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Approximation of Explicit Surfaces by Fairness Bicubic Variational Splines

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Abstract

In this paper we present an approximation method of surfaces by a new type of splines, which we call *fairness bicubic splines*, from a given Lagrangian data set. An approximating problem of explicit surfaces is obtained by minimizing a quadratic functional in a parametric space of bicubic splines. The existence and uniqueness of this problem are shown as long as a convergence result of the method is established. We analyze some numerical and graphical examples in order to show the validity of our method.

Keywords: Smoothing; Variational surface; Fairness spline; Bicubic spline.

AMS Classifications: 65D07, 65D10, 65D17.

1 Introduction

In Geology and Structural Geology the reconstruction of a curve or surface from a scattered data set is a commonly encountered problem. The theory of D^m–splines over an open bounded set has been introduced at the first time by M. Attéia [1]. We have enriched this theory and extended it to the variational spline functions [6] where the early works are therein.

Several works have used the variational approach specifically minimizing some fairness functional (see for example [4], likewise this functional also can represent the flexion energy of a thin plate [3]) on a finite element space (see [5] and [7]) in order to simplify both characterisation and computation of the solution. So we have planned to solve in this work a variational approximation problem on a finite dimensional space that is not a finite element one. This is why we focus in this paper our interest to minimize a similar fairness functional on a space of bicubic spline functions of class C^2, meanwhile in [7] we discretize in a finite element space where in case that its functions are bicubics they turns out to be of class C^1. The resulting function is called a *fairness bicubic spline*.

Some fields of applications of this problem can appear in Earth sciences, specially in Geology and Geophysics, as long as CAD and CAGD etc ...

The remainder of this paper is organised as follows. In Section 2, we briefly recall some preliminary notations and results. Section 3 is devoted to state the approximation problem and to present a method to solve it. In Section 4, we compute the resulting function, while a convergence's Theorem is proved in Section 5. In Section 6 some numerical and graphical examples are given.

2 Notations and preliminaries

We denote by $\langle \cdot \rangle$ and $\langle \cdot , \cdot \rangle$, respectively, the Euclidean norm and inner product in $\mathbb{R}^r$, for $r \geq 2$. For two real intervals (a,b) and (c,d), with $a < b$ and $c < d$, we consider the rectangle $R = [a,b] \times [c,d]$ and, for any $s \in \mathbb{N}$, let $H^s(R)$ be the usual Sobolev space of (classes of) functions u belong to $L^2(R)$, together with all their partial derivatives $D^\beta u$ with $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2$, in the distribution sense, of order $|\beta| = \beta_1 + \beta_2 \leq s$. This space is equipped with the norm

$$\|u\|_s = \left(\sum_{|\beta| \leq s} \int_R D^\beta u(x)^2 dx \right)^{1/2},$$

the semi–norms

$$|u|_\ell = \left(\sum_{|\beta| = \ell} \int_R D^\beta u(x)^2 dx \right)^{1/2}, \ 0 \leq \ell \leq s,$$

and the corresponding inner semi–products

$$(u,v)_\ell = \sum_{|\beta| = \ell} \int_R D^\beta u(x) D^\beta v(x) dx, \ 0 \leq \ell \leq s.$$

Given $R \subset \mathbb{R}^2$ we will denote by $\mathbb{P}_r(R)$ the restriction to R of the linear space of real polynomials of degree $\leq r$.

Moreover, for any $n, \ m \in \mathbb{N}^*$ let $T_n = \{x_0, ..., x_n\}$, $T_m = \{y_0, ..., y_m\}$ be some subsets of distinct points of $[a,b]$ and $[c,d]$, with $a = x_0 \leq x_1 < \cdots < x_{n-1} \leq x_n = b$ and $c = y_0 \leq y_1 < \cdots < y_{m-1} \leq y_m = d$. We denote by $S_3(T_n)$ and $S_3(T_m)$ the spaces of cubic spline functions given by

$$S_3(T_n) = \left\{ s \in C^2[a,b] \mid s|_{[x_{i-1},x_i]} \in \mathbb{P}_3[x_{i-1}, x_i], \ i = 1, ..., n \right\} \text{ and}$$
$$S_3(T_m) = \left\{ s \in C^2[a,b] \mid s|_{[y_{j-1},y_j]} \in \mathbb{P}_3[y_{j-1}, y_j], \ j = 1, ..., m \right\}.$$

