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Derivatives of subspaces V_j in the 0 and 1 MRA of B-splines

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Abstract

If we consider the B-splines's MRA of 0 and 1 orders, there is a relationship between the wavelets of one of them and the functions of the Riesz's basis of another. Besides, we have found a characterization for the two-scale sequences and Riesz's basis of these MRAs. Due to this new decomposition, the functions in V_j can be written using two expressions depending on the possible applications.
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1 Multirresolution Analysis

Definition 1 *Let's say that a chain of closed subspaces of $L^2(\mathbb{R})$ constitutes a MRA, [1], if and only if verifies*

1. $-\ \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots;$
2. $-\ \overline{\bigcup_j V_j} = L^2(\mathbb{R});$
3. $-\ \bigcap_j V_j = \{0\};$
4. $-\ f(x) \in V_j \iff f(2x) \in V_{j+1};$

5.- $\exists\ \phi_0 \in V_0$, such that $\{\phi_0(. - n);\ n \in \mathbb{Z}\}$ is a Riesz's basis of V_0. This function is called scaling function.

There are another definitions of Multirresolution Analysis where the scaling function constitutes an orthonormal basis of the closed sub-spaces of the chain. Due to this definition and using someone of his properties, its can be proved that there exists a chain of new subspaces called *Orthogonal Complements,* W_j associated to each V_j, verifying: $V_{j+1} = V_j \bigoplus W_j$, furthermore this sum is orthogonal; Another important consequence that we want to remark is: $f(x) \in V_j \iff f(x - 2^{-j}n) \in V_j$. Let $\phi_{j,n}(t) := 2^{j/2}\phi_0(2^j t - n)\ \ \forall j, n \in \mathbb{Z}$. Where $\phi_{0,0} \equiv \phi_0$.

1.1 MRA of B-splines: Definition and Properties:

In the literature of Multirresolution Analysis of splines there is one important family of these, the B-splines or basic splines:

Definition 2 *Let $n \in \mathbb{N}$, we will define the* **B-splines of order n**, *$N_n(x)$, as follows:*

$$i) N_0 = \chi_{[0,1]}, \quad ii)\ \forall n > 0, \quad N_{n+1} = N_0 * \cdots^{n+1} * N_0 = N_n * N_0 \tag{1}$$

Let us consider $n = 0, 1, 2, \ldots$, this B-spline's family verify many important properties, however we are going to remark the following ones for its application in our work:

1. $-\ N_n(x) > 0 \qquad \forall x \in (0, n+1).$
2. $-\ \mathrm{Supp}\, N_n = [0, n+1].$
3. $-\ N_n \in S^n(\mathbb{Z}) \bigcap L^2(\mathcal{R}).$
4. $-\ \sum_{k \in \mathbb{Z}} N_n(x-k) = 1.$
5. $-\ N'_{n+1}(x) = N_n(x) - N_n(x-1).$

Proposition 1 *For each n=0,1,2,... the set $\{N_n(t-m)\}_{m \in \mathbb{Z}}$ is a Riesz system in $L^2(\mathbb{R})$.*

After introducing the definition and properties of these B-splines, we are going to see when they define a MRA.

Theorem 1 *For each n=0,1,2,..., the subspaces $\mathrm{V_j^n} = \{\mathrm{S_2^n}(2^{-\mathrm{j}}\mathbb{Z})\}_{\mathrm{j} \in \mathbb{Z}}$ form a MRA. Where we denote $\mathrm{S_2^n} = \mathrm{L^2} \bigcap \mathrm{S^n}$.*

Two-scale relations:

All the Multirresolution Analysis can be characterized by a sequence that it is denoted $\mathbf{h}_n$ and it is defined by $\mathbf{h}_n :=< \phi_{1,n}, \phi_{0,0} >$; when the scaling function defines an orthonormal basis, these sequences are the coefficients of the decomposition of the functions in the basis of the correspondent subspaces V_j. In general, through this sequence, we can to obtain one first relation-ship between the functions of the different levels. In B-spline's case, another coefficients [1] are given by the expression: $N_m(x) = \sum_{k=0}^{m} 2^{-m+1} \begin{pmatrix} m \\ k \end{pmatrix} N_m(2x-k)$. Observing this expression, this formula relates B-splines of different and consecutive levels; so, these coefficients are called **two-scale sequence**.

