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Local optimality in quasiconcave bilevel programming *

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Abstract

Bilevel programming involves two optimization problems where the constraint region of the first level problem is implicitly determined by another optimization problem. In order to assure that they are well posed, when analyzing bilevel problems it is usually assumed that, for each value of the first level variables there will be a unique solution to the second level problem.

This paper is concerned with the behavior of local optimal solutions to the Quasiconcave Bilevel Programming (QCBP) problem when the previous assumption is dropped. Necessary and sufficient conditions for a local solution to the second level problem to be isolated are established in order to guarantee that a local solution to the QCBP problem is found which is an extreme point of the polyhedron defined by the common constraints.

Keywords: Bilevel programming, quasiconcave function, non-unique second level solution, local optimality, sensitivity analysis, extreme point

AMS Classification: 90C26, 90C30

1 Introduction

Bilevel programming has been applied to decentralized planning problems involving a decision process with a hierarchical structure. In terms of modeling, bilevel problems are programs which have a subset of their variables constrained to be an optimal solution of another problem parameterized by the remaining variables. They can be formulated as:

$$\begin{aligned} \text{Minimize} \quad & f_1(x_1, x_2), \quad \text{where } x_2 \text{ solves} \\ \text{Minimize} \quad & f_2(x_1, x_2) \\ \text{subject to:} \quad & (x_1, x_2) \in S, \end{aligned} \tag{1}$$

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where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ are the variables controlled by the first level and the second level decision maker, respectively; $f_1,\ f_2$: $\mathbb{R}^n \longrightarrow \mathbb{R}$, $n = n_1 + n_2$ and $S = \{x \in \mathbb{R}^n : g_j(x) \leq 0, j = 1, \ldots, m\}$ defines the common constraint region. Notice that the feasible region of the first level problem is defined implicitly by the second level optimization problem.

Let S_1 be the projection of S onto $\mathbb{R}^{n_1}$. Given $x_1 \in S_1$, the second level decision maker solves problem (2). So, the second level problem is parameterized by the variables of the first level.

$$\begin{aligned} \text{Minimize} \quad & f_2(x_1, x_2) \\ \text{subject to:} \quad & x_2 \in S(x_1) = \{x_2 : (x_1, x_2) \in S\}. \end{aligned} \tag{2}$$

Let $M(x_1)$ denote the set of optimal solutions to (2). Hence, the feasible region of the bilevel problem, called induced region, can be implicitly defined as

$$\text{IR} = \{(x_1, x_2^*) \ : \ x_1 \in S_1, \quad x_2^* \in M(x_1)\}.$$

The features of the problem, mainly its non-convexity, make it a difficult one, even when all involved functions (f_1, f_2, g_j, $j = 1, \ldots, m$) are linear. In fact, most results have been obtained in this case. In the nonlinear case, it is usually assumed that the second level objective function f_2 and the functions g_j are convex. A survey of references on bilevel problems, in both the linear and the nonlinear cases, can be found in [7].

In this paper a special case of (1) is considered in which functions f_1 and f_2 are quasiconcave and S is a polyhedron, which is assumed to be non-empty and bounded. This problem, known as the quasiconcave bilevel programming problem, includes as important particular cases those problems in which both objective functions are linear, are ratios of concave and convex functions or are multiplicative. Besides, under the usual assumption that for each value of the first level variables there exists a unique solution to the second level problem, it is possible to prove that there is an extreme point of the common constraint region S which solves the QCBP problem [3].

The purpose of this paper is to study the behavior of local optimal solutions to the QCBP problem when the mentioned uniqueness of the second level optimal solution is not assumed. We will establish necessary and sufficient conditions for a local solution to the second level problem to be a locally unique local solution (an isolated solution) in order to guarantee that a local solution to the QCBP problem is found which is an extreme point of the polyhedron S.

The paper is organized as follows. In the next section an example is given which allow us to show the inherent difficulties which arise when the set of optimal solutions to the second level problem is not a singleton. Besides some remarks are made about different approaches which have been considered in the literature to deal with this problem.

