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Orbits in the hyperbolic plane

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Abstract

The authors characterize the orbits determined by some isometries in the hyperbolic plane as rotations with a fixed center, the limit rotations with a fixed point in the infinity line and the translations along a line. These orbits are called, respectively, circumference, horocicle and hipercicle. In addition, they show a exhaustive classification of the above isometries by means of the study of their fixed points. Some methods for building the above mentioned orbits are determined and some algorithms for their implementation are described. From these algorithms we have created some programming modules with the software Mathematica for the construction of such orbits and we have solved some constructive problems related with them.

Keywords: Hyperbolic geometry, isometry, orbit.
AMS Classification: 20H15, 51F15, 20H10

1 Introduction

Let $\mathbb{C}^+$ be the open upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid Imz > 0\}$ endowed with the metric $ds = \dfrac{|dz|}{Imz}$ [3]. We denote by H^2 that set with such metric. The lines in H^2 are euclidean half-circumferences centered at a point in the boundary of H^2 which correspond to the parametrization: $x(t) = r\cos t + k_1,\ y(t) = r\sin t,\ t \in (0,\pi)$; and euclidean half-lines orthogonal to that boundary with the parametrization: $x(t) = k_2,\ y(t) = t,\ (t > 0)$. We have, making use of the Moebius transformation [4], that the group of isometries of H^2 preserving the orientation are given by

$$Iso^+(H^2) = \left\{ g : H^2 \longrightarrow H^2 \mid g(z) = \frac{az+b}{cz+d};\ a,b,c,d \in \mathbb{R};\ ad-bc = 1 \right\}.$$

Let $D = \{z \in \mathbb{C};\ |z| < 1\}$, i.e., the image of $\mathbb{C}^+$ by the Cayley transformation $f_c : \mathbb{C}^+ \longrightarrow D$, defined by $f_c(z) = \dfrac{z-i}{z+i}$. By means of f_c, the metric on H^2 is transformed in $ds = 2\dfrac{|dz|}{1-|z|^2}$ for D, and with that metric, it will be denoted by D^2. We can check that [1]

$$Iso^+(D^2) = \left\{ s : D^2 \longrightarrow D^2 \mid s(z) = \frac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}};\ \alpha, \beta \in \mathbb{C},\ \alpha\overline{\alpha} - \beta\overline{\beta} = 1 \right\}$$

2 Rotations, limit rotations and translations

Definition 1 *$g : H^2 \to H^2$ ($s : D^2 \to D^2$) is a rotation centered at $z_0 \in H^2$ ($z_0 \in D^2$) provided it is an isometry preserving the orientation and it just fixes the point z_0. z_0 is called rotation center.*

Definition 2 *$g : H^2 \to H^2$ ($s : D^2 \to D^2$) is a limit rotation centered at z_0 belonging to the infinity line, whether it is an isometry preserving the orientation and it just fixes the point z_0. z_0 is called limit rotation center.*

Definition 3 *A transformation $g : H^2 \to H^2$ (resp. $s : D^2 \to D^2$) is a translation on H^2 (resp. D^2) either on or with respect to a line l whether it is an isometry which preserves the orientation and fixes two points in the infinity line, which are those obtained as the intersection of infinity line with l, called translation line.*

The following results can be found in [2]

Theorem 1 *Let $g \in Iso^+(H^2)$ and $s \in Iso^+(D^2)$.*

a) g is a rotation on H^2 iff $|a + d| < 2$.

b) s is a rotation on D^2 iff $|Re\alpha| < 1$.

Theorem 2 *Let $g \in Iso^+(H^2)$ and $s \in Iso^+(D^2)$.*

a) g is a limit rotation or the identity in H^2 iff $|a + d| = 2$.

b) s is a limit rotation or the identity in D^2 iff $|Re\alpha| = 1$.

Theorem 3 *Let $g \in Iso^+(H^2)$ and $s \in Iso^+(D^2)$.*

a) g is a translation on H^2 iff $|a + d| > 2$.

b) s is a translation on D^2 iff $|Re\alpha| > 1$.

