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A HODIE method for 2D parabolic problems of convection-diffusion type*

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Abstract

In this work we construct a numerical method to solve a two dimensional convection-diffusion parabolic problem for which the diffusion term can be very small. To deduce the method we use the Peaceman-Rachford scheme to discretize in time and a finite difference scheme of HODIE type, defined on a piecewise uniform Shihskin mesh, for the spatial discretization. The numerical results show that the method is uniformly convergent with respect to the diffusion parameter, having order two in both time and spatial variables. Therefore, the method is more efficient that the schemes used so far to solve this type of problems.

Keywords: Peaceman-Rachford scheme, HODIE scheme, parabolic problem, uniform convergence.

AMS Classification: 65N12, 65N30, 65N06.

1 Introduction

We consider the linear parabolic 2D problem

$$\begin{cases} u_t + L_\varepsilon u = f(x_1, x_2, t), & (x_1, x_2, t) \in \Omega \times (0, T], \\ u(x_1, x_2, 0) = u_0(x_1, x_2), & (x_1, x_2) \in \overline{\Omega}, \\ u(x_1, x_2, t) = 0, & (x_1, x_2, t) \in \partial\Omega \times (0, T], \end{cases} \tag{1}$$

where $\Omega = (0,1)^2$ and the differential operator L_ε is defined by

$$L_\varepsilon u \equiv -\varepsilon \Delta u + \mathbf{a} \cdot \nabla u + bu. \tag{2}$$

We suppose that the diffusion parameter ε can be very small, $0 < \varepsilon \leq 1$ and that $\mathbf{a} = (a_1, a_2), b, f$ and u_0 are sufficiently smooth functions satisfying

$$a_i(x_1, x_2) \geq \alpha_i > 0, \; i = 1, 2, \quad b(x_1, x_2) \geq 0. \tag{3}$$

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It is known, see [11] for example, that under sufficient compatibility conditions among the data, the solution of (1) has a regular layer in the output boundary

$$\Omega_o = \{(x_1, x_2) \in \partial\Omega; \mathbf{a} \cdot \mathbf{n} > 0\},$$

where $\mathbf{n}$ is the normal exterior to $\partial\Omega$, and a corner layer in the neighborhood of (1,1). This behavior motivates that, to solve efficiently (for all values of ε) this problem, it is necessary to use robust numerical methods (see [5] for a rigorous definition), i.e, the error associated to the numerical method can be bounded independently of the diffusion parameter ε. In recent years, many simple numerical methods with this property (finite differences or finite elements) defined on piecewise uniform Shishkin meshes, see [5, 11], have been developed to solve both stationary and time dependent singularly perturbed problems. Nevertheless, we know of few methods having order bigger than one (see [6, 7]), used to solve singularly perturbed parabolic problems. In this paper we want to show that it is possible to find a method of order 2 combining an alternating direction technique to discretize in time, namely the Peaceman-Rachford scheme, [10] and the HODIE technique defining a finite difference scheme to discretize in space, [2]. This same idea was used in [3, 4] to find some numerical uniformly convergent methods having order 1, for both convection-diffusion and reaction-diffusion problems of type (1).

To simplify the construction of the method, we decompose the differential operator L_ε in the form $L_\varepsilon = L_{1,\varepsilon} + L_{2,\varepsilon}$, where

$$L_{i,\varepsilon} \equiv -\varepsilon \frac{\partial^2}{\partial x_i^2} + a_i \frac{\partial}{\partial x_i} + b_i, \;\; i = 1, 2, \tag{4}$$

and $b = b_1 + b_2$, $b_i \geq \tilde{b}_i \geq 0$. The operators $L_{i,\varepsilon}$, $i = 1, 2$ are a family of 1D differential operators depending on the variables $x_2 \in (0, 1)$ and $x_1 \in (0, 1)$ respectively. Also, it is necessary to decompose the right-hand side of the continuous problem (1) in an appropriate way. Following [3] we take $f(x_1, x_2, t) = f_1(x_1, x_2, t) + f_2(x_1, x_2, t)$, where

$$\begin{aligned} f_2(x_1, x_2, t) &= f(x_1, 0, t) + x_2(f(x_1, 1, t) - f(x_1, 0, t)), \\ f_1(x_1, x_2, t) &= f(x_1, x_2, t) - f_2(x_1, x_2, t). \end{aligned}$$

Throughout the paper C will denote any positive constant independent of ε and the mesh sizes.