If we consider the called **Haar's Wavelet** or 0-spline wavelet, denoted by $\psi_{0,0}^0$, and the scaling function of the 1 B-spline's MRA then:

$$\tilde{\phi}_{0,0}^1(t) = \begin{cases} t & \text{si } t \in [0,1] \\ -t+2 & \text{si } t \in [1,2] \end{cases} \qquad \psi_{0,0}^0(t) = \begin{cases} 1 & \text{si } t \in [0,1/2] \\ -1 & \text{si } t \in [1/2,1] \end{cases}$$

So, we are working with two Multirresolution Analysis verifying the following condition

$$D\left(\tilde{\phi}_{0,0}^1\right)(t) = \left(\frac{d}{dt} N_1\right)(t) = \sqrt{2}\psi_{-1,0}^0(t) = 2^{1/2}\, \psi_{0,0}^0(2^{-1}t) \tag{2}$$

2 Derivatives of scaling function and wavelet functions in 0 and 1 B-splines's MRA

If we calculate the derivatives of the wavelet functions and the scaling functions associated to them, we have that the derivative of the **scaling function** is

$$\frac{d\phi^1_{0,0}}{dt}(t) = \sum_k a^1_k\, g'(t-k)$$
$$= \sum_k a^1_k\, \psi^0_{0,0}\left(\frac{t}{2} - \frac{k}{2}\right) = \sqrt{2}\left[\sum_p a^1_{2p}\, \psi^0_{-1,p}(t) \;+\; \sum_p a^1_{2p+1}\, \psi^0_{-1,p}(t-1)\right] \quad (3)$$

Analogously, we can obtain the expressions that are verified for these functions in any level of the MRA. Thus, $D(\tilde{\phi}^1_{j,n}(t)) = 2^j\, 2^{j/2}\, 2^{1/2}\, \psi^0_{-1,0}(2^j t - n) = 2^{(3j+1)/2}\, \psi^0_{-1,0}(2^j t - n)$. And, the derivative of the **Wavelet function** at the level j can be expressed by $D\left(\psi^1_{j,n}(t)\right) = 2^{7j/2} \sum_m \sum_p (-1)^m\, \overline{h}^1_{1-m}\, a^1_p\, \psi^0_{-1,n}(2^{j+1}t - p - m)$.

2.1 Relationship between sequences $\mathbf{h}^1$ and $\mathbf{h}^0$

We denote $\mathbf{h}^1 := \{< \phi^1_{0,0},\, \phi^1_{1,n} >\}_n$, $\mathbf{h}^0 := \{< \phi^0_{0,0},\, \phi^0_{1,n} >\}_n$

From the fact that N_1 is a Riesz's base of V^1_0 and deriving the decomposition obtained of $\phi^i_{0,0} \in V^i_0 \subset V^i_1$,where$\{\phi^i_{1,n}\}_n$, $i = 0, 1$ let be an orthonormal base,it is got:

$$\left(\frac{d}{dt}\phi^1_{0,0}\right)(t) = 2^{3/2} \sum_m h^1_m \left(\frac{d}{dt}\phi^1_{0,0}\right)(2t - m) = 2^2 \sum_{p,k} h^1_{p-k}\, a_k \psi^0_{0,0}(t - 2^{-1}p) \quad (4)$$

We compare the two previous expressions putting them in function of the same elements of the orthonormal basis, it is got for each $m \in \mathbf{Z}$:

$$2^{1/2} \sum_k \sum_q (-1)^q\, \overline{h^0}_{1-q}\, h^0_{m-2(q+k)}\, a_k = \sum_k \sum_q (-1)^{m-q}\, \overline{h^0}_{1-m+q}\, h^1_{q-k}\, a_k \quad (5)$$

