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ON MODELLING VISCOELASTICITY OF FIBRED BIOLOGICAL TISSUES

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Abstract. The main goal of this paper is to present a fully three-dimensional finite strain anisotropic viscohyperelastic model for soft biological tissues. The structural model is formulated by employing Simos constitutive framework based on irreversible thermodynamics with internal variables. This model is based in a local additive decomposition of the stress tensor into initial and non-equilibrium part (Kelvin-Voight generalized model) where we consider different viscoelastic behavior of the matrix material and families of the fibers. To describe the constitutive behavior of biological soft tissue, we consider a material constructed from two family fibers continuously distributed in a compliant solid isotropic matrix. Since the mechanical response of biological tissues is almost isochoric, we employ uncoupled volumetric and deviatoric response over any range of deformation. This is achieved by a local multiplicative decomposition of the deformation gradient into volume-preserving and dilatational parts that permits to model the incompressible properties of soft biological tissues.

Keywords: Visco-hyperlasticity, fibred tissues, finite strains, recursive algorithm.

AMS classification: 74C20,74D10,74S05.

§1. Motivation and Introduction

The main characteristics of biological soft tissues are that they sustain large deformations, rotations and displacements, have a highly non-linear behavior and possess strongly anisotropic mechanical properties with time and rate dependency. The typical anisotropic behavior is often caused by a number of collagen fiber families which are arranged in a soft matrix material named ground substance [3]. The time-rate dependent material behavior of soft biological tissues has been well-documented and quantified in the literature. This has included ligaments, tendons, blood vessels and articular cartilage. This behavior can arise from fluid flow in or out of the tissue, from inherent viscoelasticity of the solid phase, or from viscous interactions between tissue components of phases [1]. In this paper we present a fully three-dimensional finite strain anisotropic viscohyperelastic model for fibred biological tissues. The structural model is formulated within the framework of nonlinear continuum mechanics [9] and is well-suited for a finite element formulation based on irreversible thermodynamics with internal variables [6]. This model is based in a local additive decomposition of the stress tensor into initial and non-equilibrium part as resulted from a structure of the free energy that generalizes linear viscoelastic models (Kelvin-Voight generalized model). Also, we used a local multiplicative decomposition of the deformation gradient into volume-preserving and dilatational parts [8] that permits to model the incompressible properties of soft biological tissues. To simulate the viscoelastic properties of soft biological tissues, we consider different viscoelastic behavior of the matrix material and families of the fibers [3]. A numerical integration procedure that is second-order accurate and takes place entirely in the reference configuration is used, this fact implies that incremental objectivity is trivially satisfied [7].

§2. Viscohyperelastic Model

Following Simo [6], we postulate an uncoupled free energy function $\Psi(\mathbf{C},\mathbf{m}_0,\mathbf{n}_0,\mathbf{Q})$ of the form

$$\Psi = \Psi^0_{vol}(J) + \Psi^0_{dev}(\bar{\mathbf{C}},\mathbf{m}_0\otimes\mathbf{m}_0,\mathbf{n}_0\otimes\mathbf{n}_0) - \sum_{i=1}^{n}\frac{1}{2}\bar{\mathbf{C}}:\mathbf{Q}_i + \Xi(\sum_{i=1}^{n}\mathbf{Q}_i) \tag{1}$$

where Ψ^0_{vol} and Ψ^0_{dev} are the volumetric and deviatoric part of the initial elastic stored energy function Ψ^0, $\mathbf{m}_0$ and $\mathbf{n}_0$ are the direction of the fibers, $\mathbf{Q}_i$ play the role of internal variables (not accessible to direct observation and corresponding to the reference configuration) and Ξ is a certain function of the internal variables. Note that we have considered a material reinforced by two families of fibers continuously distributed in a compliant solid isotropic matrix [9]. Restricting our attention to the isothermal case and exploiting the Clausius-Duhem inequality $\mathscr{D}_{int} = -\dot{\Psi} + \frac{1}{2}\mathbf{S}:\dot{\mathbf{C}} \geq 0$ [5] lead to

