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Flux splitting solvers for shallow water equations with source terms *

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Abstract

In this work we study some finite volume methods for shallow water equations with source terms. We can find flux difference and flux splitting solvers for hyperbolic conservation laws. In this work we analyze flux splitting methods of Steger Warming and Vijayasundaram, and construct the numerical source term, in such a way to verify an enhanced consistency property.

Keywords: Finite Volume Method, upwinding, shallow water, source terms.

Subject Classifications: AMS (MOS) : 65N06, 76B15, 76M20, 76N99.

1 Introduction.

This paper deals with the numerical solution of 1D Shallow Water Equations for channels with variable depth and width. These equations for channels of rectangular cross-section are a couple of conservation laws linking the depth h and the discharge q, which in condensated form read as follows

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x} F(W) = G_1(x, W) + G_2(x, W), \quad \text{in }]0, L[\times]0, T[. \tag{1}$$

Here $W = \begin{pmatrix} h \\ q \end{pmatrix}$ is the unknown, while F is the flux function,

$$F(W) = \begin{pmatrix} q \\ \dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \end{pmatrix}. \tag{2}$$

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Also, G_1 and G_2 are the source terms that respectively arise due to variable depth and width of the channel. These are defined by

$$G_1(x,W) = \begin{pmatrix} 0 \\ ghH'(x) \end{pmatrix} \qquad G_2(x,W) = \begin{pmatrix} -q\dfrac{b'(x)}{b(x)} \\ \\ -\dfrac{q^2}{h}\dfrac{b'(x)}{b(x)}. \end{pmatrix}, \tag{3}$$

where $H(x)$ is a function which describes the bottom of the channel with respect to a reference height ($H(x) = \bar{h} - z_b(x)$), and $b(x)$ is a function yielding the width of the channel. Both H and b are considered known.

Source terms affect only to the law of conservation of momentum. As we have written the system, it corresponds to the second equation. However, the source term due to variable width, that is G_2, does not have null his first component. That is because this first component of G_2 are not due to a really source term, it comes from the flux function.

The numerical solution of Shallow Water Equations with source terms faces the problem that low-accuracy solvers yield quite inacurate solutions, exhibiting in particular large errors in the computation of wave speed (Cf. [2]).

This difficulty is overcome if the numerical scheme solves some steady solution at least with order two. This is the Bermúdez-Vázquez enhanced consistency condition (Cf. [5]). The extension of usual solvers for homogeneous conservation laws to Shallow Water Equations with source terms has been reached for several flux-difference (Cf. [1]) schemes. However, in [4], Vázquez Cendón remarks the difficulty of the extension of flux splitting solvers to non homogeneus Shallow Water Equations.

In this work we study the methods of Steger Warming and Vijayasundaram. These two methods present two different difficulties for the extension to Shallow Water Equation with source term.

2 Numerical flux.

In this section we introduce the definition of the methods of Steger Warming and Vijayasundaram. We write system (1) as

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}F(W) = G(x,W), \tag{4}$$

where the flux function F can be written as $F(W) = A(W)W$,

For the approximation of the solution of (4) we consider the following numerical scheme:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\Big(\phi(W_i^n, W_{i+1}^n) - \phi(W_{i-1}^n, W_i^n)\Big) +$$

$$+\Delta t\,\mathcal{G}(x_{i-1}, x_i, x_{i+1}, W_{i-1}, W_i, W_{i+1}) \tag{5}$$

where $\mathcal{G}$ is the numerical source term and the function ϕ, the numerical flux.