3 Construction of the circumference

Definition 4 *A circumference is said to be set of images of a point $Q \in H^2$ (respectively D^2) by mean of the reflection according to all the lines throug a given point $A \in H^2$ (respectively D^2), which is called center the circumference.*

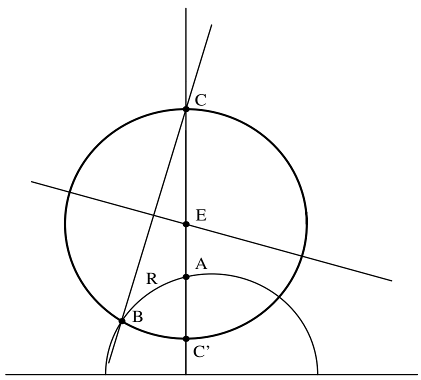
Remark 4 *a) The circumferences centered at $A(a,b)$ are, by this definition, the orbits of the group of all eliptic isometries with a fixed point A (rotations with center at A).*

b) The circumference is a euclidean circumference contained in $\mathbb{C}^+$ (respectively D).

c) A circumference is a orthogonal curve to each line which contains A.

Problem 1 *Given R and $A(a,b)$, build the circumference centered a A with radius R.*

Determination in H^2



Draw the line $x = a$, calculate the point C on this line whose second coordinate is greaterthan the second coordinate of A and whose distance to A is R, obtaining $C(a, be^R)$. Let us consider any line through A and on it we determine a point $B(c,d)$ whose distance to A is R. Finally, we draw the euclidean mediatriz of the euclidean segment line line $\overline{BC}$, whose equation is

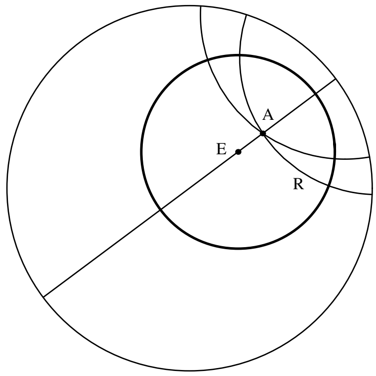
$$y - \frac{be^R + d}{2} = \frac{a-c}{d-be^R}\left(x - \frac{a+c}{2}\right)$$

the intersection of that mediatriz with $x = a$ gives us the center E of the euclidean circumference:

$$\left(a, \frac{d^2 - b^2e^{2R} + (a-c)^2}{2(d-be^R)}\right)$$

Determination in D^2

Given the radius R and the center A in D^2, let $\widetilde{A} = f_c^{-1}(A)$. We determinate three equidistant points R of $\widetilde{A}$ from their images by f_c, in D^2, calculate the center E and the radius of the euclidean circumference contained in D.



4 Construction of the horocycle

Definition 5 *The horocycle with center a point P of the infinity line is the set of all images of a fixed point Q by the reflections respect the asimptotic lines in the point P.*

Remark 5 *a) This definition characterizes the horocycles as the orbits for the group of parabolic isometries (limit rotations) with point P belonging to the infinity line.*

b) The horocycle is obtained from the circumference as a limit case. We have the following posibilities:

In H^2 : If we move the center of the circumference through a given line to a point $P(p, 0)$ of the infinity line, the horocycle is a euclidean circumference contained in $\mathbb{C}^+$ and tangent to the abscises axis at that point. If we move the center to the infinity point of the infinity line, the horocycle is a euclidean line contained in $\mathbb{C}^+$ whosw equation is $y = k$.

In D^2 : If we move the center of the circumference through a given line to a point in $fr(D^2)$, the horocycle is a euclidean circumference contained in the unit disk D tangent to the frontier of D, at the point.