In section 3 we review relevant results from nonlinear sensitivity analysis theory and introduce some basic concepts useful in proving the main results of the paper, given in section 4. Finally, in section 5 concluding remarks are drawn.

2 What does it happen when non-unique optima solutions to the second level problem are allowed?

Problems caused by the existence of multiple optima when solving the second level problem for given $x_1 \in S_1$ have already been considered for the linear bilevel problem (see [1, 2]). The following simple QCBP problem (3) allows us to show that the first level decision maker could not reach his optimal decision without 'force' the decision of the second level decision maker. Let us consider problem (3) in which x_1 is the variable controlled by the first level decision maker and x_2 is the variable controlled by the second level one.

$$\begin{array}{ll} \text{Minimize} & x_1 + x_2, \quad \text{where } x_2 \text{ solves} \\ \text{Minimize} & \dfrac{-x_1 + 2x_2 + 7}{x_1 + x_2 + 2} \\ \text{subject to:} & \\ & 3x_1 - 5x_2 \leq 15 \\ & 3x_1 - x_2 \leq 21 \\ & 3x_1 + x_2 \leq 27 \\ & 3x_1 + 4x_2 \leq 45 \\ & x_1 + 3x_2 \leq 30 \\ & x_1, x_2 \geq 0 \end{array} \tag{3}$$

Notice that for $x_1 = 1$ the second level problem has multiple optima, $M(1) = [0, \frac{29}{3}]$. So, the optimization problem of the first level is ill-posed. The first level decision maker's problem would be

$$\begin{array}{ll} \text{Minimize} & f_1(x_1, x_2^*) \\ \text{subject to:} & x_1 \in S_1, x_2^* \in M(x_1), \end{array} \tag{4}$$

which is not well defined at points where $M(x_1)$ is not a singleton.

For evaluating $f_1(1, x_2^*)$ it is necessary to give a rule for selecting $x_2^* \in M(1)$. Moreover, the best value for the first objective function is $f_1 = 1$ obtained when $x_1 = 1$ and $x_2 = 0$. However the first level decision maker can not force this value because the second level decision maker is indifferent to each x_2 in the interval $[0, \frac{29}{3}]$.

On the other hand, if the first level objective function was $f_1 = -x_1 + 3x_2$, then the first level decision maker could reach his minimum $f_1 = 5$, obtained when $x_1 = 5$, since the second level problem given $x_1 = 5$ has a unique optimal solution $x_2 = 0$.

Different approaches have been proposed in the literature to assure that the bilevel problem is well posed. The most common one is to assume that, for each value of the first level variables x_1, there is a unique solution to the second level problem, that is, the set $M(x_1)$ is a singleton for all $x_1 \in S_1$. Some authors have considered the weaker assumption that $M(x_1^*)$ must be a singleton only if the selection x_1^* provides an optimal solution for the bilevel problem.

Other approaches focus on the way of selecting $x_2^* \in M(x_1)$, in order to evaluate $f_1(x_1, x_2)$, when $M(x_1)$ is not a singleton. Among the rules that have been proposed [5], it is worth mentioning the optimistic or weak approach and the pessimistic or strong approach. The first one assumes that there exists cooperation between levels, so that the second level decision maker always select the variables x_2 to provide the best value of f_1, i.e.

$$x_2^* = \text{argmin}\{f_1(x_1, x_2) : x_2 \in M(x_1)\}.$$

In the second one, the first level decision maker behaves as if the second level decision maker always selects the optimal decision which gives the worst value of f_1, i.e.

$$x_2^* = \text{argmax}\{f_1(x_1, x_2) : x_2 \in M(x_1)\}.$$

Finally, other approaches perturbs the second level problem [4] so that, under not too restrictive assumptions, the optimal solution of the regularized problem is uniquely determined.

In this paper, we investigate the possibility of finding a continuous function $x_2(x_1)$ such that for each $x_1 \in S_1$ a point $x_2 \in S(x_1)$ will be uniquely determined which will be optimal in a local sense. Under the assumptions that guarantee the existence of this function, we will prove that there exists an extreme point of S which is a local solution to the QCBP problem.