Before constructing the numerical method, we recall some results showing the asymptotic behavior of the exact solution of (1). Firstly, from [8, 9] we know that under sufficient regularity and compatibility conditions for $\mathbf{a}$, b, f and u_0, for λ a not integer real number the solution $u(x_1, x_2, t) \in C^{l+\lambda, l+\lambda, (l+\lambda)/2}(\overline{\Omega} \times [0, T])$, i.e., u is a l-Hölder function (the derivatives up to order l are Hölder continuous functions). In second place, in [3, 12] it was proved that the exact solution of (1) can be written as $u = u_0 + w$, where u_0 is the regular component of u and w is its singular component; moreover, $w = u_1 + u_2 + u_3$ where u_1 and u_2 are the layer functions associated to the regular layers in $x_1 = 1$ and $x_2 = 1$ respectively and u_3 is the corner layer function. These functions satisfy

$$\begin{array}{rcl} \left|\dfrac{\partial^{k_s+k_t} u_0(x_1,x_2,t)}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_t}}\right| & \leq & C, \\ \left|\dfrac{\partial^{k_s+k_t} u_1(x_1,x_2,t)}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_t}}\right| & \leq & C\varepsilon^{-k_1}\exp\left(-\dfrac{\alpha_1(1-x_1)}{\varepsilon}\right), \\ \left|\dfrac{\partial^{k_s+k_t} u_2(x_1,x_2,t)}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_t}}\right| & \leq & C\varepsilon^{-k_2}\exp\left(-\dfrac{\alpha_2(1-x_2)}{\varepsilon}\right), \\ \left|\dfrac{\partial^{k_s+k_t} u_3(x_1,x_2,t)}{\partial x_1^{k_1}\partial x_2^{k_2}\partial t^{k_t}}\right| & \leq & C\varepsilon^{-k_s}\min\left\{\exp\left(-\dfrac{\alpha_1(1-x_1)}{\varepsilon}\right),\exp\left(-\dfrac{\alpha_2(1-x_2)}{\varepsilon}\right)\right\}, \end{array} \tag{5}$$

with $k_s = k_1 + k_2,\ k_s + 2k_t \leq l$.

2 The numerical scheme

To obtain the totally discrete method, we begin by discretizing the time variable with the Peaceman-Rachford scheme using a constant step size Δt. This scheme is defined as

$$\left\{\begin{array}{l} u^0 = u_0(x_1,x_2), \\ \left\{\begin{array}{ll} (I + (\Delta t/2)L_{1,\varepsilon})u^{n+1/2} = (I - (\Delta t/2)L_{2,\varepsilon})u^n + & \\ \quad +(\Delta t/2)(f_1(x_1,x_2,t_{n+1/2}) + f_2(x_1,x_2,t_n)), & x_2 \in (0,1), \\ u^{n+1/2}(0,x_2) = u^{n+1/2}(1,x_2) = 0, & \end{array}\right. \\ \left\{\begin{array}{ll} (I + (\Delta t/2)L_{2,\varepsilon})u^{n+1} = (I - (\Delta t/2)L_{1,\varepsilon})u^{n+1/2} + & \\ \quad +(\Delta t/2)(f_1(x_1,x_2,t_{n+1/2}) + f_2(x_1,x_2,t_{n+1})), & x_1 \in (0,1), \\ u^{n+1}(x_1,0) = u^{n+1}(x_1,1) = 0, & \end{array}\right. \end{array}\right. \tag{6}$$

where $u^n(x_1,x_2)$ is the approximation to $u(x_1,x_2,t)$ in the time level $t_n = n\Delta t$.