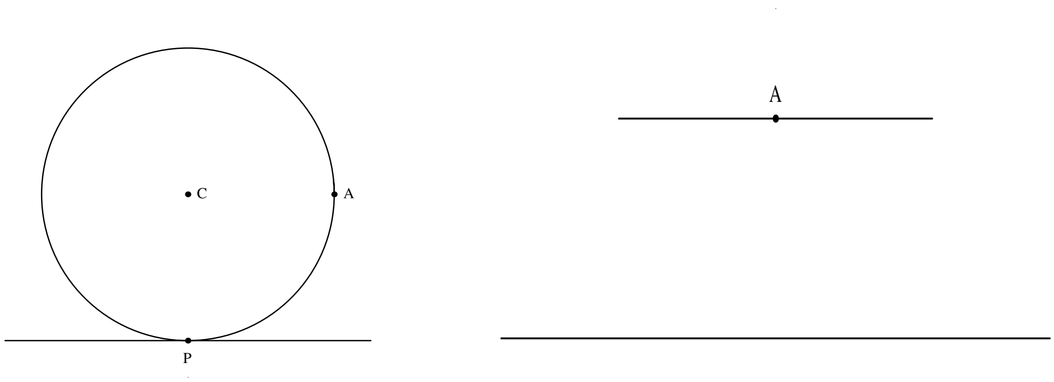
c) It is easy to chek that the horocycle orthogonally intersects all the lines of the asimptotic pencil at P, being equal the distance between two with center at P.

Problem 2 *Given the points $A(a, b)$ in the hyperbolic plane and P in the infinity line, build the horocycle through A and P.*

Determination in H^2

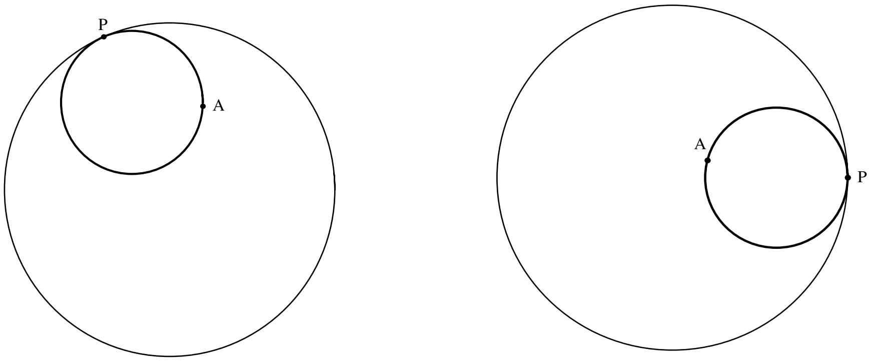
1.- If $P(p, 0)$ is a point of the infinity line, then the center of the euclidean circumference is $C(p, q)$; where $q = \dfrac{(a-p)^2 + b^2}{2b}$, which coincides with its radius.

2.- If P is infinity, the horocycle through A with center P is the euclidean line contained in $\mathbb{C}^+$ with equation $y = b$.



Determination in D^2

Let $P(p,0) \in fr(D^2)$, we determine $f_c^{-1}(A) = \widetilde{A} \in H^2$, and $f_c^{-1}(P) = \widetilde{P}$ which is a point in the infinity line. Next we build the horocycle in H^2 and, by means of f_c, we obtain the horocycle in D^2.



Problem 3 *Calculate the horocycles through two given points $A(a,b)$ and $B(c,d)$.*

Determination in H^2

1.- If $b \neq d$, taking into account that the equation of the euclidean line which is orthogonal to the euclidean segment $\overline{AB}$ and which contains its midle point is