3 Local optimality and sensitivity result

As far as we know, the concept of local solution in bilevel programming was first introduced by Falk and Liu [6]. For the sake of self-containedness, we briefly list in this section some useful definitions and a proposition presented in [6].

Definition 1. *A point (x_1, x_2) is a semi-local solution to (1) if $(x_1, x_2) \in S$ and x_2 is a local solution to (2) with x_1 fixed.*

Definition 2. *A point (x_1^*, x_2^*) is said to be a [strict] local solution to (1) if* *(a)* (x_1^*, x_2^*) *is a semi-local solution,*

(b) there exists a neighborhood U *of* (x_1^*, x_2^*) *such that* $[f_1(x_1^*, x_2^*) < f_1(x_1, x_2)]$ $f_1(x_1^*, x_2^*) \leq f_1(x_1, x_2)$ *for all semi-local solutions* $(x_1, x_2) \in U$*.*

Theorem 3. *(Proposition 2.7 in [6]) Suppose Karush-Kuhn-Tucker (KKT), Strong Second-Order Sufficient (SSOS) and Linear Independence (LI) conditions hold at* x_2^0 *with multipliers* u^0 *for (2) with* $x_1 = x_1^0$*, and that functions* f_2 *and* $g_j, j = 1, \ldots, m$ *are* $C^{(2}$ *in a neighborhood of* $(x_1^0,\, x_2^0)$*.*

Then, for x_1 *in a neighborhood of* x_1^0*, there exists a unique continuous function* $z(x_1) = (x_2(x_1),\, u(x_1))$ *satisfying KKT, SSOS and LI at* $x_2(x_1)$ *with* $u(x_1)$ *for (2) with* x_1*, such that* $z(x_1^0) = (x_2^0,\, u^0)$ *and* $x_2(x_1)$ *is a locally unique local solution of (2) with* x_1*.*

This theorem states that for a fixed strategy x_1^0 of the first level decision maker, if SSOS and LI hold at a local optimal strategy x_2^0 of the second level decision maker given x_1^0, then for each strategy x_1 of the first level decision maker near x_1^0 the second level decision maker has a locally unique optimal strategy x_2 near x_2^0.

In other words, if the first level decision maker makes a decision x_1^0 and x_2^0 is the local optimal strategy of the second level decision maker with respect to x_1^0, it follows from theorem 3 that there exists a $x_2(x_1)$ defined in the neighborhood M of x_1^0 such that $x_2(x_1^0) = x_2^0$ and $x_2(x_1)$ is the local optimal strategy of the second level decision maker with respect to the strategy x_1 in M. Hence, the set of semi-local solutions can be projected into x_1-space and we can define $F(x_1) = f_1(x_1, x_2(x_1))$ for $x_1 \in M$; then the bilevel problem locally reduce to the unconstrained minimization problem $\text{Minimize}_{x_1} F(x_1)$.

4 Properties of a bilevel local solution to the QCBP problem

Throughout the remainder of the paper we restrict our attention to QCBP problems for which the following strong regularity condition is verified:

For each value of $x_1 \in S_1$*, problem (2) has at least a local solution, and SSOSC and LI conditions are satisfied at all local solutions.*

Taking into account the previous results, we propose to consider the following definition of induced region of the QCBP problem:

$$\text{IR} = \left\{ (x_1,\, x_2) \in S \;:\; x_2 \text{ is a locally unique (isolated) local solution to } \min_{y \in S(x_1)} f_2(x_1, y) \right\}$$

as the feasible region of the QCBP problem. Next we prove some properties on the geometry of IR.

Lemma 4. *IR lies on the boundary of S.*

Proof. Let $(\overline{x}_1, \overline{x}_2)$ be a point of IR. Since $\overline{x}_2$ is an isolated local solution to (2) for $x_1 = \overline{x}_1$, there exists a neighborhood U around it such that $\overline{x}_2$ is the unique local solution, hence $f_2(\overline{x}_1, \overline{x}_2) < f_2(\overline{x}_1, x_2) \;\; \forall x_2 \in U$.

