

An urn model with non-deterministic total replacement

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Abstract

A strong law is obtained for the process $\{X_n\}$ that represents the proportion of balls of each colour in a generalized Pólya urn with non-deterministic total replacement. We prove that this process fits the Robbins-Monro scheme of stochastic approximation and, by means of the ODE method, we obtain its a.s. limit.

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1 Introduction

Let us consider an urn that initially contains $T_0 > 0$ balls of two colours, for example black and white. The replacement policy consists in drawing one ball from the urn, putting it back into the urn and, if the ball is white, adding a white balls and b black balls and, otherwise, c white balls and d black balls. In the probabilistic literature it is usual to assume that

$$a + b = c + d = s, \quad (1)$$

so that the total number of balls in each stage, $\{T_n\}_{n \geq 0}$, is a deterministic process

$$T_n = T_0 + ns, \text{ for all } n \geq 0.$$

Let $\{X_n\}$ be the bidimensional stochastic process that represents the proportion of balls of each colour in the urn at stage n . This process evolves in the 1-simplex

$$\Delta_1 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0\}.$$

The asymptotic behaviour of this process is well-known (see, [4], [5]) and some generalizations have been studied taking advantage of the deterministic behaviour of $\{T_n\}$ (see, for example, [6], [1], [2]).

When condition (1) is not assumed the process $\{T_n\}$ is non-deterministic. This is the case in [3], where the exact distribution of the number of balls of each colour in each stage is obtained, but strong laws for $\{X_n\}$ are not established. Our aim is to obtain a strong law for $\{X_n\}$. In section 2 we model the urn process as a recurrence equation and we prove that it fits the Robbins-Monro scheme of stochastic approximation. In section 3 we apply the EDO method to obtain a.s convergence of $\{X_n\}$. Since the EDO associated with the recurrence equation evolves in the Δ_1 manifold, we can restate the EDO as a differential algebraic equation (DAE) and apply the qualitative theory of DAEs to solve it.

2 Recurrence equation of the urn model

Let us consider the replacement matrix

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are non-negative numbers, and $b, c > 0$. If we identify number 1 with white balls and number 2 with black balls, then the element c_{ij} of C represents the number of balls of colour j that will be added to the urn when a ball of colour i has been extracted.

Let $\{A_n\}$ be a bidimensional stochastic process such that, for each n , the distribution of A_n conditioned to X_{n-1} is

$$A_n = \begin{cases} (1, 0) & \text{with probability } X_{1n-1} \\ (0, 1) & \text{with probability } X_{2n-1} \end{cases}$$

Let $S_n = A_n C \mathbf{1}^t$, with $n \geq 1$ and $\mathbf{1} = (1, 1)$. The sequence $\{S_n\}$ indicates the total number of balls added to the urn in each stage, so that

$$T_n = T_0 + \sum_{k=1}^n S_k, \quad n \geq 1.$$

Now, we can describe the evolution of the process by means of the recurrence equation

$$\begin{aligned} X_{n+1} &= \frac{T_n X_n + A_{n+1} C}{T_{n+1}} \\ &= X_n + \frac{A_{n+1} C - S_{n+1} X_n}{T_{n+1}}. \end{aligned} \quad (2)$$

The process $\{(X_n, A_n)\}$ will be referred to as a generalized Pólya urn model (GPUM), and we will consider the natural filtration $\{\mathcal{F}_n\}_{n \geq 1}$, where $\mathcal{F}_n = \sigma((X_i, A_i) \mid 1 \leq i \leq n)$.

Lemma 1

Let $\{\tilde{T}_n\}$ be a sequence such that

$$\tilde{T}_n = \sum_{k=1}^n E[S_k | \mathcal{F}_{k-1}].$$

Then,

$$\frac{1}{n}|T_n - \tilde{T}_n| \rightarrow 0, \quad a.s.$$

Proof. Let $m = \max\{a + b, c + d\}$. Since the sequence $\{S_n\}$ is uniformly bounded by m , the lemma follows straightforward from Theorem 2.19 in [7]. \square

Theorem 1

Let $\{(X_n, A_n)\}$ be a GPUM. Then, the process $\{X_n\}$ fits the recurrence equation

$$X_{n+1} = X_n + \gamma_n Y_n, \tag{3}$$

that satisfies the conditions

[A1] $\sup_n E|Y_n|^2 < \infty$.

[A2] $Y_n = F(X_n) + \varepsilon_n + \beta_n$, where

[A2.1] $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a continuous function.