$$\begin{aligned}\mathbf{S} &= 2\frac{\partial\Psi(\mathbf{C},\mathbf{m}_0,\mathbf{n}_0,\mathbf{Q}_i)}{\partial\mathbf{C}} = Jp\mathbf{C}^{-1} + J^{-\frac{2}{3}}DEV\left[2\frac{\partial\Psi^0_{dev}(\bar{\mathbf{C}},\mathbf{m}_0,\mathbf{n}_0)}{\partial\bar{\mathbf{C}}} - \sum_{i=1}^{n}\mathbf{Q}_i\right] \\ &= \mathbf{S}^0_{vol} + \mathbf{S}^0_{dev} - J^{-\frac{2}{3}}DEV\left[\sum_{i=1}^{n}\mathbf{Q}_i\right],\end{aligned} \tag{2}$$

$$\mathscr{D}_{int} = -\sum_{i=1}^{n}\frac{\partial\Psi(\mathbf{C},\mathbf{m}_0,\mathbf{n}_0,\mathbf{Q}_i)}{\partial\mathbf{Q}_i}:\dot{\mathbf{Q}}_i = \sum_{i=1}^{n}\left[\frac{1}{2}\bar{\mathbf{C}} - \frac{\partial\Xi(\mathbf{Q}_i)}{\partial\mathbf{Q}_i}\right]:\dot{\mathbf{Q}}_i \geq 0. \tag{3}$$

The stress $\mathbf{Q}_i$ may be interpreted as non-equilibrium stresses in the sense of non-equilibrium thermodynamics, which remain unaltered under superposed spatial rigid body motions [7]. This fundamental requirement is the same invariance property classically placed on the second Piola-Kirchhoff tensor $\mathbf{S}$ and automatically ensures frame indifference of the constitutive relationship (2). Motivated by Holzapfel and Gasser [3] and in order to consider different contribution of the matrix material and families of the fibers on the non-equilibrium part, we divide the internal variables in

$$\mathbf{Q}_i = \sum_{\substack{j=1,\\ j\neq 3}}^{8}\mathbf{Q}_{ij}, \tag{4}$$

where $\mathbf{Q}_{i1}$ and $\mathbf{Q}_{i2}$ are the isotropic contribution due to the matrix material associated to $\bar{I}_1$ and $\bar{I}_2$ invariants of $\bar{\mathbf{C}}$ [9] and $\mathbf{Q}_{i4},\ldots,\mathbf{Q}_{i8}$ are the anisotropic contribution due to the two family of fibers associated to $\bar{I}_4,\ldots,\bar{I}_8$ invariants [9] with

$$\begin{aligned}&\bar{I}_1 = \mathrm{tr}\,\bar{\mathbf{C}}, && \bar{I}_4 = \mathbf{m}_0.\bar{\mathbf{C}}.\mathbf{m}_0, && \bar{I}_6 = \mathbf{n}_0.\bar{\mathbf{C}}.\mathbf{n}_0, \\ &\bar{I}_2 = \frac{1}{2}(\mathrm{tr}(\bar{\mathbf{C}})^2 - \mathrm{tr}\,\bar{\mathbf{C}}^2), && \bar{I}_5 = \mathbf{m}_0.\bar{\mathbf{C}}^2.\mathbf{m}_0, && \bar{I}_7 = \mathbf{n}_0.\bar{\mathbf{C}}^2.\mathbf{n}_0, \quad \bar{I}_8 = \mathbf{m}_0.\bar{\mathbf{C}}.\mathbf{n}_0.\end{aligned} \tag{5}$$