If $\overline{x}_2$ belonged to the interior of $S(\overline{x}_1)$ (which is a non-empty and compact polyhedron), then there would exist $x_2^1,\ x_2^2 \in U$, and a $\lambda \in (0,\, 1)$ such that $\overline{x}_2 = \lambda x_2^1 + (1-\lambda)x_2^2$. Since f_2 is quasiconcave on $S(x_1)$ then $f_2(\overline{x}_1, \overline{x}_2) \geq \min\{f_2(\overline{x}_1, x_2^1),\, f_2(\overline{x}_1, x_2^2)\}$, which would contradict that $\overline{x}_2$ is an isolated local minimum.

Moreover, $\overline{x}_2$ is an extreme point of $S(\overline{x}_1)$. Otherwise, there would be a non-empty face $S^j(\overline{x})$ of $S(\overline{x}_1)$ such that $\overline{x}_2$ would belong to the relative interior of $S^j(\overline{x})$. In the same way as before, this fact contradicts that $\overline{x}_2$ is an isolated local minimum. As a consequence, $(\overline{x}_1, \overline{x}_2)$ belongs to the boundary of S. □

It readily follows from lemma 4 that for each $(x_1, x_2) \in$ IR, there exists a face $S^j \neq S$, such that $(x_1, x_2) \in \text{ri } S^j$, where ri S^j denotes the relative interior of S^j.

Lemma 5. *Let $(x_1^0, x_2(x_1^0)) \in ri\ S^j \cap IR$. Then, $S^j \cap V = \{(x_1^0, x_2(x_1^0))\}$, where $V = \{(x_1^0, x_2) : x_2 \in S(x_1^0)\}$.*

Proof. Suppose that there exists another point $(x_1^0, x_2^0) \in S^j \cap V$.

Since $(x_1^0, x_2(x_1^0)) \in \text{ri } S^j$ and $(x_1^0, x_2^0) \in S^j$, there exists a $\mu > 1$ such that $(x_1^0, x_2) = \mu(x_1^0, x_2(x_1^0)) + (1-\mu)(x_1^0, x_2^0) \in S^j$.

If we set $\lambda = (\mu-1)/\mu$, then $0 < \lambda < 1$ and we can write $(x_1^0, x_2(x_1^0)) = (1-\lambda)(x_1^0, x_2) + \lambda(x_1^0, x_2^0)$, contradicting that $x_2(x_1^0)$ is an extreme point of $S(x_1^0)$. □

Lemma 6. *Let S^j be a non-empty face of S and let $(x_1^0, x_2(x_1^0)) \in IR$. If $(x_1^0, x_2(x_1^0)) \in ri\ S^j$, then $S^j \subset IR$.*

Proof. Let $(y_1, y_2) \in S^j$. Since $(x_1^0, x_2(x_1^0)) \in \text{ri } S^j$, there exists a neighborhood $\tilde{U}$ around it such that $U = \tilde{U} \cap \text{aff } S^j \subset \text{ri } S^j$, where aff S^j denotes the affine hull of face S^j.

Since $x_2(x_1^0)$ is a locally unique local solution to (2) with x_1^0, there exists a neighborhood V_1 around x_1^0 and a unique function $z(x_1) = (x_2(x_1),\, u(x_1))$ such that $x_2(x_1)$ is a locally unique local solution to (2) with $x_1 \in V_1$. Then, by the continuity of z, there exists a neighborhood V_2 around x_1^0 such that $\{(x_1, x_2(x_1)) : x_1 \in V_2\} \subset U$.