Remark 1 *We want to have an important mention about this result. We have put our interest in the B-splines's MRA case, but unless the relations of the derivatives with these splines, we have only used, properties that all MRA, with an analogous condition about the derivative of the scaling function and wavelet function. So that, the relation (5) could be generalized.*

2.2 Z-Transform relations:

However, the previous relations are more interesting for the possible applications, if we work with the Z-transforms of these functions, because this transform permit us to consider analytic functions instead of sequences. Moreover, with the new expressions that we are going to obtain, a filter interpretation is possible. The following results and notations are necessary:

$$(\mathcal{U}\mathbf{a})_n := \begin{cases} a_k & \text{si } n = 2k \\ 0 & \text{if } n = 2k-1 \end{cases} \Rightarrow \mathcal{U}(z) = A(z^2) \qquad \widetilde{a}_n := a_{-n} \Rightarrow \widetilde{A}(z) = A(1/z)$$

$$a_{-1,n} := (-1)^n\, a_n \Rightarrow A_{-1}(z) = A(-z) \qquad a_{1,n} := (-1)^n\, a_{1-n}$$

$$\tau_k a_n := a_{n-k} \Rightarrow T(z) = z^{-k} A(z) \qquad \Rightarrow A_1(z) = -\tfrac{1}{z}\, A(-\tfrac{1}{z})$$

If we operate in adequate way in both members of the (5) and we use the above results, it is became

$$2^{1/2}\, z^{-1}\; A(z^2)\, \overline{H^0(\overline{z})}\, H^0(-z^{-2}) = H^1(z)\, A(z)\, \overline{H^0(-1/\overline{z})} \tag{6}$$

3 Characterization of functions through Z-transforms

A) $\underline{x(t) \in V_0^1}$: Let $h_n^x :=< x,\, x_{1,n} >$, and let we denote $\mathbf{g}^0 := \{< g,\, \phi_{1,p}^1 >\}_p$ and $\mathbf{g}^1 := \{< g(2.),\, \phi_{1,p}^1 >\}_p$, where it is can be seen $\mathbf{g}^1 = \{< g,\, \phi_{0,p}^1 >\}_p$, $\mathbf{g}^0 = \{2^{1/2} \sum_p g_p^1\, h_{q-2p}^1\}_q$ and so $G^0(z) = 2^{1/2}\, G^1(z^2)\, H^1(z)$. As we have an orthonormal basis:

$$\begin{aligned} h_n^x =< x,\, x_{1,n} > &= 2^{1/2} \sum_{k,p} a_k\, \overline{a_p}\; < g(t-k),\, g(2t-n-p) > \\ &= 2^{1/2} \sum_{k,p,q,l} a_k \overline{a}_p\, g_q^0\, g_l^1 \delta_{2k+q,n+p+l} = \left(\widetilde{\overline{\mathbf{a}}} * \mathbf{g}^1 * \widetilde{\widetilde{\mathcal{U}\mathbf{a}} * \mathbf{g}^0} \right)(n) \end{aligned} \tag{7}$$

These scalar products, cannot be in general obtained, but we are working with functions with compact support, the B-splines, so, there are a finite numbers of these products and we can calculate them.

B-splines: In this case, we can calculate the scalar products. Taking Z-Transforms and substituting these products in our expression, we would have got

$$H^x(z) = \frac{\sqrt{2}}{24}\, A(z^2)\; \overline{A(\frac{1}{\overline{z}})}\, \left[\frac{1}{z^3} + \frac{6}{z^2} + \frac{10}{z} + z + 6 \right] \tag{8}$$