Let $\{\varphi_1, \ldots, \varphi_{n+3}\}$ and $\{\psi_1, \ldots, \psi_{m+3}\}$ be respectively the B-spline basis of $S_3(T_n)$ and $S_3(T_m)$. We consider the space $S_3(T_n, T_m)$ of bicubic spline functions given by

$$S_3(T_n, T_m) = \text{span}\{\varphi_i(x)\psi_j(y) \mid 1 \leq i \leq n+3, \ 1 \leq j \leq m+3\}.$$

Finally, we have that $S_3(T_n, T_m)$ is a Hilbert subspace of $H^3(R)$ equipped with the same norm, semi–norm and inner semi–product of such space and, moreover, that verifies

$$S_3(T_n, T_m) \subset H^3(R) \cap C^2(R). \tag{1}$$

3 Fairness bicubic spline

Let $\Upsilon_0 \subset \mathbb{R}^3$ be a explicit surface defined by a function f belonging to $C^2(R)$. For each $r \in \mathbb{N}^*$ let $A^r = \{a_1, \ldots, a_r\}$ be a subset of distinct points of R such that

$$\sup_{p \in R} \min_{i=1,\ldots,r} \langle p - a_i \rangle = \mathrm{o}\left(\frac{1}{r}\right), \ r \to +\infty. \tag{2}$$

Let L^r be the operator defined from $H^3(R)$ into $\mathbb{R}^r$ by $L^r v = (v(a_1), \ldots, v(a_r))^T$ and suppose that

$$\mathrm{Ker} L^r \cap \mathbb{P}_2(R) = \{0\}. \tag{3}$$

Now, we consider the following problem: Find an approximating explicit surface Υ of Υ_0 defined by a function σ of $S_3(T_n, T_m)$ that fits the data points $\{f(a_i), \ i = 1, \ldots, r\}$ and minimises all the semi-norms of order less than 3 in $S_3(T_n, T_m)$.

For any $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{R}^3$ with τ_1, τ_2 belonging to $\mathbb{R}_+$ and $\tau_3 > 0$, let J_τ^r be the functional defined on $H^3(R)$ by

$$J_\tau^r(v) = \langle L^r(v - f) \rangle^2 + \sum_{j=1}^{3} \tau_j \, |v|_j^2 \, .$$

REMARK 3.1 The first term of $J_\tau^r(v)$ indicates how well v approaches f in a discrete least discrete squares sense. The second term can represent some different conditions as for example: fairness conditions (see [4] and [5]), a classical smoothness measure, etc., while the parameter vector τ weights the importance given to each condition. □

Then, for any $r \geq 3$ we consider the following minimisation problem: Find $\sigma_\tau^{N,r}$ such that

$$\begin{cases} \sigma_\tau^{N,r} \in S_3(T_n, T_m), \\ \forall v \in S_3(T_n, T_m), \quad J_\tau^r(\sigma_\tau^{N,r}) \leq J_\tau^r(v). \end{cases} \tag{4}$$

Theorem 3.1 *The problem (4) has a unique solution, called the fairness bicubic spline in $S_3(T_n, T_m)$ relative to A^r, L^r and τ, which is also the unique solution of the following variational problem: Find $\sigma_\tau^{N,r}$ such that*

$$\begin{cases} \sigma_\tau^{N,r} \in S_3(T_n, T_m), \\ \forall v \in S_3(T_n, T_m), \quad \left\langle L^r \sigma_\tau^{N,r}, L^r v \right\rangle_r + \sum\limits_{j=1}^{3} \tau_j (\sigma_\tau^{N,r}, v)_j = \langle L^r f, L^r v \rangle_r \, . \end{cases}$$

4 Computation

Now well, we are going to show how to obtain in practice any *fairness bicubic spline* but we assume that we know the parameter values associated to given data points. Therefore, the set $\sigma_\tau^{N,r}(R)$, for $N, r \in \mathbb{N}^*$ and a given value of the parameter vector τ, provides a solution for Problem (4).

For any $n,\ m \in \mathbb{N}^*$ we consider $N = (n+3)(m+3)$ and $\{v_1, \dots, v_N\}$ a basis of the space $S_3(T_n, T_m)$. Thus, $\sigma_\tau^{N,r}$ can be written as $\sigma_\tau^{N,r} = \sum_{i=1}^{N} \alpha_i\, v_i$, with $\alpha_i \in \mathbb{R}$ unknown, for $i = 1, \dots, N$. Applying Theorem 3.1 we obtain a linear system of order N as follows

$$\sum_{i=1}^{N} \alpha_i \left(\langle L^r v_i, L^r v_j \rangle + \sum_{s=1}^{3} \tau_s (v_i, v_j)_s \right) = \langle L^r f, L^r v_j \rangle\, , \forall j = 1, \dots, N,$$

that is equivalent to

$$C\ \alpha = b \tag{5}$$

with $C = (c_{ij})_{1 \le i,j \le N}$, $\alpha = (\alpha_1, \dots, \alpha_N)^T$ and $b = (b_1, \dots, b_N)^T$, where for $i, j = 1, \dots, N$ one has

$$\begin{cases} c_{ij} & = & \langle L^r v_i, L^r v_j \rangle + \sum\limits_{s=1}^{3} \tau_s \left(v_i, v_j\right)_s , \\ b_j & = & \langle L^r f, L^r v_j \rangle\, . \end{cases}$$