So, we formulate the evolution equations separately for each contribution. We consider the following set of rate equations governing the evolution of the internal variables $\mathbf{Q}_{ij}$ (cf. [6])

$$\dot{\mathbf{Q}}_{ij} + \frac{1}{\tau_{ij}}\mathbf{Q}_{ij} = \frac{\gamma_{ij}}{\tau_{ij}} DEV\left[2\frac{\partial \Psi^{0(j)}_{dev}}{\partial \bar{\mathbf{C}}}\right], \qquad \lim_{t\to -\infty} \mathbf{Q}_{ij} = \mathbf{0}, \tag{6}$$

with $\gamma_{ij} \in [0,1]$ and $\tau_{ij} > 0$. The evolution equations (6) are linear and, therefore, explicitly lead to the following convolution representation

$$\mathbf{Q}_{ij}(t) = \frac{\gamma_{ij}}{\tau_{ij}} \int_{-\infty}^{t} \exp[\frac{-(t-s)}{\tau_{ij}}] DEV\left[2\frac{\partial \Psi^{0(j)}_{dev}}{\partial \bar{\mathbf{C}}(s)}\right] ds. \tag{7}$$

Substitution (7) into (2) then yields the following equivalent expression

$$\begin{aligned}\mathbf{S} = Jp\mathbf{C}^{-\mathbf{1}} + J^{-\frac{2}{3}} \sum_{\substack{j=1\\ j\neq 3}}^{8} \Bigg[\left(1-\sum_{i=1}^{n}\gamma_{ij}\right) DEV\left\{2\frac{\partial \Psi^{0(j)}_{dev}(\bar{\mathbf{C}},\mathbf{m}_0,\mathbf{n}_0)}{\partial \bar{\mathbf{C}}}\right\} \\ + \sum_{i=1}^{n}\left(\gamma_{ij}\int_{-\infty}^{t}\exp\left[\frac{-(t-s)}{\tau_{ij}}\right]\frac{d}{ds}\cdot\left\{DEV\left[2\frac{\partial \Psi^{0(j)}_{dev}}{\partial \bar{\mathbf{C}}(\mathbf{s})}\right]\right\}ds\right)\Bigg].\end{aligned} \tag{8}$$

Note that $\mathbf{Q}_i$ attains its equilibrium values as $t/\tau_{ij} \longrightarrow \infty$. The corresponding value of the equilibrium stress is a fraction of the initial stress; that is

$$\lim_{t/\tau_{ij}\longrightarrow\infty} \mathbf{S} = Jp\mathbf{C}^{-\mathbf{1}} + J^{-\frac{2}{3}} \sum_{\substack{j=1\\ j\neq 3}}^{8} \left[\left(1-\sum_{i=1}^{n}\gamma_{ij}\right) DEV\left\{2\frac{\partial \Psi^{0(j)}_{dev}(\bar{\mathbf{C}},\mathbf{m}_0,\mathbf{n}_0)}{\partial \bar{\mathbf{C}}}\right\}\right]. \tag{9}$$

§3. Integration Algorithm

The basic idea in the numerical integration of constitutive equations is to evaluate the convolution integral in (8) through a recurrence relation. A related procedure was first suggested by Herrmann and Peterson [2] and Taylor et al. [10] and modified by Simo [6]. The key idea is to transform the convolution representation discussed in the preceding sections into a two-step recurrence formula involving internal variables stored at the quadrature points of a finite-element method [7]. First at all, we introduce the following internal algorithmic history variables by expression [7]

$$\mathbf{H}^{(ij)} = \int_{-\infty}^{t} \exp\left[\frac{-(t-s)}{\tau_{ij}}\right]\frac{d}{ds}\cdot\left\{DEV\left[2\frac{\partial \Psi^{0(j)}_{dev}}{\partial \bar{\mathbf{C}}(\mathbf{s})}\right]\right\}ds. \tag{10}$$

Let $[T_0, T] \subset \mathbb{R}$, with $T_0 < T$, be the time interval of interest. Without lost of generality, we take $T_0 = -\infty$. Further, let $[T_0, T] = \bigcup_{n\in\mathbb{I}}[t_n, t_{n+1}]$, be a partition of the interval $[T_0, T]$ with $\mathbb{I}$ the integers and $\Delta t_n = t_{n+1} - t_n$ the associated time increment. From an algorithmic standpoint, the problem is in the usual strain-driven format and we assume that at certain times t_n and t_{n+1} all relevant kinematics quantities are known. Using the semigroup property of the exponential function, the property of additivity of the integral over the interval of integration and the midpoint rule to approximate the integral over $[t_n, t_{n+1}]$ to arrive at update formula [7]

$$\mathbf{H}_{n+1}^{(ij)} = \exp\left[\frac{-\Delta t_n}{\tau_{ij}}\right]\mathbf{H}_n^{(ij)} + \exp\left[\frac{-\Delta t_n}{2\tau_{ij}}\right](\overline{\mathbf{S}}_{n+1}^{0(j)} - \overline{\mathbf{S}}_n^{0(j)}). \tag{11}$$