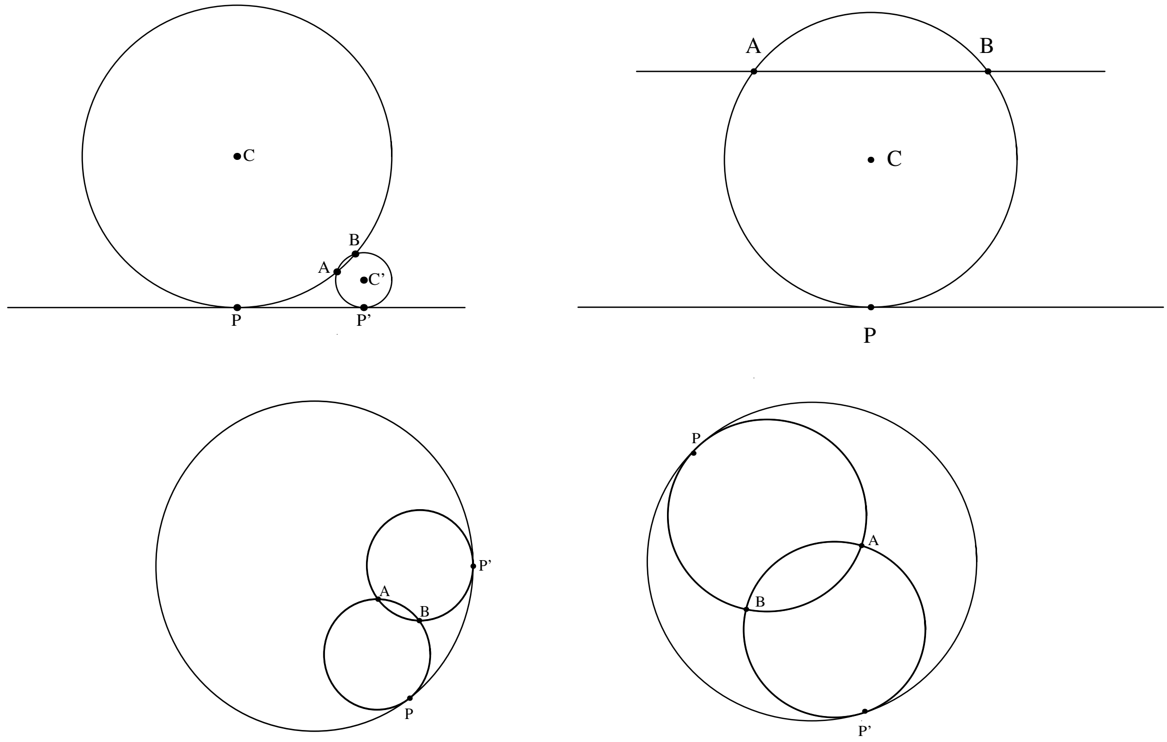
$$(c-a)x + (d-b)y - \frac{d^2-b^2}{2} - \frac{c^2-a^2}{2} = 0$$

and that the point A belongs to the euclidean circumference, the solution of the system of indeterminates p and q

$$\begin{cases} (c-a)p + (d-b)q - \dfrac{d^2-b^2}{2} - \dfrac{c^2-a^2}{2} = 0 \\ p^2 - 2ap + a^2 - 2bq + b^2 = 0 \end{cases}$$

determines the centers C and C' of such euclidean circumferences.

2.- If $b = d$, we obtain the horocycle, which is a euclidean circumference whose center and radius are determined by the system of 1.- taking $b = d$, and the horocycle given by the euclidean line $y = b$.



Determination in D^2

We calculate $f_c^{-1}(A) = \widetilde{A}(\widetilde{a}, \widetilde{b})$, and $f_c^{-1}(B) = \widetilde{B}(\widetilde{c}, \widetilde{d})$ and build the horocycles in H^2. then we obtain, by means of f_c, the corresponding horocycles in D^2.

5 Construction of the hypercycle

Definition 6 *We define the hypercycle determined by a point Q and a line l in the hyperbolic plane as the set of images of Q by means of reflections respect of all the orthogonal lines to l. The line l is called translation line.*

Remark 6 *a) The hypercycles are, by this definition, the orbits of the group of all hyperbolic isometries fixing the two points of the intersection of the line l and the infinity line.*

b) In H^2: If the translation line meets the infinity line in two points $A(a, 0)$ and $B(b, 0)$, the hypercycle is an arc of euclidean line through such points. If the translation line meets the infinity line in a point $A(a, 0)$ and another one the infinity point, the hypercycle is a euclidean half-line whose origin point is A.

c) In D^2: An hypercycle is an arc of euclidean circumference through two points of the infinity line determined by the translation line.