Finally, we point out that the matrix C is symmetric, positive definite and of band type. In practice we use the following notations:

$$\begin{array}{rcl} A & = & (L^r v_i)_{1 \le i \le N}\, , \\ B_s & = & ((v_i, v_j)_s)_{1 \le i,j \le N}\, , \ s = 1, 2, 3, \end{array}$$

hence the system (5) is equivalent to

$$(A^T A + \tau_1 B_1 + \tau_2 B_2 + \tau_3 B_3)\alpha = A^T L^r f.$$

5 Convergence

Under adequate conditions, we are going to prove that the *fairness bicubic spline* $\sigma_\tau^{N,r}$ in $S_3(T_n, T_m)$ relative to A^r, $L^r f$ and τ, converges to f when N and r tend to $+\infty$.

Theorem 5.1 *Suppose that the hypotheses (1) and (2) hold and that*

$$\tau_3 = \mathrm{o}\left(r^2\right), \ r \to +\infty, \tag{6}$$

$$\forall\ i = 1, 2,\ \tau_i = \mathrm{o}(\tau_3),\ r \to +\infty, \tag{7}$$

and

$$\frac{r\,h^8}{\tau_3} = \mathrm{o}(1),\;\; r \to +\infty. \tag{8}$$

Then, one has

$$\lim_{N,r\to+\infty} \left\| f - \sigma_\tau^{N,r} \right\| = 0.$$

PROOF The scheme of the proof is the following.

Step 1. We obtain that

$$\exists C > 0,\, \exists \lambda > 0,\, \forall\, r \geq \lambda,\, \|\sigma_\tau^{N,r}\|_3 \,\leq\, C,$$

which means that the family $\left(\sigma_\tau^{N,r}\right)_{r,N\in\mathbb{N}^*}$ is bounded in $S_3(T_n, T_m)$. It follows that there exists one sub-sequence $\left(\sigma_{\tau_l}^{N_l,r_l}\right)_{l\in\mathbb{N}}$, with $\tau_l = \tau(r_l)$, $\lim\limits_{l\to+\infty} r_l = +\infty$, $\lim\limits_{l\to+\infty} N_l = +\infty$ and one element $f^* \in H^3(R)$ such that

$$\sigma_{\tau_l}^{N_l,r_l} \text{ converges weakly to } f^* \text{ in } H^3(R). \tag{9}$$

Step 2. Let us now prove that $f^* = f$. We suppose that $f^* \neq f$. From the continuous injection of $H^3(R)$ into $C(R)$ it follows that there exists $\theta > 0$ and an open rectangle R_0 of R such that

$$\forall p \in R_0,\, |f^*(p) - f(p)| \,>\, \theta.$$

As such injection is also compact then from (9) we obtain

$$\exists l_0 \in \mathbb{N},\, \forall l \geq l_0,\, \forall p \in R_0,\, |\sigma_{\tau_l}^{N_l,r_l}(p) - f^*(p)| \,\leq\, \frac{\theta}{2}.$$

Hence, for all $l \geq l_0$ and all $p \in R_0$ we have

$$|\sigma_{\tau_l}^{N_l,r_l}(p) - f(p)| \,\geq\, |f^*(p) - f(p)| - |\sigma_{\tau_l}^{N_l,r_l}(p) - f^*(p)| \,>\, \frac{\theta}{2}. \tag{10}$$

Now well, for l sufficiently great and using (2) we deduce that there exists a point $a^{r_l} \in A^r \cap R_0$ such that

$$|\sigma_{\tau_l}^{N_l,r_l}(a^{r_l}) - f(a^{r_l})| = \mathrm{o}(1),\, l \to +\infty,$$

which is a contradiction with (10). Consequently $f^* = f$.