Following the convolution representation (8), the algorithmic approximation for the second Piola-Kirchhoff stress takes the form

$$\mathbf{S} = J_{n+1} p_{n+1} \mathbf{C}_{n+1}^{-1} + J_{n+1}^{-\frac{2}{3}} \sum_{\substack{j=1 \\ j\neq 3}}^{8} \left[\left(1 - \sum_{i=1}^{n} \gamma_{ij}\right)\overline{\mathbf{S}}_{n+1}^{0(j)} + \sum_{i=1}^{n}\left(\gamma_{ij}\left\{DEV_{n+1}[\mathbf{H}_{n+1}^{(ij)}]\right\}\right)\right], \tag{12}$$

where $\overline{\mathbf{S}}_{n+1}^{0(j)}$ is the term in the initial stress response corresponding to $\overline{I}_j$, i.e., $\overline{\mathbf{S}}_{n+1}^{0(1)}$ and $\overline{\mathbf{S}}_{n+1}^{0(2)}$ are due to the matrix material and $\overline{\mathbf{S}}_{n+1}^{0(4)} \ldots \overline{\mathbf{S}}_{n+1}^{0(8)}$ are due to the two family fibers.

§4. Consistent algorithmic tangent moduli

We wish to use the proposed constitutive model in a Finite Element Method (FEM) [4]. The tangent moduli plays a crucial role in the numerical solution of the boundary value problem by Newton-type iterative methods as FEM. For instance, use of these consistently linearized moduli is essential in order to preserve the quadratic rate of the asymptotic convergence that characterizes Newtons method [4]. In order to obtain a more easy recursive update procedure, we rewrite the update formula (11) as follows [7]

$$\tilde{\mathbf{H}}_n^{(ij)} = \exp\left[\frac{-\Delta t_n}{\tau_{ij}}\right]\mathbf{H}_n^{(ij)} - \exp\left[\frac{-\Delta t_n}{2\tau_{ij}}\right]\overline{\mathbf{S}}_n^{0(j)}, \tag{13}$$

$$\mathbf{H}_{n+1}^{(ij)} = \tilde{\mathbf{H}}_n^{(ij)} + \exp\left[\frac{-\Delta t_n}{2\tau_{ij}}\right]\overline{\mathbf{S}}_{n+1}^{0(j)}. \tag{14}$$

With this notation

$$\mathbf{S}_{n+1} = J_{n+1} p_{n+1} \mathbf{C}_{n+1}^{-1} + J_{n+1}^{-\frac{2}{3}} \sum_{\substack{j=1 \\ j\neq 3}}^{8} \left[(1 - \gamma_j + \nu_j)\overline{\mathbf{S}}_{n+1}^{0(j)} + \sum_{i=1}^{n} \gamma_{ij}\{DEV_{n+1}[\tilde{\mathbf{H}}_n^{(ij)}]\}\right]. \tag{15}$$

Using the standard definition for elasticity tensor ($\mathbb{C} = 2\partial \mathbf{S}(\mathbf{C})/\partial \mathbf{C}$) from (15) we obtain