Matrix A is diagonalizable if $h > 0$, concretly $A = X\Lambda X^{-1}$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \lambda_1 = \frac{q}{h} + \sqrt{\frac{1}{2}gh}, \quad \lambda_2 = \frac{q}{h} - \sqrt{\frac{1}{2}gh}, \quad X = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}. \tag{6}$$

This naturally yields the flux decomposition $F(W) = F^+(W) + F^-(W)$, where

$$F^{\pm}(W) = (A)^{\pm}(W)W, \quad \text{with } (A)^{\pm}(W) = X\Lambda^{\pm}X^{-1}, \tag{7}$$

and

$$\Lambda^+ = \begin{pmatrix} \max(\lambda_1, 0) & 0 \\ 0 & \max(\lambda_2, 0) \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} \min(\lambda_1, 0) & 0 \\ 0 & \min(\lambda_2, 0) \end{pmatrix}. \tag{8}$$

In accordance with this decomposition, the flux-splitting schemes are built from numerical flux functions of the form

$$\phi(U, V) = \phi^+(U, V) + \phi^-(U, V) \tag{9}$$

with

$$\phi^+(U, V) = B_1(U, V)U, \quad \phi^-(U, V) = B_2(U, V)V, \tag{10}$$

where B_1 and B_2 are 2×2 matrices that should be specified for each actual scheme. ϕ^+ and ϕ^- respectively represent upwinding to the left and to the right.

A consistent scheme for the homogeneous equation is obtained if B_1 and B_2 verify

$$F^+(W) = B_1(W, W)W, \quad F^-(W) = B_2(W, W)W. \tag{11}$$

The schemes of Steger-Warming (Cf. [3]) and Vijayasundaram (Cf. [6]), are defined respectively by $B_1(U, V) = A^+(U)$, $B_2(U, V) = A^-(V)$ and $B_1(U, V) = A^+((U + V)/2)$, $B_2(U, V) = A^-((U + V)/2)$.

3 Numerical source.

In this section we construct the numerical source function associated to the schemes of Steger Warming and Vijayasundaram.

The construction of $\mathcal{G}$ must reflect an upwinding of the source term according to the construction of the numerical flux function. In this case, matrices of re-scaling must be introduced, with the aim of achieving the condition of enhanced consistency. Following this idea we start from the following expression :

$$G = PAA^{-1}P^{-1}G = P\Big(\frac{A}{2} + \frac{|A^*|}{2} + \frac{A}{2} - \frac{|A^*|}{2}\Big)A^{-1}P^{-1}G =$$

$$= \underbrace{\frac{1}{2}\Big(G + P|A^*|A^{-1}P^{-1}G\Big)}_{\mathrm{G}_L} + \underbrace{\frac{1}{2}\Big(G - P|A^*|A^{-1}P^{-1}G\Big)}_{\mathrm{G}_R} \tag{12}$$

where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \quad \text{and} \quad A^* = P^*AP^{*-1} \quad \text{with} \quad P^* = \begin{pmatrix} c^* & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

For the sake of brevity we only define the numerical source function corresponding to $G = G_1$:

$$\mathcal{G}(x_{i-1}, x_i, x_{i+1}, W_{i-1}, W_i, W_{i+1}) = \frac{1}{2}\Big((G)_{i,L} + (G)_{i,UL}\Big) + \frac{1}{2}\Big((G)_{i,R} - (G)_{i,UR}\Big), \tag{14}$$

with the following definition for $(G)_{i,L}$, $(G)_{i,R}$, $(G)_{i,UL}$ and $(G)_{i,UR}$.

Steger Warming:

$$(G)_{i,L} = \begin{pmatrix} 0 \\ g\dfrac{h_{i-1}+h_i}{2}\dfrac{H_i - H_{i-1}}{\Delta x} \end{pmatrix}, \quad (G)_{i,R} = \begin{pmatrix} 0 \\ g\dfrac{h_i+h_{i+1}}{2}\dfrac{H_{i+1} - H_i}{\Delta x} \end{pmatrix} \tag{15}$$