Step 3. As $H^3(R)$ is compactly injected in $H^2(R)$, using (9) and taking into account that $f^* = f$ we have $f = \lim\limits_{l\to+\infty} \sigma_{\tau_l}^{N_l,r_l}$ in $H^2(R)$. Then

$$\lim_{l\to+\infty} ((\sigma_{\tau_l}^{N_l,r_l}, f))_2 = \|f\|_2^2\,. \tag{11}$$

Using again (9) and that $f^* = f$ we obtain

$$\lim_{l\to+\infty} (\sigma_{\tau_l}^{N_l,r_l}, f)_3 = \lim_{l\to+\infty} \left(((\sigma_{\tau_l}^{N_l,r_l}, f))_3 - ((\sigma_{\tau_l}^{N_l,r_l}, f))_2 \right) = |f|_3^2\,. \tag{12}$$

Moreover, for all $l \in \mathbb{N}$ we have

$$\left|\sigma_{\tau_l}^{N_l,r_l} - f\right|_3^2 = \left|\sigma_{\tau_l}^{N_l,r_l} - f\right|_3^2 + |f|_3^2 - 2(\sigma_{\tau_l}^{N_l,r_l}, f)_3$$

we deduce from (12) and (11) that $\lim\limits_{l\to+\infty} \left\|\sigma_{\tau_l}^{N_l,r_l} - f\right\|_3 = 0$.

Step 4. Finally, reasoning by contradiction we prove that the result is true. □

6 Numerical and graphical examples

We consider the explicit surface Υ_0 defined in the rectangle $R = [0,1] \times [0,1]$ by the following function

$$f(x,y) = \cos[6\pi\left((x-0.5)^2 + (y-0.5)^2\right)]\left(1 + (x-0.5)^2 + (y-0.5)^2\right).$$

The graph of this function appears in Figure 1.

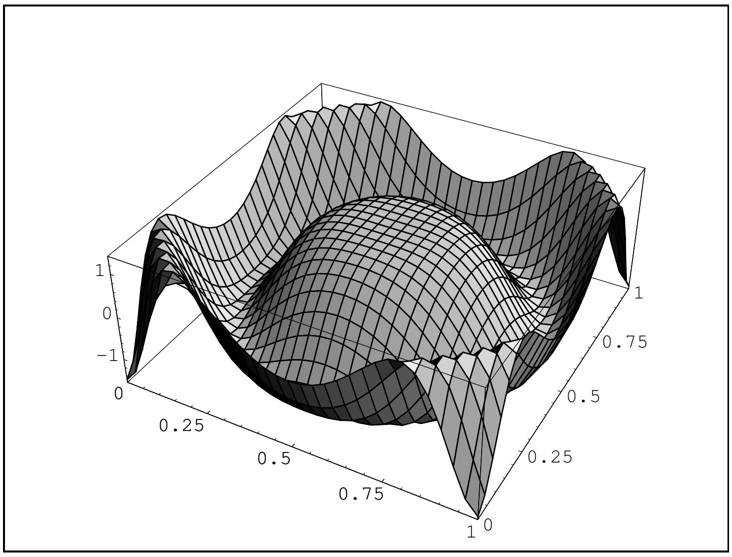

Figure 1: Original surface Υ_1

By using our smoothness method we have computed an approximating surface Υ of Υ_0 defined by a *fairness bicubic spline* $\sigma_\tau^{N,r}$ from some scattered data points. The space of bicubic spline functions of class C^2 has been constructed by one partition of 7×7 equal rectangles which means that we have taken $n = m = 7$ so $\dim S_3(T_n, T_m) = 100$ that is the order of the linear system given in (5).

Likewise, for any $\tau \in \mathbb{R}^3_+$, $\tau_3 > 0$, we have computed the following estimation of the relative error E_r given by

$$E_r = \left(\frac{\sum_{i=1}^{10000} |\sigma_\tau^{N,r}(a_i) - f(a_i)|^2}{\sum_{i=1}^{10000} |f(a_i)|^2} \right)^{1/2}$$

where $a_1, \ldots, a_{10000}$ are random points in R.

Now, we are going to show graphically the importance of the parameter vector τ. Figures 2 and 3 show the weight of the parameter vector τ in the approximating surface in agreement with the interpretation given in Remark 3.1 and so the effectiveness of this approximation method.