To discretize in space we first construct the Shishkin mesh in a standard way, taking into account that there are regular layers in the output boundary. Then, the mesh is the tensor product of Shishkin meshes used in the case of positive convection-diffusion 1D singularly perturbed problems, i.e, $\Omega_\varepsilon^N = I_{1,\varepsilon}^N \times I_{2,\varepsilon}^N$, where

$$\begin{array}{rl} I_{k,\varepsilon}^N = & \{x_{k,i} = iH_k,\ i = 0,\ldots,N/2,\ x_{k,i} = x_{k,N/2} + (i - N/2)h_k, \\ & \quad i = N/2+1,\ldots,N,\ h_{k,i} = x_{k,i} - x_{k,i-1},\ i = 1,\ldots,N\}, \end{array} \tag{7}$$

with $N \geq 4$ a positive even integer, $H_k = 2(1-\sigma_k)/N,\ h_k = 2\sigma_k/N$ and the transition parameters σ_k are defined by

$$\sigma_k = \min\{1/2, \sigma_{k,0}\varepsilon\ln N\}, \tag{8}$$

with $\sigma_{k,0} = 4/\alpha_k$ (see [2] for a theoretical justification of this choice). We assume that $\sigma_k = \sigma_{k,0}\varepsilon\ln N$ (the interesting case in practice) and therefore the spatial mesh is piecewise uniform.

Now, on the meshes $I^N_{k,\varepsilon}$, $k=1,2$, we discretize the 1D problems of (6) using a HODIE finite difference scheme (see [2] for details of the construction). The coefficients $r_j^{k,\bullet}$ and $q_j^{k,\bullet}$, defining theses schemes, are calculated by imposing that polynomials of degree less than or equal to 2 are in the kernel of local error operator and that they satisfy the normalization condition $q_j^{k,1}+q_j^{k,2}=1, \quad 1\le j\le N-1, \quad k=1,2$. Finally, for the analysis it will be required that the scheme be of positive type, i.e., the coefficients satisfy

$$\begin{array}{l} r_j^{k,-}<0,\ r_j^{k,+}<0,\ r_j^{k,c}>0, \\ r_j^{k,-}+r_j^{k,+}+r_j^{k,c}\ge 0, \quad h_{k,j+1}r_j^{k,+}-h_{k,j}r_j^{k,-}>c>0. \end{array}$$

Under these restrictions, in [1] it is proved that the coefficients are well defined and therefore the scheme can be written as follows.

