{H}$ is in Birkhoff normal form up to order $2l$ we get

$$\mathcal{H} = H_2(\Psi_1, \Psi_2) + H_4(\Psi_1, \Psi_2) + \cdots H_{2l}(\Psi_1, \Psi_2) \tag{2}$$

where H_{2k}, $1 \leq k \leq l$, is a homogeneous polynomial of degree k in Ψ_i with real coefficients. In particular

$$H_2 = \omega_1 \Psi_1 - \omega_2 \Psi_2, \qquad 0 < \omega_1, \quad 0 < \omega_2$$

$$H_4 = \tfrac{1}{2}(A\Psi_1^2 - 2B\Psi_1\Psi_2 + C\Psi_2^2).$$

Let $D_{2k} = H_{2k}(\omega_2, \omega_1)$, Arnold's theorem [1] ensures the stability of the origin if for some $k = 1, 2, \ldots, l$ $D_{2k} \neq 0$.

The very first application of the theorem is due to Leontovich [6] and Deprit & Deprit-Bartolomé [3] to study the stability of the Lagrange's equilateral solutions of the restricted three-body problem for all the mass ratios except for the resonant cases 1:2 and 1:3. They are precisely the cases where the Arnold's theorem fails.

For the resonant cases, several results were established by Sokolski [10, 11] and Markeev [7] for resonances of order less than five. Recently, Cabral & Meyer [2] gave an extension of the Arnold's theorem that provides an stability criterium for both resonant and non resonant cases:

Theorem 1 *Let us consider the Hamiltonian (1) in normal form up to order s and ω_1 and ω_2 satisfying a resonance condition of order r ($n\omega_1 = m\omega_2$ and $m + n = r$) such that $r < s$ and*

$$\mathcal{H} = H_2(\Psi_1, \Psi_2) + H_4(\Psi_1, \Psi_2) + \cdots + H_{2l}(\Psi_1, \Psi_2) + H_s(\Psi_1, \Psi_2, n\psi_1 + m\psi_2), \tag{3}$$

where $s = 2l + 1$ or $s = 2l + 2$, H_s beeing a homogeneous polynomial of degree s in $\sqrt{\Psi_1}$ and $\sqrt{\Psi_2}$ with coefficients which are finite Fourier series in the single angle $n\psi_1 + m\psi_2$. Let us define

$$\Psi_(\psi) = H_s(\omega_2, \omega_1, \psi),$$

where $\psi = n\psi_1 + m\psi_2$ and $D_{2k} = H_{2k}(\omega_2, \omega_1)$. If for some $k = 2, \ldots, l$ we have that $D_{2k} \neq 0$, then Arnold's theorem guarantees the stability of the origin. Therefore, suppose that $D_{2k} = 0$ for $k = 2, \ldots, l$. Then, if $\Psi_(\psi) \neq 0$ for all ψ, the origin is stable and if $\Psi(\psi)$ has a simple zero, then the origin is unstable.

This result has a geometrical counterpart suggested by Elipe *et al.* [5]. The idea is very simple and it consists to look at the orbits around the origin in the phase space after the normalization procedure. Then, closed orbits around the equilibrium imply stability while asymptotic orbits imply instability. The key lies on a suitable representation of the normal form of the Hamiltonian in terms of an appropriate set of variables that underlie the topological structure of the phase space.

2 Normal forms

Definition 1 *Let be*

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \dots$$

the Hamiltonian function where

$$\mathcal{H}_2 = \frac{\omega_1}{2}(q_1^2 + p_1^2) - \frac{\omega_2}{2}(q_2^2 + p_2^2),$$
$$\mathcal{H}_s = \sum_{i+j+k+l=s} a_{ijkl} q_1^i q_2^j p_1^k p_2^l, \quad s > 2.$$

We say that $\mathcal{H}$ is in the Birkhoff normal form up to order N if $D\mathcal{H}_s = 0$ for $s \leq N$, where

$$D = \omega_1 \left(q_1 \frac{\partial -}{\partial p_1} - p_1 \frac{\partial -}{\partial q_1} \right) - \omega_2 \left(q_2 \frac{\partial -}{\partial p_2} - p_2 \frac{\partial -}{\partial q_2} \right).$$

That is, $\mathcal{H}$ is in normal form up to order N if, for each $s \leq N$, we get $\{\mathcal{H}_2; \mathcal{H}_s\} = 0$, where $\{-;-\}$ stands for the Poisson bracket.