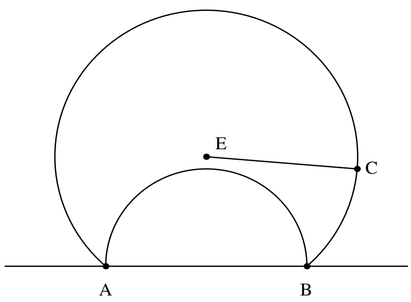
d) If the point Q belongs to the translation line, the hypercycle coincides with that line.

e) The distance between any point of the translation line and the hypercycle, measured on the orthogonal line is the same, is constant.

f) Any two hypercycles of the same pencil are "paralels" in the sense: that the distance of one of then to another one is constant.

Problem 4 *Given two points A and B of the infinity line and a third point C of the hyperbolic plane, build the unique hypercycle through A, B and C.*

Detemination in H^2

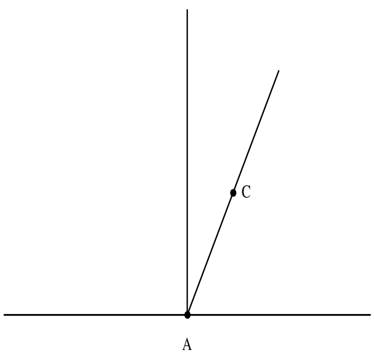


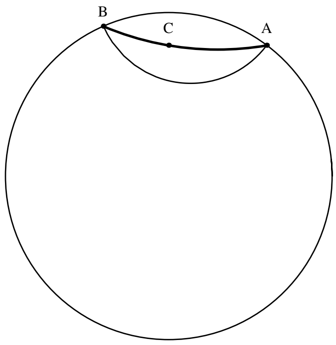
1.- If A and B are points of the infinity line with coordenates $(a, 0)$ and $(b, 0)$ respectively, with $a < b$ and $C(c, d) \in H^2$, the hypercycle through A, B and C is the intersection with $\mathbb{C}^+$ of the euclidean circumference whose center E and radius r are obtained solving the system

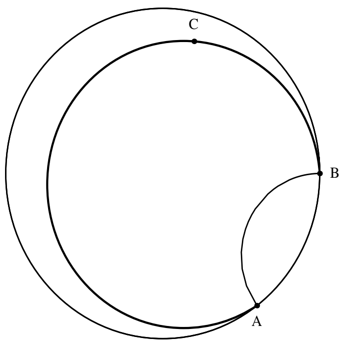
$$\begin{cases} (a-p)^2 + q^2 = r^2 \\ (b-p)^2 + q^2 = r^2 \\ (c-p)^2 + (d-q)^2 = r^2 \end{cases}$$

The center E has coordenates: $\left(\dfrac{a+b}{2}, \dfrac{(a-c)(b-c)+d^2}{2d}\right)$,

and its radius is: $\dfrac{\sqrt{((a-c)^2+d^2)\,((b-c)^2+d^2)}}{2d}$.







2.- If $A(a,0)$, $B=\infty$ and $C(c,d) \in H^2$, the hypercycle is the euclidean half-line of equation: $y = \dfrac{d}{c-a}(x-a)$, if $c \neq a$, or $x = a$, if $c = a$.

Determination in D^2

Let $\widetilde{A} = f_c^{-1}(A)$, $\widetilde{B} = f_c^{-1}(B)$ and $\widetilde{C} = f_c^{-1}(C)$ and σ the hypercycle in H^2 determined by $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$. Then $f_c(\sigma)$ is the hypercycle in D^2 detemined by A, B and C.

6 Conclusions

The main contributions of this work to the study of the orbits of some isometries in the hyperbolic plane are the following ones:

1.- The systematic characterization of the isometries as rotation, limit rotation and translation through the study of their fixed point.

2.- The determination of precise algoritmhs for the construction of the orbits, circumference, horocycle and hypercycle from diferent initial dates.

3.- The implementation of these algorithms in a set of programing modules with the software *Mathematica* included in a pack that we have designed for the numeric and graphic resolution of a wide group of constructive problems in the hyperbolic plane, in H^2 and D^2.

4.- This results form part of a complete study of the diferent isometries, as well as a wide group of constructive problems in the hyperbolic plane.

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