For $n=0,1,\dots,\frac{T}{\Delta T}$,

$$\left\{\begin{array}{l}
U^0_{x_{1,i},x_{2,j}}=u_0(x_{1,i},x_{2,j}),\ 0\le i,j\le N,\\
\bar{L}^N_{2,\varepsilon}U^0_{x_{1,i},x_{2,j}}=L_{2,\varepsilon}u_0(x_{1,i},x_{2,j}),\ 0\le i,j\le N,\\
\\
\left\{\begin{array}{l}
L^N_{1,\varepsilon}U^{n+1/2}_{x_{1,j},x_{2,k}}\equiv r_j^{1,-}U^{n+1/2}_{x_{1,j-1},x_{2,k}}+r_j^{1,c}U^{n+1/2}_{x_{1,j},x_{2,k}}+r_j^{1,+}U^{n+1/2}_{x_{1,j+1},x_{2,k}}=\\
=q_j^{1,1}(U^n_{x_{1,j-1},x_{2,k}}+(\Delta t/2)(-\bar{L}^N_{2,\varepsilon}U^n_{x_{1,j-1},x_{2,k}}+\\
+f_1(x_{1,j-1},x_{2,k},t_{n+1/2})+f_2(x_{1,j-1},x_{2,k},t_n)))+\\
+q_j^{1,2}(U^n_{x_{1,j},x_{2,k}}+(\Delta t/2)(-\bar{L}^N_{2,\varepsilon}U^n_{x_{1,j},x_{2,k}}+\\
+f_1(x_{1,j},x_{2,k},t_{n+1/2})+f_2(x_{1,j},x_{2,k},t_n))),\quad j=1,\cdots,N-1,\\
U^{n+1/2}_{0,x_{2,k}}=U^{n+1/2}_{N,x_{2,k}}=0,
\end{array}\right. \qquad x_{2,k}\in I^N_{2,\varepsilon},\\
\bar{L}^N_{1,\varepsilon}U^{n+1/2}_{x_{1,j},x_{2,k}}=-\bar{L}^N_{2,\varepsilon}U^n_{x_{1,j},x_{2,k}}-2\dfrac{U^{n+1/2}_{x_{1,j},x_{2,k}}-U^n_{x_{1,j},x_{2,k}}}{\Delta t}+f_1(x_{1,j},x_{2,k},t_{n+1/2})+\\
+f_2(x_{1,j},x_{2,k},t_n),\\
\\
\left\{\begin{array}{l}
L^N_{2,\varepsilon}U^{n+1}_{x_{1,k},x_{2,j}}\equiv r_j^{2,-}U^{n+1}_{x_{1,k},x_{2,j-1}}+r_j^{2,c}U^{n+1}_{x_{1,k},x_{2,j}}+r_j^{2,+}U^{n+1}_{x_{1,k},x_{2,j+1}}=\\
=q_j^{2,1}(U^{n+1/2}_{x_{1,k},x_{2,j-1}}+(\Delta t/2)(-\bar{L}^N_{1,\varepsilon}U^{n+1/2}_{x_{1,k},x_{2,j-1}}+\\
+f_1(x_{1,k},x_{2,j-1},t_{n+1/2})+f_2(x_{1,k},x_{2,j-1},t_{n+1})))+\\
+q_j^{2,2}(U^{n+1/2}_{x_{1,k},x_{2,j}}+(\Delta t/2)(-\bar{L}^N_{1,\varepsilon}U^{n+1/2}_{x_{1,k},x_{2,j}}+\\
+f_1(x_{1,k},x_{2,j},t_{n+1/2})+f_2(x_{1,k},x_{2,j},t_{n+1}))),\quad j=1,\cdots,N-1,\\
U^{n+1}_{x_{1,k},0}=U^{n+1}_{x_{1,k},N}=0,
\end{array}\right. \qquad x_{1,k}\in I^N_{1,\varepsilon},\\
\bar{L}^N_{2,\varepsilon}U^{n+1}_{x_{1,k},x_{2,j}}=-\bar{L}^N_{1,\varepsilon}U^{n+1/2}_{x_{1,k},x_{2,j}}-2\dfrac{U^{n+1}_{x_{1,k},x_{2,j}}-U^{n+1/2}_{x_{1,k},x_{2,j}}}{\Delta t}+f_1(x_{1,k},x_{2,j},t_{n+1/2})+\\
+f_2(x_{1,k},x_{2,j},t_{n+1}),
\end{array}\right. \tag{9}$$

where

$r_j^{1,-} = \Delta t/2(-2\varepsilon + q_j^{1,1}(-(2h_{1,j} + h_{1,j+1})a_1(x_{1,j-1}, x_2)+$
$+h_{1,j}(h_{1,j} + h_{1,j+1})b_1(x_{1,j-1}, x_2)) - q_j^{1,2}h_{1,j+1}a_1(x_{1,j}, x_2))/(h_{1,j}(h_{1,j} + h_{1,j+1})) + q_j^{k,1},$
$r_j^{1,+} = \Delta t/2(-2\varepsilon + h_{1,j}a_1(x_{1,j}, x_2) - q_j^{1,1}h_{1,j}((a_1(x_{1,j-1}, x_2) + a_1(x_{1,j}, x_2)))/$
$/(h_{1,j+1}(h_{1,j} + h_{1,j+1})),$
$r_j^{1,c} = \Delta t/2(-\tilde{r}_j^{1,-} - \tilde{r}_j^{1,+} + q_j^{1,1}b_1(x_{1,j-1}, x_2) + q_j^{1,2}b_1(x_{1,j}, x_2)) + q_j^{k,2},$
$q_j^{1,2} = 1 - q_j^{1,1}.$

with

$$q_j^{1,1} = \begin{cases} a_1(x_{1,j}, x_2)/(a_1(x_{1,j-1}, x_2) + a_1(x_{1,j}, x_2)), & \text{if } H_1||a_1||_\infty \geq 2\varepsilon, \\ (h_{1,j} - h_{1,j+1})/(3h_{1,j}), & \text{if } H_1||a_1||_\infty < 2\varepsilon, \end{cases}$$

for $1 \leq j \leq N/2$ and $q_j^{1,1} = 0$ for $N/2 < j < N$ and the coefficients $r_j^{2,\bullet}, q_j^{2,\bullet}$ are defined analogously.