$$\begin{aligned}\mathbb{C}_{n+1} = \mathbb{C}_{vol\ n+1}^{0} + \sum_{\substack{j=1 \\ j\neq 3}}^{8} \Big[&(1 - \gamma_j + \nu_j)\mathbb{C}_{dev\ n+1}^{0(j)} \\ &- \frac{2}{3} J_{n+1}^{\frac{4}{3}} \sum_{i=1}^{n} \gamma_{ij} \Big\{DEV_{n+1}[\tilde{\mathbf{H}}_n^{(ij)}] \otimes \overline{\mathbf{C}}_{n+1}^{-1} + \overline{\mathbf{C}}_{n+1}^{-1} \otimes DEV_{n+1}[\tilde{\mathbf{H}}_n^{(ij)}] \\ &- \left(\tilde{\mathbf{H}}_n^{(ij)} : \overline{\mathbf{C}}\right)\left(\mathbb{I}_{\mathbf{C}_{n+1}}^{-1} - \frac{1}{3}\overline{\mathbf{C}}_{n+1}^{-1} \otimes \overline{\mathbf{C}}_{n+1}^{-1}\right)\Big\}\Big].\end{aligned}$$

	C_1	C_2	C_3	C_4	D	
Set I	10	10	100	1	0.0036844	
Set II	10	10	10	1	0.0036844	
Set III	10	10	0.1	1	0.0036844	
	γ_{11}	$\tau_{11}(s)$	γ_{12}	$\tau_{12}(s)$	γ_{14}	$\tau_{14}(s)$
Example I	0.3	10.0	0.3	10.0	0.3	10.0
Example II	0.05	10.0	0.05	10.0	0.3	10.0
Example III	0.05	0.1	0.05	0.1	0.3	10.0
Example IV	0.3	10.0	0.3	10.0	0.05	10.0
Example V	0.3	10.0	0.3	10.0	0.05	0.1
Example VI	0.6	10.0	0.6	10.0	0.3	10.0
Example VII	0.3	10.0	0.3	10.0	0.6	10.0

Table 1: Viscoelastic material parameters

§5. Influence of Viscoelastic Parameters in Strain-Stress Response

In order to study the influence of the viscoelastic parameters in the stress-strain response, we considered a transversely isotropic and hyperelastic material with its constitutive behaviour defined by the the initial elastic stored energy function

$$\begin{aligned}\Psi(\mathbf{C})^0 &= \Psi_{vol}(J) + \Psi^0_{dev}(\bar{\mathbf{C}}, \mathbf{m}\otimes\mathbf{m}) \\ &= \frac{1}{D}(\ln(J))^2 + C_1(\bar{I}_1 - 3) + C_2(\bar{I}_1 - 3)^2 + \frac{C_3}{C_4}\left[e^{C_4(\bar{I}_4-1)} - 1\right],\end{aligned}$$

where $C_3 \geq 0$ is a stress-like material parameter and $C_4 \geq 0$ is a dimensionless parameter. Three sets of elastic material constant were chosen (Table 1) and we considered only one internal variable ($i = 1$).

For a strain rate of $3.6\%s^{-1}$, uniaxial relaxation test was simulated up to a stretch ratio $\lambda = 1.36$. The viscoelastic parameters are as summarized in Table 1. Example I where the viscoelastic parameters are equal for matrix and fibers, correspond to isotropic viscoelastic material. In the examples II and III the viscoelastic parameters of the matrix are assumed very small with respect to the fibers. For the examples IV and V the viscoelastic parameters of the fibers are assumed very small with respect to the matrix. Finally, in examples VI and VII τ_{1j} is increased for the matrix and fibers respectively. Fig.1.a illustrates the evolution of the stress response with the time for the set I of constants. As can be seen, changing the viscoelastic parameters of the matrix contribution not produce changes in the stress evolution and the thermodynamic equilibrium stress, examples I, II, III and VI. On the contrary, when we decrease the free-energy factor γ_{14} (example IV), there are increase of the initial stress (6%) and the equilibrium stress (29%). In addition, when the relaxation time is decrease until $\tau_{14} = 0.1$ s (example V) the equilibrium stress is achieved very fast. This fact is due to the contribution of the matrix to stress response is very small with respect to fibers one. The evolution of the stress response with the time for the set II of constants is shown in Fig.1.b. In this case, the elastic parameters of the family fibers are assumed to be of the same order as the stiffness parameters of the matrix material. So, changes in the viscoelastic parameters of the matrix or fibers part similarly affect to the equilibrium stress. On the contrary, when the set III of constants was used, the elastic parameters of the family fiber are assumed to be very small with respect to the matrix one. This causes a material response which is almost isotropic elastic according the Mooney-Rivlin material and provokes that changes in the viscoelastic parameters of the family fibers not affect to the stress response (Fig.1.c).

