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BIVARIATE APPROXIMATION BY DISCRETE SMOOTHING PDE SPLINES

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Abstract. This paper deals with the construction and characterization of discrete PDE splines. For this purpose, we need a PDE equation (usually an elliptic PDE), certain boundary conditions and a set of points to approximate. We give two results about the convergence of a discrete PDE spline to a function of a fixed space in two different cases: (1) when the approximation points are fixed; (2) when the boundary points are fixed. We provide a numerical and graphic example of approximation by discrete PDE splines.

Keywords: Spline, smoothing, interpolation.

AMS classification: 65D05, 65D07, 65D10.

§1. Introduction

The objective of this paper is to find a function verifying certain conditions regarding the boundary of a domain while at the same time the function has to approximate a data point set in its interior. Consequently, this method can be conceived as a surface generation technique (see [4], [1] and [10] for similar works). The method entails minimizing a functional in an adequate space. This functional has information concerning the equation in terms of a semi-norm and information regarding the data set point in terms of discrete least squares. This paper is the logical continuation of [5] and [6]. The remainder of it is organized as follows: Section 2 explains the notations; Section 3 defines and characterizes discrete PDE splines; Section 4 studies the convergence of a discrete PDE spline to an adequate function; Section 5 gives a description of the computation of the method, and we finish with some numerical and graphical examples of various discrete PDE splines that illustrate the behaviour of the different parameters of the method. Similar proofs of the results can be found in [7, Chapter 5] with some variations.

§2. Preliminaries

We shall use the following notations: the Euclidean norm and inner product in $\mathbb{R}^m$ will be denoted by $\langle\cdot\rangle_m$ and $\langle\cdot,\cdot\rangle_m$ respectively, for any $m \in \mathbb{N}$, $m \geq 2$; $H^n(\Omega)$ represents the usual Sobolev space of order n of (classes of) functions $u \in L^2(\Omega)$, together with all their partial derivatives $\partial^{\mathbf{i}}u$, in the distribution sense, of order $|\mathbf{i}| \leq n$, where, for all $\mathbf{i} = (i_1, i_2) \in \mathbb{N}^2$, $|\mathbf{i}| = i_1 + i_2$ and $\partial^{\mathbf{i}}u(\mathbf{x}) = \frac{\partial^{|\mathbf{i}|}u}{\partial x_1^{i_1}\partial x_2^{i_2}}$, for any $\mathbf{x} = (x_1, x_2) \in \Omega$ and, finally, $H_0^n(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^n(\Omega)$. Obviously $H_0^0(\Omega) = L^2(\Omega)$.

The space $L^2(\Omega)$ is equipped with the inner product $(u,v)_{0,\Omega} = \int_\Omega u(\mathbf{x})v(\mathbf{x})\,d\mathbf{x}$ and the corresponding norm $|u|_{0,\Omega} = (u,u)_{0,\Omega}^{1/2}$. The Sobolev space $H^n(\Omega)$ is equipped with the inner product $((u,v))_{n,\Omega} = \sum_{|\mathbf{i}|\le n} \int_\Omega \partial^{\mathbf{i}}u(\mathbf{x})\partial^{\mathbf{i}}v(\mathbf{x})\,d\mathbf{x}$, the norm $\|u\|_{n,\Omega} = ((u,u))_{n,\Omega}^{1/2}$, the semi-inner products $(u,v)_{l,\Omega} = \sum_{|\mathbf{i}|=l} \int_\Omega \partial^{\mathbf{i}}u(\mathbf{x})\partial^{\mathbf{i}}v(\mathbf{x})\,d\mathbf{x}$, with $0 \le l \le n$, and the corresponding semi-norm $|u|_{l,\Omega} = (u,u)_{l,\Omega}^{1/2}$, for all $0 \le l \le n$.

§3. Formulation of the problem

Let Ω be an open, bounded, polyhedral domain of $\mathbb{R}^2$. Thus, Ω has a Lipschitz boundary. Let $L : H^{2n}(\Omega) \to L^2(\Omega)$ be a differential operator given by

$$Lu(\mathbf{x}) = \sum_{|\mathbf{i}|,|\mathbf{j}|\le n} (-1)^{|\mathbf{j}|}\partial^{\mathbf{j}}(p_{\mathbf{ij}}(\mathbf{x})\partial^{\mathbf{i}}u(\mathbf{x})),\ \mathbf{x} \in \Omega, \tag{1}$$