[A2.2] $\{\varepsilon_n\}$ is a sequence of martingale differences relative to the σ -algebra $\{\mathcal{F}_n\}$.

[A2.3] $\beta_n \rightarrow 0, a.s.$

[A3] $\sum \gamma_n = \infty, \sum \gamma_n^2 < \infty, \gamma_n > 0, \forall n, \gamma_n \rightarrow 0$.

Proof. Let $\mathbf{s} = (a + b, c + d)$. From Lemma 1 we have

$$\tilde{T}_n = T_0 + \sum_{k=1}^n X_{k-1} \mathbf{s}^t.$$

From (2), we can write

$$X_{n+1} = X_n + \frac{A_{n+1}C - S_{n+1}X_n}{\tilde{T}_{n+1}} + (A_{n+1}C - S_{n+1}X_n)\left(\frac{1}{T_{n+1}} - \frac{1}{\tilde{T}_{n+1}}\right).$$

Now, we have that

$$\begin{aligned} & E[X_{n+1} | \mathcal{F}_n] \\ &= X_n + \frac{1}{\tilde{T}_{n+1}} \left(X_n(C - X_n \mathbf{s}^t) + E \left[(A_{n+1}C - S_{n+1}X_n) \left(\frac{\tilde{T}_{n+1}}{T_{n+1}} - 1 \right) | \mathcal{F}_n \right] \right). \end{aligned}$$

As

$$X_{n+1} = E[X_{n+1} | \mathcal{F}_n] + (X_{n+1} - E[X_{n+1} | \mathcal{F}_n]), \quad (4)$$

the process $\{X_n\}$ satisfies the recurrence relation

$$X_{n+1} = X_n + \gamma_{n+1}(F(X_n) + \varepsilon_{n+1} + \beta_{n+1}).$$

where

$$\begin{aligned} F(X_n) &:= X_n(C - X_n \mathbf{s}^t), \\ \gamma_{n+1} &:= \frac{1}{\tilde{T}_{n+1}}, \\ \varepsilon_{n+1} &:= \tilde{T}_{n+1}(X_{n+1} - E[X_{n+1} | \mathcal{F}_n]), \\ \beta_{n+1} &:= E \left[(A_{n+1}C - S_{n+1}X_n) \left(\frac{\tilde{T}_{n+1}}{T_{n+1}} - 1 \right) | \mathcal{F}_n \right]. \end{aligned}$$

It is easy to check that conditions **[A1]**, **[A2.1]**, **[A2.2]** and **[A3]** hold. Thus, we only have to focus on condition **[A2.3]**.

We consider, for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, the norm $\|\mathbf{x}\| = |x_1| + |x_2|$.

Let $m = \max\{a + b, c + d\}$. Then,

$$\|(A_{n+1}C - S_{n+1}X_n)\| \leq 2m,$$

and

$$|X_n \mathbf{s}^t - S_{n+1}| \leq 2m.$$

Since $T_{n+1} = T_n + S_{n+1}$, S_{n+1} is a positive random variable and T_n is \mathcal{F}_n -measurable, it follows that

$$\begin{aligned} \|\beta_{n+1}\| &\leq E \left[\|(A_{n+1}C - S_{n+1}X_n)\| \left| \frac{\tilde{T}_n - T_n + X_n \mathbf{s}^t - S_{n+1}}{T_n} \right| \mid \mathcal{F}_n \right] \\ &\leq 2m \left| \frac{\tilde{T}_n - T_n}{T_n} \right| + 4m^2 \frac{1}{T_n} \end{aligned}$$

As $T_n > n$, the second addend converges to 0, and from Lemma 1 the first one also converges to 0. Therefore $\|\beta_n\|$ converges to 0 and the result follows. \square

Remark 1 The recurrence relation (3) with conditions **[A1]**-**[A3]** is usually referred to as the Robbins-Monro algorithm of stochastic approximation.

3 Almost sure convergence of the urn model

In order to obtain the a.s. convergence of the process $\{X_n\}$ of a GPUM, we are going to use the ODE method. This method relates a recurrence equation with an ODE (see [8]). Then, under some assumptions, the asymptotic behaviour of the ODE corresponds with that of the recurrence equation. From Theorem 5.2.1 in [8], and the remark that follows it, we can state the following theorem.

Theorem 2

Let $\{X_n\}$ be a stochastic process that fits the recurrence equation (3) and the conditions [A1], [A2] and [A3] of Theorem 1.