§6. Conclusions

We have presented an anisotropic Kelvin-Voight type visco-hyperelastic constitutive model capable to model fiber-reinforced composite materials undergoing finite strains as soft biological tissues. The structural model was formulated by employing Simos constitutive framework based on irreversible thermodynamics with internal variables [6], where we have considered different viscoelastic behavior of the matrix material and the different families of fibers. Motivated by [3] and in order to consider different contributions of the matrix and fibers on the non-equilibrium part, we considered the internal variables to correspond to separated contributions of the matrix and fibers. A numerical integration procedure that is second-order accurate and takes place entirely in the reference configuration was used; this fact implies that incremental objectivity is trivially satisfied. To our knowledge, the anisotropic visco-hyperelastic model based local additive decomposition of the stress tensor into initial and non-equilibrium parts (Kelvin-Voight generalized model) in fibred materials has not been recorded previously in the literature.

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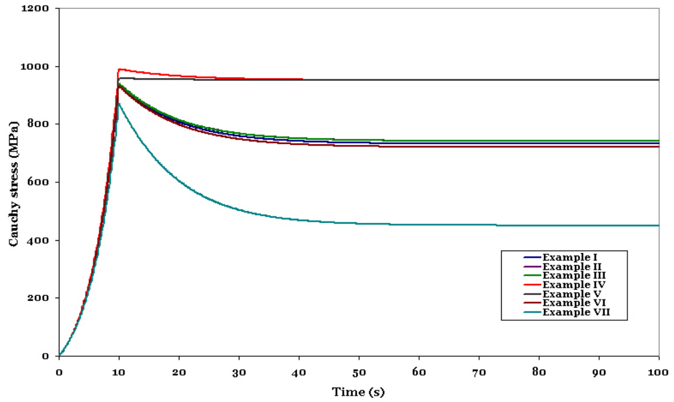
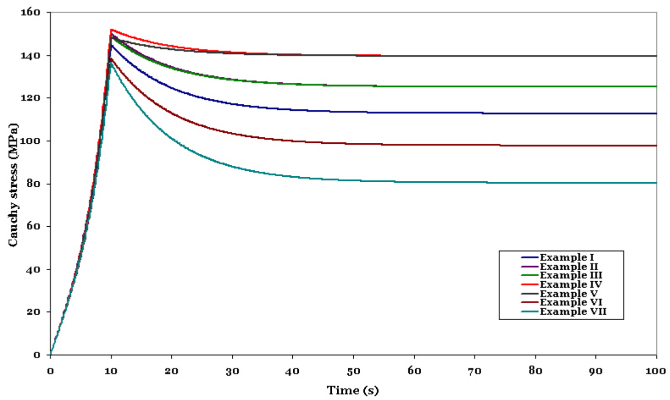
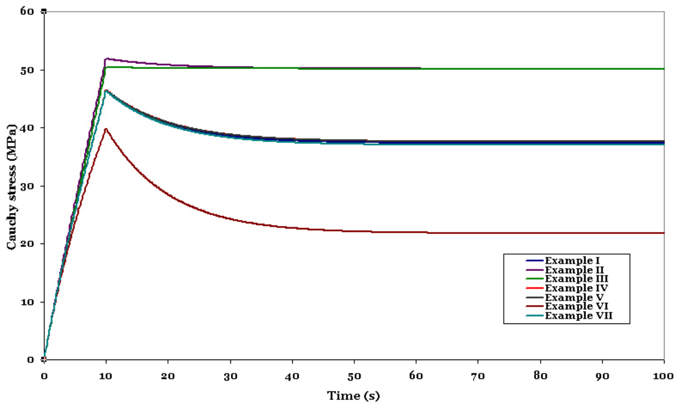


(a) Set I



(b) Set II



(c) Set III

Figure 1: Results of the influence of viscoelastic parameters in strain-stress response

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