We define $(G)_{i,UL}$ and $(G)_{i,UR}$ as functions of P and P^* by

$$(G)_{i,UL} = \frac{1}{\Delta x}\Bigg[P|A^*(W_i)|A^{-1}\left(\frac{W_{i-1}+W_i}{2}\right)P^{-1}\begin{pmatrix} 0 \\ g\dfrac{h_{i-1}+h_i}{2}H(x_i) \end{pmatrix} -$$

$$- P|A^*(W_{i-1})|A^{-1}\left(\frac{W_{i-1}+W_i}{2}\right)P^{-1}\begin{pmatrix} 0 \\ g\dfrac{h_{i-1}+h_i}{2}H(x_{i-1}) \end{pmatrix}\Bigg] \tag{16}$$

$$(G)_{i,UR} = \frac{1}{\Delta x}\Bigg[P|A^*(W_{i+1})|A^{-1}\left(\frac{W_i+W_{i+1}}{2}\right)P^{-1}\begin{pmatrix} 0 \\ g\dfrac{h_i+h_{i+1}}{2}H(x_{i+1}) \end{pmatrix} -$$

$$- P|A^*(W_i)|A^{-1}\left(\frac{W_i+W_{i+1}}{2}\right)P^{-1}\begin{pmatrix} 0 \\ g\dfrac{h_i+h_{i+1}}{2}H(x_i) \end{pmatrix}\Bigg] \tag{17}$$

Vijayasundaram:

$$(G)_{i,L} = \begin{pmatrix} 0 \\ g\dfrac{h_{i-1/2}+h_i}{2}\dfrac{H_i - H_{i-1}}{\Delta x} \end{pmatrix}, \quad (G)_{i,R} = \begin{pmatrix} 0 \\ g\dfrac{h_i+h_{i+1/2}}{2}\dfrac{H_{i+1} - H_i}{\Delta x} \end{pmatrix}, \tag{18}$$

where by $h_{j+1/2}$ we represent $(h_j + h_{j+1})/2$. We define $(G)_{i,UL}$ and $(G)_{i,UR}$ as

$(G)_{i,UL} =$

$$= \frac{1}{\Delta x}\Bigg[P\Big|A^*\left(W_{i-1/2}\right)\Big|A^{-1}\left(W_{i-1/2}\right)P^{-1}\begin{pmatrix} 0 \\ g\dfrac{h_{i-1}+h_i}{2}(H(x_i) - H(x_{i-1})) \end{pmatrix} \tag{19}$$

$(G)_{i,UR} =$

$$= \frac{1}{\Delta x}\left[P\left|A^*\left(W_{i+1/2}\right)\right|A^{-1}\left(W_{i+1/2}\right)P^{-1}\begin{pmatrix} 0 \\ g\frac{h_i + h_{i+1}}{2}(H(x_{i+1}) - H(x_i)) \end{pmatrix}\right. \tag{20}$$

To define the numerical source function, we should give the values of c, and c^*. These values are given in such a way that the scheme calculates in an exact way the steady solution

$$\begin{pmatrix} h \\ q \end{pmatrix} \equiv \begin{pmatrix} H \\ 0 \end{pmatrix}. \tag{21}$$

Specifically, we define $c = 2$ and $c^* = 2$. This is equivalent to

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad P^* = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}$$

Then we have:

Theorem : *The schemes defined by* (9), (10), (15), (16), (17), (18), (19), (20) *and* (22) *calculates in an exact way the steady solution* (21). □

4 Arbitrary section.

In this section we study 1D Shallow Water equations with arbitrary section. Therefore the unknowns are $Q(x,t)$ and $A(x,t)$, the discharge through the cross sectional and the area, respectively.