Lemma 1 *Let $N \geq N_0$, where N_0 is the smallest integer such that*

$$||a_k||_\infty < \frac{\alpha_k N_0}{4 \log N_0}, \quad ||b_k||_\infty \Delta t \leq 2, \quad k = 1, 2.$$

Then, if $N^{-1} \leq (\Delta t/4) \min_{k=1,2} \alpha_k$, the HODIE operators defined in (9) are of positive type, satisfy the discrete maximum principle and they are ε-stables in the maximum norm.

Proof. The proof is straightforward from the definition of $r_j^{k,\bullet}$ and $q_j^{k,\bullet}, k = 1, 2$.

Theorem 1 *Let u be the solution of (1) and $\{U^n\}$ the numerical solution of (9). Then under the same hypotheses of Lemma 1, if we take $q, \Delta t$ such that $0 < q < 1$ y $N^{-q} \leq c(\Delta t)^2$, there exists C such that*

$$\max_{t_n} \|U_{i,j}^n - u(x_{1,i}, x_{2,j}, t_n)\|_\infty \leq C(\Delta t + N^{-2+q} \log^2 N), \quad 0 \leq i, j \leq N. \tag{10}$$

Proof. We only give the outlines of the proof (see [1] for a detailed proof of this result). Firstly, we must study the stability and the consistency, uniformly in ε, of the Peaceman-Rachford scheme, and therefore in a classical way the convergence of time semidiscretization can be proved. After that, taking into account the bounds (5) for the derivatives of the exact solution of (1), we study the accuracy of the spatial discretization. Using appropriate Taylor expansions we can obtain bounds of the local discretization error depending on the diffusion parameter. Nevertheless, from these bounds and the uniform stability of the HODIE scheme it is not possible to deduce the uniform convergence of the scheme. Therefore, it seems necessary to use the barrier function technique. Applying in an appropriate way the discrete maximum principle, the ε-uniform convergence of the HODIE scheme can be obtained. Finally, it is possible to find a recurrence relation between the errors corresponding to two consecutive time levels; from this and the hypotheses imposed on the discretization parameters, the required result of convergence is deduced.

Remark 1 *From the previous theorem we only deduce the first order of ε-uniform convergence in the time variable. Nevertheless, as we will see in the next section, that it is not the real situation. We think that this reduction of order is due to the technique of analysis, but up to now we do not dispose of a complete proof of this. Also, the strong restriction $N^{-q} \leq c(\Delta t)^2$, between the discretization parameters, is not necessary in practice; we only need that $N^{-1} \leq (\Delta t/4) \min_{k=1,2} \alpha_k$ to preserve that the HODIE scheme be uniformly stable in maximum norm and it satisfies a discrete maximum principle.*

3 Numerical results

In this section we present the results obtained in the numerical approximation of two different examples of type (1). The first one is

$$u_t - \varepsilon \Delta u + u_{x_1} + u_{x_2} = f, \quad (x_1, x_2, t) \in (0,1)^2 \times (0,2], \tag{11}$$

where f and u_0 are chosen so that the exact solution is

$$u(x_1, x_2, t) = (1 - e^{-t}) x_1 x_2 (e^{-(1-x_1)/\varepsilon} - 1)(e^{-(1-x_2)/\varepsilon} - 1).$$

For every N, ε and Δt the error at each mesh point in time level t_n is given by

$$E_{\varepsilon,N,\Delta t}(i,j,n) = |u(x_{1,i}, x_{2,j}, t_n) - U_{i,j}^n|,$$

and the maximum errors are $E_{\varepsilon,N} = \max_{i,j,n} E_{\varepsilon,N,\Delta t}(i,j,n)$. We denote by $E_N = \max_{\varepsilon} E_{\varepsilon,N}$ the corresponding ε-uniform error. From these values, the numerical orders of convergence can be approximated by

$$p = \frac{\log\left(E_{\varepsilon,N}/E_{\varepsilon,2N}\right)}{\log 2}, \quad p_{uni} = \frac{\log\left(E_N/E_{2N}\right)}{\log 2}.$$