The topology of the phase space after normalization depends on the structure of those monomials that belong to the normal form. In order to characterize them, and following Elipe [4], it is preferred to introduce canonical complex variables defined as

$$q_k = \frac{1}{\sqrt{2}}(u_k + iv_k), \qquad p_k = \frac{i}{\sqrt{2}}(u_k - iv_k), \quad k = 1, 2.$$

Now, $\mathcal{H}_s$ is an s degree homogeneous polynomial in the variables u_k, v_k and the operator D is given by

$$D = i\omega_1 \left(u_1 \frac{\partial -}{\partial u_1} - v_1 \frac{\partial -}{\partial v_1} \right) - i\omega_2 \left(u_2 \frac{\partial -}{\partial u_2} - v_2 \frac{\partial -}{\partial v_2} \right).$$

Thus, a monomial $u_1^{\alpha_1} u_2^{\alpha_2} v_1^{\beta_1} v_2^{\beta_2}$ is in normal form if it satisfies

$$(\omega_1, -\omega_2) \cdot (\alpha_1 - \beta_1, \alpha_2 - \beta_2) = 0.$$

The inner product above vanishes in the trivial case $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. So, a monomial that is a product of powers of $u_1 v_1$ and $u_2 v_2$ is in normal form. Note that $u_1 v_1$ and $u_2 v_2$ corresponds to $i\Psi_1$ and $i\Psi_2$, where Ψ_1 and Ψ_2 are the momenta in Poincaré variables.

If, in addition, ω_1 and ω_2 satisfy the resonant condition of order r, $n\omega_1 = m\omega_2$, $(\text{mcd}\,(m, n) = 1$ and $m + n = r)$, the monomials which are powers of $u_1^n u_2^m$ and $v_1^n v_2^m$ are also in the normal form. In this case, the normal form is not solely a function of the momenta in Poincaré variables, but also of the angles, although in the combination $n\psi_1 + m\psi_2$. Indeed, depending on the order of the resonance we have

$$u_1^n u_2^m = \Psi_1^{n/2} \Psi_2^{m/2} \left[\cos(n\psi_1 + m\psi_2) - i \sin(n\psi_1 + m\psi_2) \right],$$
$$v_1^n v_2^m = (-i)^{n+m} \Psi_1^{n/2} \Psi_2^{m/2} \left[\cos(n\psi_1 + m\psi_2) + i \sin(n\psi_1 + m\psi_2) \right].$$

Thus, angles appear at orders $s = k(m+n) + 2j$, $(k \geq 1,\ j \geq 0)$.

Taking into account the previous considerations, we can define, for a resonance of order r, the following four variables (invariants)

$$I_1 = u_1 v_1, \quad I_2 = u_2 v_2, \quad I_3 = u_1^n u_2^m, \quad I_4 = v_1^n v_2^m,$$

so that each term of the normal form can be expressed as a function of these four variables. In particular,

$$\mathcal{H}_s = \sum_{2(\alpha_1+\alpha_2)+l(\alpha_3+\alpha_4)=s} a_{\alpha_1\alpha_2\alpha_3\alpha_4} I_1^{\alpha_1} I_2^{\alpha_2} I_3^{\alpha_3} I_4^{\alpha_4}.$$

Note that the four invariants are not independent, but satisfy the relation

$$I_1^n I_2^m = I_3 I_4. \tag{4}$$

Besides, $i\omega_1 I_1 - i\omega_2 I_2$ is an integral provided that the Hamiltonian is in Birkhoff normal form.

3 Extended Lissajous variables

The discussion in the previous section suggests that the reduced phase space is generated by four variables not all independent, but satisfying the functional equation (4). Taking into account the formal integral H_2, the reduced phase space is regarded as a two dimensional surface for each $H_2 =$ cte. For this reason, it is convenient to introduce a new set of variables with a twofold objective. On the one hand, to introduce new real invariants and on the other hand, to serve as a parameterization of the reduced phase space. These are the so called *extended Lissajous variables* [4], specially useful to handle oscillators in resonance.

Under the assumption $n\,\omega_1 = m\,\omega_2$ we consider the transformation

$$q_1 = \sqrt{\frac{\Phi_1+\Phi_2}{m}} \sin m(\phi_1+\phi_2), \qquad P_1 = \sqrt{\frac{\Phi_1+\Phi_2}{m}} \cos m(\phi_1+\phi_2),$$
$$q_2 = \sqrt{\frac{\Phi_1-\Phi_2}{n}} \sin n(\phi_1-\phi_2), \qquad P_2 = \sqrt{\frac{\Phi_1-\Phi_2}{n}} \cos n(\phi_1-\phi_2).$$