where $p_{\mathbf{ij}} \in C^{|\mathbf{j}|}(\Omega)$ and $p_{\mathbf{ij}} = p_{\mathbf{ji}}$, for all $|\mathbf{i}|,|\mathbf{j}| \le n$. We now consider the symmetric bilinear form associated with L defined on $H^n(\Omega)\times H^n(\Omega)$ by $(u,v)_L = \sum_{|\mathbf{i}|,|\mathbf{j}|\le n}(p_{\mathbf{ij}}\partial^{\mathbf{i}}u, \partial^{\mathbf{j}}v)_{0,\Omega}$, and we assume that

$$\sum_{|\mathbf{i}|,|\mathbf{j}|\le n-1} \boldsymbol{\xi}^{\mathbf{i}} p_{\mathbf{ij}}(\mathbf{x})\boldsymbol{\xi}^{\mathbf{j}} \ge 0, \quad \forall \mathbf{x} \in \Omega, \tag{2}$$

and that there exists $\nu > 0$ such that

$$\sum_{|\mathbf{i}|,|\mathbf{j}|= n} \boldsymbol{\xi}^{\mathbf{i}} p_{\mathbf{ij}}(\mathbf{x})\boldsymbol{\xi}^{\mathbf{j}} \ge \nu\langle\boldsymbol{\xi}\rangle_2^{2n}, \quad \forall \mathbf{x} \in \Omega, \tag{3}$$

for all $\boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathbb{R}^2$, where $\boldsymbol{\xi}^{\mathbf{i}} = \xi_1^{i_1}\xi_2^{i_2}$, for any $\mathbf{i} = (i_1,i_2) \in \mathbb{N}^2$. Due to (3), the differential operator L is said to be strongly elliptic on Ω. According to the hypotheses (2)–(3) the bilinear form $(\cdot,\cdot)_L$ defines a semi–inner product on $H^n(\Omega)$ whose associated semi–norm is denoted by $|u|_L = (u,u)_L^{1/2}$.

Suppose we are given two integers $n \ge 2$ and $r > 0$; the functions $f \in L^2(\Omega)$ and $h_j \in C(\overline{\Omega})$, for $j = 0,\dots,n-1$, an ordered set $A^r = \{\mathbf{a}_1,\dots,\mathbf{a}_m\}$ of $m = m(r) \ge 0$ ($m \in \mathbb{N}^*$) distinct points of Ω, an ordered set $B^N = \{\mathbf{b}_1,\dots,\mathbf{b}_N\}$ of $N \in \mathbb{N}^*$ distinct points of $\partial\Omega$, none of which is a geometric vertex of $\overline{\Omega}$, a data vector $\boldsymbol{\beta} = (\beta_1,\dots,\beta_m) \in \mathbb{R}^m$, a subset $\mathscr{H}$ of real positive numbers that admits 0 as an accumulation point, for any $h \in \mathscr{H}$, a triangulation $\mathscr{T}_h$ of $\overline{\Omega}$ by means of simplices or rectangles of diameter $h_K \le h$. Moreover, we suppose given, for any $h \in \mathscr{H}$, a finite element space X_h made up over $\mathscr{T}_h$ such that

$$X_h \text{ has finite dimension } I = I(h), \tag{4}$$

$$X_h \subset H^n(\Omega) \cap C^{n-1}(\overline{\Omega}). \tag{5}$$

For each $l = 1,\dots,N$ and $j = 0,1,\dots,n-1$, we denote $\phi_{n(l-1)+j+1} : H^n(\Omega) \to \mathbb{R}$ the linear map given by $\phi_{n(l-1)+j+1} = \frac{\partial^j v}{\partial \mathbf{n}^j}(\mathbf{b}_l)$, and let $\boldsymbol{\tau}^N : H^n(\Omega) \to \mathbb{R}^{Nn}$ be the linear operator $\boldsymbol{\tau}^N(v) = (\phi_k(v))_{k=1,\dots,Nn}$. For all $k = 1,\dots,Nn$, we suppose that

$$\phi_k \text{ is a degree of freedom of } X_h. \tag{6}$$

We define the operator $\boldsymbol{\rho} : H^n(\Omega) \to \mathbb{R}^m$, given by $\boldsymbol{\rho}(v) = (v(\mathbf{a}_i))_{1,\ldots,m}$, the vector space $H_0^{Nh} = \{u \in X_h : \boldsymbol{\tau}^N u = \mathbf{0}\}$ and the convex set $H^{Nh} = \{u \in X_h : \boldsymbol{\tau}^N u = \mathbf{y}\}$, being $\mathbf{y} = (y_i)_{i=1\ldots,Nn}$ with $y_{n(l-1)+j+1} = h_j(\mathbf{b}_l)$, for $l = 1,\ldots,N$ and $j = 0,\ldots,n-1$. Later, we suppose that

$$\ker \boldsymbol{\rho} \cap \mathbb{P}_{n-1}(\Omega) = \{\mathbf{0}\}, \tag{7}$$

where $\mathbb{P}_{n-1}(\Omega)$ is the space of polynomial functions defined over $\mathbb{R}^2$ of degree $\le n-1$ with respect to the set of variables.