We denote by $\sigma(x,z)$ the breadth of the channel in (x,z). So, the area $A(x,t)$ is

$A = \int_{z_b}^{z_b+h} \sigma(x,z)dz.$ If $W = \begin{pmatrix} A \\ Q \end{pmatrix}$ then, we have the following equations:

$$\frac{\partial W}{\partial t}(x,t) + \frac{\partial F}{\partial x}(x, W(x,,t)) = \widetilde{G}(x, W(x,t)).$$

where if $\sigma_h(x,t) = \sigma(x, z_b(x) + h(x,t))$ and $\sigma_{z_b}(x) = \sigma(z_b(x))$ then F and $\widetilde{G}$ are:

$$F(x,W) = \begin{pmatrix} Q \\ \frac{Q^2}{A} + \frac{g}{2\sigma_h}A^2 \end{pmatrix}, \quad \text{and} \quad \widetilde{G}(x,W) = V(x,W) + G(x,W),$$

with

$$V(x,W) = \begin{pmatrix} 0 \\ \frac{g}{2}\left(\frac{A^2}{\sigma_h}\right)_x - g\frac{A}{\sigma_h}(A)_x \end{pmatrix}, \quad \text{and} \quad G(x,W) = G_1 + G_2,$$

$$G_1 = \begin{pmatrix} 0 \\ -g\dfrac{\sigma_{z_b}}{\sigma_h} z_b' A_1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0 \\ g\dfrac{A}{\sigma_h}\displaystyle\int_{z_b}^{z_b+h} \dfrac{\partial \sigma}{\partial x}(x,z)dz \end{pmatrix}.$$

There are two essentially differences between these equations and the ones decribed in previos sections: The flux function F depends on x and a new source term appears, V.

The term V appears when the equations are written in conservative form, it is not really a source term. On the other hand, V is the derived of F respect to x.

The discretizations of F and G are easy extensions of theses ones of previous sections. We propose a centered discretization of V, that is equivalent to build numerical flux functions which contains the expressions:

$$F(W_{i+1}) - F(W_i) - V_{i+1/2}, \quad F(W_i) - F(W_{i-1}) - V_{i-1/2}$$

and

$$F(W_{i+1/2}) - F(W_i) - V_{i+1/2}, \quad F(W_i) - F(W_{i-1/2}) - V_{i-1/2}$$

for the methods of Steger Warming and Vijayasundaram respectively.

The discretization of V is $\mathcal{V} = 1/2(V_{i-1/2} + V_{i+1/2})$, where

$$V_{i\pm1/2} = (\pm)\frac{1}{\Delta x}\begin{pmatrix} 0 \\ \dfrac{g}{2}\left(\dfrac{A_{i\pm1}^2}{\sigma_{h,i\pm1}} - \dfrac{A_i^2}{\sigma_{h,i}}\right) - g\dfrac{A_i + A_{i\pm1}}{\sigma_{h,i} + \sigma_{h,i\pm1}}(A_{i\pm1} - A_i) \end{pmatrix}$$

for the method of Steger Warming, and

$$V_{i\pm1/2} = (\pm)\frac{1}{\Delta x}\begin{pmatrix} 0 \\ \dfrac{g}{2}\left(\dfrac{A_{i\pm1}^2}{\sigma_{h,i\pm1}} - \dfrac{A_i^2}{\sigma_{h,i}}\right) - g\dfrac{A_i + A_{i\pm1/2}}{\sigma_{h,i} + \sigma_{h,i\pm1/2}}(A_{i\pm1} - A_i) \end{pmatrix}.$$

for the method of Vijayasundaram.

5 Numerical test.

We present a test which involves source terms corresponding to variable bottom, variable width and friction effects.

In [5] it is reported a limit solution of shallow-water equations with source terms, for small Froude number and "short" domains. We compare this analytical solution with Vijayasundaram's method.