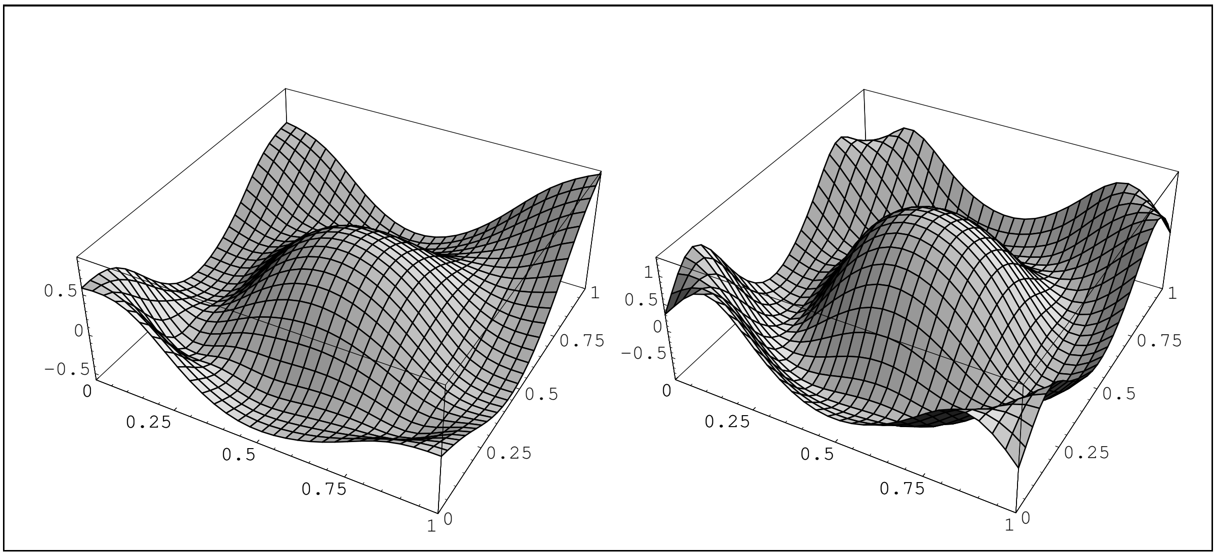

Figure 2: Two approximating surfaces parameterised by the *Fairness bicubic spline* $\sigma_\tau^{N,r}$ for respectively $r = 450$, $\tau = \{10^{-1}, 10^{-1}, 10^{-3}\}$, $E_r = 0.542214$ and $r = 450$, $\tau = \{10^{-2}, 10^{-2}, 10^{-4}\}$, $E_r = 0.300362$.

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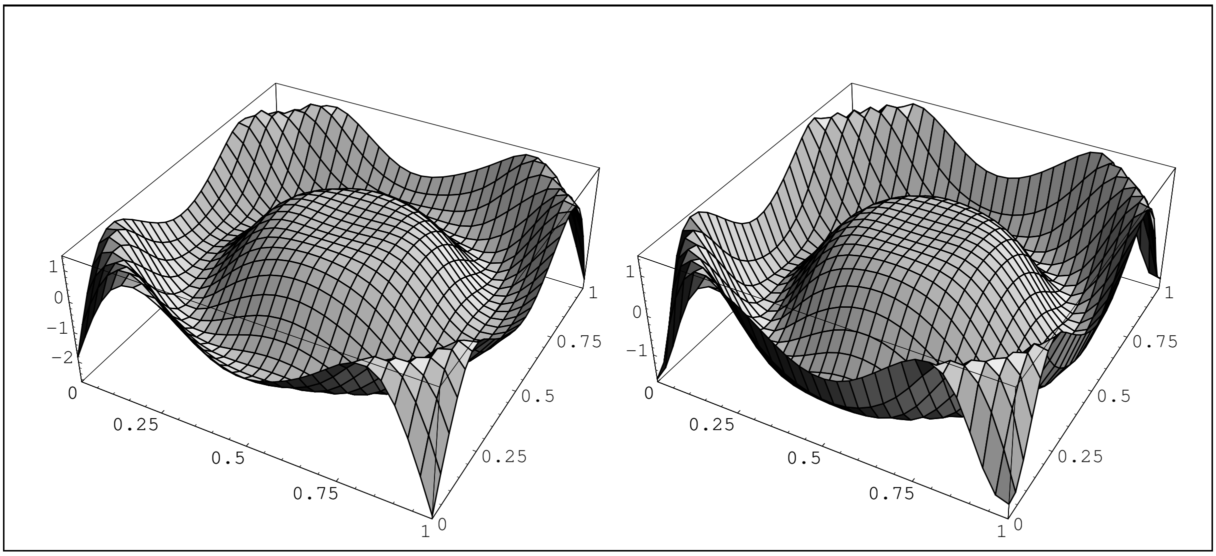

Figure 3: Original surface Υ_1 and an approximating surface parameterised by the *Fairness bicubic spline* $\sigma_\tau^{N,r}$ for respectively $r = 750$, $\tau = \{10^{-3}, 10^{-3}, 10^{-7}\}$, $E_r = 0.0759871$ and $r = 1000$, $\tau = \{10^{-7}, 10^{-7}, 10^{-10}\}$, $E_r = 0.0277037$.

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