B) $\underline{\{x(t-n); n \in \mathbf{Z}\} \text{ orthonormal basis of } V_0^1}$: Now, we have that $\delta_{n,m} =< x_n,\, x_m >= \sum_p a_p\, \overline{a}_q < g(t-p),\, g(t-(m+q)) >= \frac{2}{3} \sum_p a_p\, \overline{a}_{p-s} + \frac{1}{6} \sum_p a_p \left(\overline{a}_{p-(s+1)} + \overline{a}_{p-(s-1)} \right)$. If we again consider its Z-transform, we obtain

$$1 = A(z)\; \overline{A(\frac{1}{\overline{z}})}\, \left[\frac{2}{3} + \frac{1}{6z} + \frac{z}{6} \right] \tag{9}$$

Thus, we have obtained a **Necessary condition** for the candidates to be a scaling function of a B-spline MRA. Getting $x \equiv \phi^1_{0,0}$

$$H^1(z) = \frac{\sqrt{2}}{4}\,\frac{A(z^2)}{A(z)}\,\frac{z^4+6z^3+10z^2+6z+1}{z^2(z^2+4z+1)} \tag{10}$$

4 Derivatives spaces

We remind that our initial hypothesis is given by (4). We want to introduce a new decomposition of these spaces $V^1_j = V^1_{j-1} \bigoplus W^1_{j-1}$, which allows us to obtain their elements in function of the elements of the another MRA, $\mathcal{A}_0$. Firstly, we are going to define a new subspaces that they are necessary, for that the translations of the wavelets being in adequate spaces. It is know that the wavelet functions are in the orthogonal complements W_j and in these spaces $\psi(t) \in W_j \iff \psi(t-2^{-j}n) \in W_j$ it's verified. However, we are interested in other translations, so we cannot assure that the new function be in these complements. Thus, we need the following definition

Definition 3 *We will call* **Shifted spaces of the orthogonal complements** *to*

$$\overline{W^m_j} := \left\langle \{\psi^m_{j,p}(t-2^{-j-1}) \,/\, \psi^m_{j,p} \ \in W^m_j\} \right\rangle$$

Then, due to this definition, we can see in (3) that $\psi^0_{-1,p}(.-1) \in \overline{W^0_j}$. After introducing those complements, we are going to find one relation between the derivative subspaces de $\mathcal{A}^0$ and the ones of $\mathcal{A}^1$.

Theorem 2 *In any pair of Multirresolution Analysis verifying (2) is got*

$$D\left(V^1_0\right) = W^0_{-1} + \overline{W^0_{-1}}$$

Proof: We are going to seeing for double inclusion,
$\underline{D\left(V^1_0\right) \subset W^0_{-1} + \overline{W^0_{-1}}}$: Let $v^1_0(t) \in V^1_0 \Rightarrow v^1_0(t) = \sum_k \alpha_k\, \phi^1_{0,0}(t-k)$, deriving this expression:

$$\begin{aligned}\frac{d\,v^1_0(t)}{dt} &= \sum_k \alpha_k \frac{d\,\phi^1_{0,0}(t-k)}{dt} \\ &= \sqrt{2}\sum_k \alpha_k \left[\sum_p a_{2p}\,\psi^0_{-1,p}[(t-k)] + \sum_p a_{2p+1}\,\psi^0_{-1,p}[(t-k-1)]\right]\end{aligned} \tag{11}$$

For proving, we must make difference between $k = 2m$, or $k = 2m+1$ $m \in \mathbf{Z}$. In both cases the result is $\psi^0_{-1,p}(.-2m-1) \in \overline{W^0_{-1}}$ and $\psi^0_{-1,p}(.-2m) \in W^0_{-1}$.
$\underline{W^0_{-1} + \overline{W^0_{-1}} \subseteq D(V^1_0)}$: We are going to make this proof in two previous steps:

$\underline{W^0_{-1} \subseteq D(V^1_0)}$: As $\psi^0_{-1,n}(t) = \psi^0_{0,0}(2^{-1}t - n) = 2^{-1/2}\,(\frac{d}{dt}N_1)(t-2n) = \frac{d}{dt}(N_1(t-2n))$ and$N_1 \in V^1_0 \Longrightarrow N_1(t-n) \in V^1_0 \;\; \forall n \in \mathbf{Z} \Longrightarrow N_1(t-2n) \in V^1_0$. Then, $\psi^0_{-1,n} \in D(V^1_0)$.