If $\mathbf{u} \in \mathbb{R}^2$ is a globally asymptotically stable point to the ODE with restrictions

$$\begin{cases} \dot{\mathbf{x}} = F(\mathbf{x}), \\ \mathbf{x} \in \Delta_1 \end{cases} \quad (5)$$

then

$$X_n \rightarrow \mathbf{u}, \quad a.s.$$

□

Now, in order to study (5), we point out that its equilibrium points and solutions are the same as those of the DAE

$$\begin{cases} \dot{\mathbf{x}} = F(\mathbf{x}) - z\mathbf{1}^t \\ 0 = 1 - \mathbf{x}\mathbf{1}^t \end{cases} \quad (6)$$

where $\mathbf{x} \in \mathbb{R}^2$ and $z \in \mathbb{R}$.

The stability of equilibrium points for DAEs depends on the spectral abscissa of a regular matrix pencil, (A, B) , where the spectral abscissa is

$$\alpha(A, B) := \max\{\operatorname{Re}(\lambda) : \det(\lambda A + B) = 0\}.$$

Now, from Theorem 4.3 in [9], an equilibrium point of the DAE (6) is asymptotically stable if and only if $\alpha(A, B) < 0$, with

$$A = \begin{pmatrix} I_2 & \mathbf{0}^t \\ \mathbf{0} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -JF(\mathbf{u}) & \mathbf{1}^t \\ -\mathbf{1} & 0 \end{pmatrix}$$

where JF is the the jacobian matrix of the function F .

We are going to apply the previous comments to our problem.

First, we calculate the equilibrium points by solving

$$F(x_1, x_2) = 0, \quad x_1 + x_2 = 1,$$

which gives

$$\phi(x_1) = (-a - b + c + d)x_1^2 + (a - 2c - d)x_1 + c = 0.$$

We have to distinguish two cases:

i) If $a + b = c + d$, then there is a unique root

$$x_1 = \frac{c}{-a + 2c + d} = \frac{c}{c + b} \in [0, 1],$$

and thus a unique equilibrium point

$$(x_1^*, x_2^*) = \left(\frac{c}{c + b}, \frac{b}{c + b} \right).$$

ii) If $a + b \neq c + d$, then there are two roots of the polynomial. As $\phi(0) = c > 0$ and $\phi(1) = -b < 0$, only one of them is in $[0, 1]$, namely

$$x_1^* = \frac{-\beta - \sqrt{\beta^2 - 4\alpha c}}{2\alpha},$$

where we have denoted $\alpha := -a - b + c + d$, $\beta := a - 2c - d$. Denoting $x_2^* = 1 - x_1^*$, the unique equilibrium point in the 1-simplex is (x_1^*, x_2^*) .

For $\mathbf{x} = (x_1, x_2) \in \Delta_1$ we have that $F(\mathbf{x})\mathbf{1}^t = 0$. The Jacobian matrix

$$JF(\mathbf{x}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a + b \\ c + d \end{pmatrix} (x_1, x_2) - (x_1, x_2) \begin{pmatrix} a + b \\ c + d \end{pmatrix} I_2$$

has

$$\lambda_1(x_1) = -c - d + (-a - b + c + d)x_1 = -c - d + \alpha x_1$$

as eigenvalue with eigenvector $\mathbf{1}$; the other eigenvalue is

$$\lambda_2(x_1) = a - 2c - d + 2(-a - b + c + d)x_1 = 2\alpha x_1 + \beta. \quad (7)$$

To compute these eigenvalues we have already used that $(x_1, x_2) \in \Delta_1$. Evaluating (7) in the equilibrium point $x_1 = x_1^*$, we obtain

$$\lambda_2(x_1^*) = -(c + b) < 0$$

if $a + b = c + d$ and

$$\lambda_2(x_1^*) = -\sqrt{\beta^2 - 4\alpha c} < 0$$

otherwise.

From these results, we have established, in any case, the existence of a unique stable point to the ODE with restrictions (5), and therefore, it is also a globally asymptotically stable point. Now, we can apply Theorem 2 to obtain the following result.

Theorem 3

Let $\{(X_n, A_n)\}$ be a GPUM.

a) If $a + b = c + d$ then $X_n \rightarrow (\frac{c}{c+b}, \frac{b}{c+b}), a.s.$

b) If $a + b \neq c + d$ then $X_n \rightarrow (u_1, 1 - u_1), a.s.$

where $\alpha := -a - b + c + d$, $\beta := a - 2c - d$ and

$$u_1 = \frac{-\beta - \sqrt{\beta^2 - 4\alpha c}}{2\alpha}.$$

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