In table 3 we show the results obtained for this problem. As we can see, they clearly show the second order (except by the logarithmic factor) ε-uniform convergence of the scheme. The second example is

$$\begin{cases} u_t - \varepsilon \Delta u + (1 - x_1 x_2/2) u_{x_1} + (1 + x_1 x_2/2) u_{x_2} = \\ \qquad\qquad = e^{-t} t^2 x_1 x_2 (1 - x_1)(1 - x_2), (x_1, x_2, t) \in (0,1)^2 \times (0,2], \\ u(x_1, x_2, 0) = 0, \ (x_1, x_2) \in [0,1]^2, \quad u(x_1, x_2, t) = 0, \ (x_1, x_2, t) \in \partial(0,1)^2 \times (0,2]. \end{cases} \tag{12}$$

The exact solution is now unknown and we estimate the errors using the double mesh principle as follows. Taking step size $\Delta t/2$, we calculate a second solution $V_{i,j}^n$ on the Shishkin mesh with $2N$ mesh points for each spatial direction. Then, we construct the interpolating quadratic polynomial $P(x_{1,i}, x_{2,j})$ of $\{V_{i,j}^n\}$ in the finest mesh and we compare with the corresponding values of $U_{i,j}^n$ in the coarse mesh, i.e., $E_{\varepsilon,N,\Delta t}(i,j,n) \simeq |P(x_{1,i}, x_{2,j}) - U_{i,j}^n|$.

ε	N=32 Δt=0.1	N=64 Δt=0.05	N=128 Δt=0.025	N=256 Δt=0.0125
2^0	3.706E-5 2.001	9.204E-6 2.007	2.291E-6 2.000	5.727E-7
2^{-4}	5.795E-3 1.619	1.887E-3 2.006	4.699E-4 2.000	1.175E-4
2^{-8}	6.205E-3 1.409	2.337E-3 1.509	8.209E-4 1.537	2.829E-4
2^{-12}	6.086E-3 1.405	2.298E-3 1.510	8.073E-4 1.587	2.687E-4
2^{-16}	6.076E-3 1.405	2.295E-3 1.510	8.056E-4 1.589	2.679E-4
2^{-20}	6.076E-3 1.405	2.295E-3 1.510	8.056E-4 1.589	2.678E-4
E_N p_{uni}	6.205E-3 1.409	2.337E-3 1.509	8.209E-4 1.537	2.829E-4

Table 1: Maximum errors and numerical orders for (11)

Table 3 shows the results for this case; from this table we observe again that the second order is achieved. In this second example, we do not have the compatibility conditions required to prove the theoretical results. Therefore, we conjecture that in practice these compatibility conditions are too strong and that similar results can be obtained for more general problems.

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ε	N=32 Δt=0.2	N=64 Δt=0.1	N=128 Δt=0.05	N=256 Δt=0.025
2^{0}	5.416E-5 2.029	1.328E-5 2.010	3.297E-6 2.000	8.242E-7
2^{-4}	1.010E-4 1.824	2.851E-5 2.011	7.071E-6 2.000	1.767E-6
2^{-8}	3.052E-4 1.874	8.326E-5 1.655	2.643E-5 2.381	5.075E-6
2^{-12}	3.417E-4 1.857	9.434E-5 1.908	2.514E-5 1.960	6.461E-6
2^{-16}	3.456E-4 1.861	9.513E-5 1.895	2.559E-5 1.949	6.628E-6
2^{-20}	3.459E-4 1.861	9.518E-5 1.894	2.561E-5 1.947	6.642E-6
E_N p_{uni}	3.459E-4 1.861	9.518E-5 1.849	2.643E-5 1.993	6.642E-6

Table 2: Maximum errors and numerical orders for (12)

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