$\underline{\overline{W^0_{-1}} \subseteq D(V^1_0)}$: Now, the basic functions are $\psi^0_{-1,0}(. - 1) = \psi^0_{0,0}(2^{-1}. - 2^{-1} - n)$. So that,

$$\left.\begin{array}{rl} \psi^0_{0,0}(2^{-1}t - 2^{-1} - n)) & = 2^{-1/2}(\frac{d}{dt}N_1)(t-1-2n) \\ N_1 \in V^1_0 & \Longrightarrow N_1(. - 1 - 2n) \in V^1_0 \end{array}\right\} \Rightarrow \psi^0_{-1,n}(t-1) \in D(V^1_0).$$

So that, as we are working with vectorial subspaces, $W^0_{-1} + \overline{W^0_{-1}} \subseteq D(V^1_0)$. Furthermore, the sum is direct since this subspaces are disjoints.

Theorem 3 *In the B-splines's MRA* $\mathbf{D(V^1_0) \subseteq V^0_0 = W^0_{-1} \bigoplus V^0_{-1}}$ *it is verified. Also, this result can be to generalize to anyone MRA verifying similar condition to (2).*

Proof:
B-splines: For that, we are first going to prove the inclusion of the each subspace. Thus, it is obvious that $\overline{W^0_{-1}} \subseteq V^0_0$, because

$$\psi^0_{-1,0}(t) \in W^0_{-1} \subseteq V^0_0 \Longrightarrow \left\{ \begin{array}{rl} \exists \;\; \{\xi_m\}_{m\in\mathbf{Z}} & \in l^2(\mathbf{Z}) \quad \text{such that} \\ \psi^0_{-1,0}(t) = & \sum\limits_m \xi_m N_0(t-m) \end{array}\right.$$

$$\psi^0_{-1,n}(t-1) = \psi^0_{0,0}[2^{-1}(t-2n-1)] = \psi^0_{-1,0}(t-2n-1)$$

and then, $\psi^0_{-1,0}(t-2n-1) = \sum_m \xi_m\, N_0(t-2n-1-m) = \sum_p \xi_{p-2n-1}\, N_0(t-p) \in V^0_0$.

As $\{\xi_p\}_p \in l^2(\mathbf{Z})$ and $N_0(t-k) \in V^0_0 \Longrightarrow \overline{W^0_{-1}} \subseteq V^0_0$.

The another inclusion, $W^0_{-1} \subseteq V^0_0$, is obvious. By that, we can assure that $D(V^1_0) \subset V^0_0$ for the B-splines's MRA

Any MRA: The constant of the expression (11), $2^{-1/2}$, can be modified, and this one leads us to any small changes in the constants of our expression. However, if we keep this constant, we have $\psi^0_{-1,0}(t) \in W^0_{-1} \subseteq V^0_0 \Longrightarrow \exists \;\; \{\xi_m\}_{m\in\mathbf{Z}} \in l^2(\mathbf{Z})$ such that $\psi^0_{-1,0}(t) = \sum\limits_m \xi_m N_0(t-m)$

Besides: $\psi^0_{-1,n}(t-1) = \psi^0_{0,0}[2^{-1}(t-2n-1)] = \psi^0_{-1,0}(t-2n-1)$. So that, we can conclude that

$$\psi^0_{-1,0}(t-2n-1) = \sum_m \xi_m\, N_0(t-2n-1-m) = \sum_p \xi_{p-2n-1}\, N_0(t-p) \in V^0_0$$

since $\{\xi_p\}_p \in l^2(\mathbf{Z})$ y $N_0(t-k) \in V^0_0$. So, it is got $\overline{W^0_{-1}} \subseteq V^0_0$, and for that $D(V^1_0) \subset V^0_0$.