Let us choose $0 < \mu_1 < 1$ such that $V_3 = \{x_1 \in S_1 : x_1 = \mu y_1 + (1-\mu)x_1^0, 0 \leq \mu \leq \mu_1\} \subset V_2$. Then, $\{(x_1, x_2(x_1)) : x_1 \in V_3\} \subset U$. Moreover, for all $\lambda \in [0,1)$, $\lambda(y_1, y_2) + (1-\lambda)(x_1^0, x_2(x_1^0)) \in \text{ri } S^j$. Thus, as a result of lemma 5, $x_2(x_1) = \mu y_2 + (1-\mu)x_2(x_1^0)$ for all $x_1 = \mu y_1 + (1-\mu)x_1^0 \in V_3$.

Hence, for all $0 \leq \mu \leq \mu_1$, $(x_1, x_2) = \mu(y_1, y_2) + (1-\mu)(x_1^0, x_2(x_1^0)) \in$ IR. In particular, $(x_1^1, x_2(x_1^1)) = \mu_1(y_1, y_2) + (1-\mu_1)(x_1^0, x_2(x_1^0)) \in \text{ri } S^j \cap$ IR and, by repeating the process, we can construct from it a new point $(x_1^2, x_2(x_1^2)) \in \text{ri } S^j \cap \text{IR}$, and so on. Therefore, we approach the point (y_1, y_2) along the line segment between (y_1, y_2) and $(x_1^0, x_2(x_1^0))$, by points belonging to IR. This implies that $(y_1, y_2) \in \text{IR}$. □

Lemma 7. *IR is piecewise linear.*

Proof. It suffices to prove that $\text{IR} = \bigcup_{j \in J} S^j$ where $J \subset \{1, \ldots, r\}$ and $S^1, \ldots, S^r$ are the non-empty faces of S. Let $(x_1, x_2(x_1)) \in \text{IR}$. Then, $(x_1, x_2(x_1)) \in \text{ri } S^j$ for some $j \in \{1, \ldots, r\}$ and, from lemma 6, $S^j \subset \text{IR}$. Hence, there exists a set of indices $J \subset \{1, \ldots, r\}$ such that $\text{IR} = \bigcup_{j \in J} S^j$. □

Lemma 8. *IR is conected.*

Proof. It readily follows from the fact that S_1 is connected and IR is the image of S_1 by the continuous map $\text{IR}(\cdot) : S_1 \to \mathbb{R}^n$ such that $x_1 \to (x_1, x_2(x_1))$. □

As a consequence of previous lemmas, we can conclude that the feasible region of the QCBP problem is comprised of the union of connected faces of S. Hence, the QCBP problem can be equivalently formulated as:

$$\begin{aligned} \text{Minimize} \quad & f_1(x_1, x_2) \\ \text{subject to:} \quad & (x_1, x_2) \in \text{IR} = \bigcup_{j \in J} S^j. \end{aligned} \tag{5}$$

This fact allows us to prove the main result of the paper regarding the local optimality to the QCBP problem.

Theorem 9. *There is an extreme point of S which is a local optimal solution to the QCBP problem.*

Proof. According to (5), the first level decision maker minimizes a continuous function over a compact set, so that there exists a minimizing solution to the QCBP problem. Let this be $(x_1^0, x_2(x_1^0))$.

Then, there exists at least one $j \in J$ such that $(x_1^0, x_2(x_1^0)) \in S^j$, and $(x_1^0, x_2(x_1^0))$ is a minimizing solution to the problem

$$\begin{aligned} \text{Minimize} \quad & f_1(x_1, x_2) \\ \text{subject to:} \quad & (x_1, x_2) \in S^j, \end{aligned} \tag{6}$$

Since f_1 is a quasiconcave and continuous function on S^j and S^j is a non-empty compact polyhedron, there exists an extreme point of S^j (therefore an extreme point of S) which is an optimal solution to problem (6), thus giving the same value of the objective function as $(x_1^0, x_2(x_1^0))$. Therefore this extreme point of S is a local optimal solution to the QCBP problem. □

5 Conclusions

In this paper, the assumption of uniqueness of the optimal solution to the second level problem in the quasiconcave bilevel programming problem has been relaxed. Under assumptions of regularity, we have proved that there exists an extreme point of the common constraint region S which is a local optimal solution to the QCBP problem. This result allow us to consider enumerative algorithms in order to solve the problem.

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