We have imposed as initial and boundary condition

$$h(x,0) = H(x), \qquad q(x,0) = 0 \quad \text{and} \quad h(0,t) = \varphi(t) + H(0), \quad q(L,t) = \psi(t)$$

with

$$\varphi(t) = 4 + 4\sin\left[\pi\left(\frac{4t}{86400} - \frac{1}{2}\right)\right]; \quad \psi(t) = 0.$$

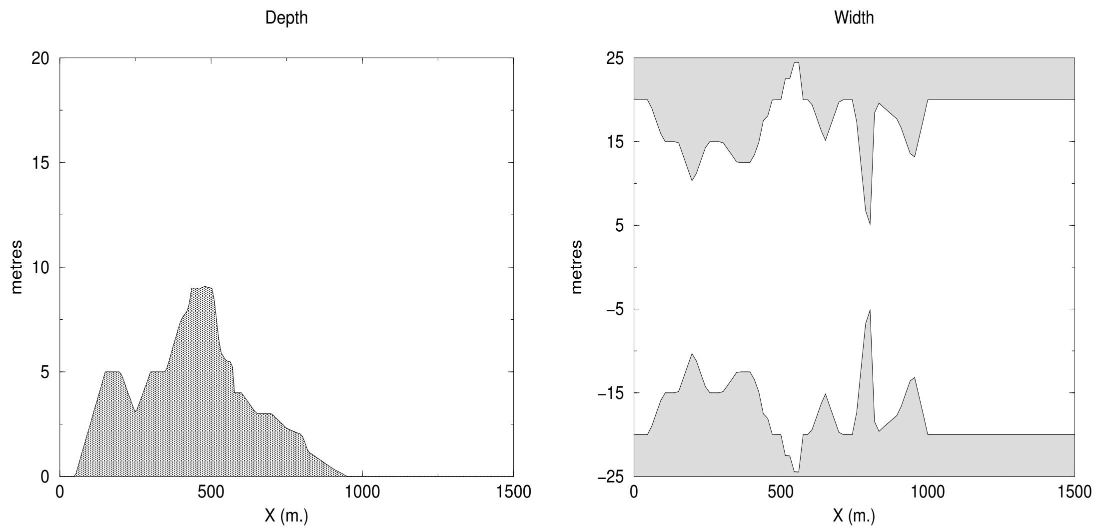

Figure 1: Test 1 : Depth function and width function.

We have taken $\Delta x = 7.5$ and a CFL condition equal to 0.8. Also, we have taken the profile bottom and width functions proposed in [5] (Figure 1). We present in Figure 2 the computed discharge, compared with the analytical solution, at time $t = 10800$.

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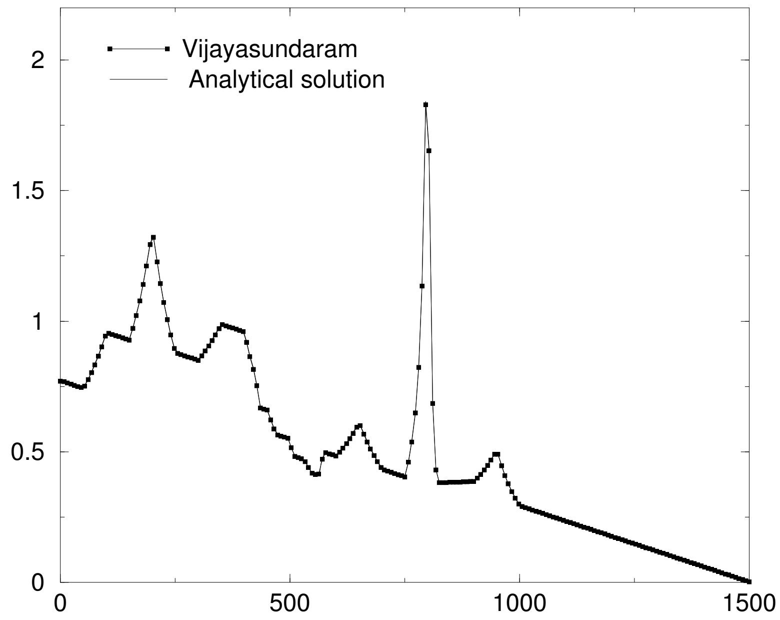

Figure 2: Test 1 : Computed versus analytical discharge at t=10800.

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