■

5 Generalization of this decomposition

Theorem 4 *There exists two subspaces A and B such that they verify*

$$A,\ B \in L^2(\mathcal{R}) \ \ with\ V_0^1 = A \bigoplus B; \quad D(A) = W_{-1}^0, \quad D(B) = \overline{W_{-1}^0}$$

Furthermore, every function belonging to V_0^1 can be reconstructed by

$$v(t) = \sum_n \left(a(2n+1)g(t-2n) \ + \ b(2n)g(t-2n-1)\right)$$

Proof:
<u>*Existence:*</u> Let $v_0^1(t) \in V_0^1 \Longrightarrow v_0^1(t) = \sum_n \alpha_n \phi_{0,0}^1(t-k)$ and it is had too $\frac{d\phi_{0,0}^1(t)}{dt} \in W_{-1}^0 + \overline{W_{-1}^0}$. So,$v_0^1(t) = a(t) + b(t)$ where $\frac{da}{dt}(t) \in W_{-1}^0$, $\frac{db}{dt}(t) \in \overline{W_{-1}^0}$. Thus, if we integrate it and using the fact that $g(t) \in V_0^1 \Longrightarrow g(t-n) \in V_0^1, \qquad \forall n \in \mathbf{Z}$:

$$\left.\begin{aligned} \frac{da(t)}{dt} &= 2^{-1/2}\sum_n a_n \psi_{0,0}^0(2^{-1}t-n) \\ &= 2^{-1/2}\sum_n a_n \frac{dg}{dt}(t-2n) \end{aligned}\right\} \Rightarrow \quad \begin{aligned} a(t) &= 2^{-1/2}\sum_n \alpha_n \, g(t-2n) \\ b(t) &= 2^{-1/2}\sum_n \beta_n \, g(t-2n-1) \end{aligned}$$

Then, we are going to define $A :=< \{\{\sqrt{3/2}g(t-2n); n \in \mathbf{Z}\} >$ and $B :=< \{\{\sqrt{3/2}g(t-2n-1); n \in \mathbf{Z}\}$; these base are orthonormal by properties of B-splines. With these definitions of our sets, it is can be proved that $A = \{a(t) \in L^2(\mathbb{R});\ \frac{da}{dt}(t) \in W_{-1}^0\}$ and $B = \{b(t) \in L^2(\mathbb{R});\ \ \frac{db}{dt}(t) \in \overline{W_{-1}^0}\}$.

<u>$V_0^1 = A \bigoplus B$</u>: By construction of A and B, it is immediately that, $\forall v(t) \in V_0^1$

$$v(t) = a(t) + b(t) = 2^{-1/2}\sum_n \alpha_n g(t-2n) + 2^{-1/2}\sum_n \beta_n g(t-2n) \tag{12}$$

It is can to see that $A \cap B = \emptyset$ and then the decomposition $V_0^1 = A \bigoplus B$, due to the fact that $W_{-1}^0 \cap \overline{W_{-1}^0} = \emptyset$ because we have a MRA and our functions is in $L^2(\mathbb{R})$.

Furthermore, the elements of these sets, A y B, can be characterizable through their samples; so, we are going to obtain a new reconstruction for the functions of V_0^1. Besides, in this reconstruction, using the 1 B-spline, not all the samples are necessary. Thanks to this fact, an optimization in the general reconstruction algorithm is offered.

<u>*Expression of $a(t)$*</u>: Let $a(t) = 2^{-1/2}\sum_n \alpha_n g(t-2n)$, we only need the even translations of $g(t)$;we have remember that $Supp\ g(t) = [0,2] \Longrightarrow Supp\ g(t-2n) = [2n,\ 2n+2]$ and these supports are disjoints; so that, taking $t_0 \in \mathbb{R}, \exists | n_0 \in \mathbf{N}$ with $g(t_0 - 2n_0) \neq 0$, and:

$$a(t_0) = 2^{-1/2}\alpha_{n_0} g(t_0 - 2n_0) \tag{13}$$

If we consider $t_0 = 2m$ or $t_0 = 2m+1$, it is got

$$\begin{aligned} a(2m) &= 2^{-1/2}\alpha_{n_0} g(2m - 2n_0) &&= 2^{-1/2}\alpha_{n_0} g(2(m-n_0)) = 0, \quad \forall m \in \mathbb{Z} \\ a(2m+1) &= 2^{-1/2}\alpha_{n_0} g(2(m-n_0)+1) &&= 2^{-1/2}\alpha_{n_0}, \quad \forall m \in \mathbb{Z} \end{aligned}$$