In these coordinates, $\mathcal{H}_2$ expresses as $\omega\Phi_2$ ($\omega_1 = m\omega$, $\omega_2 = n\omega$) and they are related to the Poincaré variables through the formulae

$$2m\Psi_1 = \Phi_1 + \Phi_2, \qquad n\psi_1 + m\psi_2 = 2mn\phi_1,$$

$$2n\Psi_2 = \Phi_1 - \Phi_2, \qquad n\psi_1 - m\psi_2 = 2mn\phi_2.$$

It is worth to note that the Birkhoff normal form is a function of the momenta Φ_1, Φ_2 and the angle ϕ_1, as it follows from the expressions for the four invariants I_1, I_2, I_3 and

I_4. Provided that I_k are complex, another set of invariants, a combination of the previous one, is introduced. They are, in terms of the Lissajous variables (see [4] for details)

$$\begin{aligned} M_1 &= \tfrac{1}{2}\Phi_1, \\ M_2 &= \tfrac{1}{2}\Phi_2, \\ S_1 &= 2^{-(m+n)/2}(\Phi_1 - \Phi_2)^{m/2}(\Phi_1 + \Phi_2)^{n/2} \sin 2mn\phi_1, \\ C_1 &= 2^{-(m+n)/2}(\Phi_1 - \Phi_2)^{m/2}(\Phi_1 + \Phi_2)^{n/2} \cos 2mn\phi_1. \end{aligned} \tag{5}$$

Now, each term of the normal form is expressed as

$$\mathcal{H}_s = \sum_{2(\alpha_1+\alpha_2)+l(\alpha_3+\alpha_4)=s} a_{\alpha_1\alpha_2\alpha_3\alpha_4} M_1^{\alpha_1} M_2^{\alpha_2} C_1^{\alpha_3} S_1^{\alpha_4},$$

proving that the normal form is a function of these four invariants. Besides, they satisfy

$$C_1^2 + S_1^2 = (M_1 + M_2)^n (M_1 - M_2)^m \tag{6}$$

together with the constraint $M_1 \geq |M_2|$, being M_2 a constant.

Note that equation (6) defines the reduced phase space as a revolution surface for each constant value of M_2. In particular, the origin is the vertex of the surface corresponding to $M_2 = 0$. Also note that the extended Lissajous variables serve to parameterize the surface (6) and that ϕ_1 is equivalent to the angle ψ in the theorem of Cabral & Meyer.

4 The geometrical criterium

Let us consider the orbits in the variety where the origin lies ($M_2 = 0$), we claim that

- if the orbits around the origin are closed then the origin is stable.
- if there are asymptotic orbits to the origin then the origin is unstable.

Since the orbits are the level contour curves of the Hamiltonian function on the surface (6), it follows that

Theorem 2 *Let be $\mathcal{H}(C_1, S_1, M_1, M_2)$ in the normal form expressed in terms of the invariants and let be $\mathcal{H}_s$ the first term of the normal form that not vanishes for $M_2 = 0$. Let be $\mathcal{G}_1$ the surface defined by $H_s(C_1, S_1, M_1, 0) = 0$ and $\mathcal{G}_2$ the surface $C_1^2 + S_1^2 = M_1^{m+n}$, then:*

- *If $\mathcal{G}_1$ and $\mathcal{G}_2$ intersect transversaly, the origin is unstable.*
- *If $\mathcal{G}_1$ and $\mathcal{G}_2$ intersect only at the origin, the origin is stable.*

Proof:

It is enough to demonstrate that the theorem is equivalent to that of Cabral and Meyer. Indeed, let us suppose that the normal form for the Hamiltonian is that of equation (3). Express the normal form in terms of the invariants and take $M_2 = 0$. It is obvious that $D_{2k} = 0$ for $k = 2, \ldots, l$ is equivalent to $H_{2k}(M_2 = 0) = 0$. On the other hand, provided H_s is a homogeneous polynomial, we have for $M_2 = 0$

$$H_s = \alpha M_1^{s/2} \Psi(2mn\phi_1) = \alpha M_1^{s/2} \Psi(\psi),$$

where $\Psi(\psi)$ is the function defined in the theorem of Cabral and Meyer. So, if we consider the intersection of the surface defined by $H_s = 0$, that is $\mathcal{G}_1$, and $\mathcal{G}_2$ we observe that:

- If $\Psi(\psi)$ has a simple zero ψ^*, then $\mathcal{G}_2$ intersects the plane $\psi = \psi^*$ and two asymptotic orbits to the origin appear, so that the origin is unstable.

- If $\Psi(\psi) \neq 0$ for all ψ the only point in common of the surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ is the vertex and the rest of the orbits, that is, the intersection of the surfaces defined by $H_s = h$ with $\mathcal{G}_2$ are all closed and the origin is stable. ■

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