As we are working with orthonormal basis of A, it is got $\alpha_{n_0} = 2^{1/2}\ a(2m+1) = < a,\ g(t_0 - 2n) >$; substituting in (13) it is lead us to $a(t_0) =\ a(2n_0+1)\ g(t_0 - 2n_0)$.

However, we can find one relation between t_0, n_0 and m, with the following argument

$$\begin{aligned} g(2m+1-n_0) \neq 0 &\iff n_0 \in \left(\tfrac{2m-1}{2}, \tfrac{2m+1}{2}\right) &&\iff n_0 = [\tfrac{2m+1}{2}] = \ m \\ g(t_0) \neq 0 &\iff t_0 - 2n_0 \in (0,2) &&\iff n_0 \in \left(\tfrac{t_0-2}{2}, \tfrac{t_0}{2}\right) \iff n_0 = [\tfrac{t_0}{2}] \end{aligned}$$

All this allow us to obtain a new expression for

$$a(t) = \sum_n a\,(2n+1)\ g\,(t-2n) \tag{14}$$

Expression of $b(t)$: Analogously:

$$b(t_0) = b\left(2[\frac{t_0}{2}]\right)\ g\left(t_0 - 2[\frac{t_0}{2}] - 1\right) \tag{15}$$

and substituting (14) and (15) in (12), a new reconstruction for the signal in V_0^1 is given

$$v(t) = \sum_n \left[a(2n+1)g(t-2n)\ +\ b(2n)g(t-2n-1)\right] \tag{16}$$

Another expression for $v(t) \in V_0^1$: Also, it is easy to prove that

1.- $a(2m) = 0 = b(2m-1) \quad \forall m \in \mathbf{Z}$,

2.- $a(2m+1) = 3/2\,\alpha_m = 3/2 < a, g(\cdot - 2m >, \quad \forall m \in \mathbf{Z}$,

3.- $b(2m) = 3/2\,\beta_m = 3/2 < b,\ g(\cdot - 2m - 1) >, \quad \forall m \in \mathbf{Z}$.

As $v(t) = a(t) + b(t)$ then $v(2m) = a(2m) + b(2m) = b(2m)$ and $v(2m+1) = a(2m+1) + b(2m+1) = a(2m+1)$. We can substituting this in the previous formula for $v(t)$, so that, we obtain a different way for the reconstruction of these signal. One advantage of this expression is that we need a less number of samples of the signal, thanks to the fact explained above. Then:

$$v(t) = \sum_n (v(2n+1)g(t-2n)\ +\ v(2n)g(t-2n-1))$$

$\blacksquare$

5.1 Generalization for any level j

Definition 4 *We define the set* A_j *as* $A_j = \left\langle \{\sqrt{2^{\frac{j}{2}-1}}3\ g(2^j t - 2n);\ n \in \mathbf{Z}\} \right\rangle$, *and the set* B_j *as* $B_j = \left\langle \{\sqrt{2^{\frac{j}{2}-1}}3\ g(2^j t - 2n - 1);\ n \in \mathbf{Z}\} \right\rangle$.

With these definitions, the previous result is generalized to $V_j^1 = A_j \bigoplus B_j, \qquad \forall j \in \mathbf{Z}$.

Remark 2 *Because we are working with a MRA*

$$\overline{(\cup_j A_j) \bigoplus (\cup_j B_j)} = \overline{\cup_j A_j} \bigoplus \overline{\cup_j B_j} = L^2(\mathbf{R})$$

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