Let L be the operator given in (1) and let us consider the problem

$$\begin{cases} Lu(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \dfrac{\partial^j u}{\partial \mathbf{n}^j}(\mathbf{x}) = h_j(\mathbf{x}), & \mathbf{x} \in \partial\Omega, \quad 0 \le j \le n-1. \end{cases} \tag{8}$$

Definition 1. We say that σ_h is a discrete PDE spline in X_h associated with L, B^N, $\mathbf{y}$, A^r, $\boldsymbol{\beta}$ and $\varepsilon > 0$, if σ_h is a solution of the problem

$$\begin{cases} \sigma_h \in H^{Nh}, \\ \forall v \in H^{Nh}, \; J(\sigma_h) \le J(v), \end{cases} \tag{9}$$

where J is the functional defined on $H^n(\Omega)$ by $J(v) = \langle \boldsymbol{\rho} v - \boldsymbol{\beta} \rangle_m^2 + \varepsilon \left(|v|_L^2 - 2(f,v)_{0,\Omega} \right)$.

Now we establish a variational characterization of the discrete PDE spline and a method of Lagrangian multipliers to solve Problem (9). The proof of these results can be consulted in [7].

Theorem 1. *Problem (9) admits a unique solution which is also the unique solution of the variational problem: find $\sigma_h \in H^{Nh}$ such that*

$$\forall v \in H_0^{Nh}, \; \langle \boldsymbol{\rho} \sigma_h, \boldsymbol{\rho} v \rangle_m + \varepsilon (\sigma_h, v)_L = \langle \boldsymbol{\beta}, \boldsymbol{\rho} v \rangle_m + \varepsilon (f, v)_{0,\Omega}.$$

We give a result that is useful if we want to obtain an expression of the discrete PDE spline.

Theorem 2. *There exists a unique $(\sigma_h, \boldsymbol{\lambda}) \in H^{Nh} \times \mathbb{R}^{Nn}$ such that for all*

$$\langle \boldsymbol{\rho} \sigma_h, \boldsymbol{\rho} v \rangle_m + \varepsilon (\sigma_h, v)_L + \langle \boldsymbol{\tau}^N v, \boldsymbol{\lambda} \rangle_{Nn} = \langle \boldsymbol{\beta}, \boldsymbol{\rho} v \rangle_m + \varepsilon (f, v)_{0,\Omega}, \tag{10}$$

for all $v \in X_h$, where σ_h is the unique solution of Problem (9).

§4. Convergence

Let $g \in H^{n+1}(\Omega)$. We are going to enunciate two results of the convergence of the discrete PDE spline associated with L, $\boldsymbol{\tau}^N g$, A^r, $\boldsymbol{\rho} g$ and ε to the function g under certain conditions, as $h \to 0$ and $r \to +\infty$, independently of N, in the first result, and as $h \to 0$ and $N \to +\infty$, independently of r, if g is the solution of the boundary problem (8), in the other one. The proof of these results can be consulted in [7].

In order to do this, we assume that the family $(X_h)_{h\in\mathscr{H}}$ verifies the following relationship (cf. P. Clément [3]): for all $h\in\mathscr{H}$, there exists a linear operator $\Pi_h : L^2(\Omega)\mapsto X_h$, and a constant $C\geq 0$, verifying:

$$\begin{aligned}&1)\quad \forall h\in\mathscr{H}, \forall l=0,\dots,n,\ \forall v\in H^{n+1}(\Omega),\ |v-\Pi_h v|_{l,\Omega}\leq Ch^{n+1-l}|v|_{n+1,\Omega};\\ &2)\quad \forall h\in\mathscr{H}, \forall v\in H^{n+1}(\Omega),\ \lim_{h\to 0}\left(\sum_{K\in\mathscr{T}_h}|v-\Pi_h v|^2_{n+1,K}\right)^{1/2}=0.\end{aligned}\tag{11}$$

Moreover, we suppose that the family $(\mathscr{T}_h)_{h\in\mathscr{H}}$ satisfies the inverse hypothesis of Ciarlet [2]. More specifically,

$$\exists \nu\geq 0,\ \forall h\in\mathscr{H},\ \forall K\in\mathscr{T}_h,\ \frac{h}{h_K}\leq\nu,\tag{12}$$

where h_K is the diameter of K. For the first result we suppose

$$\sup_{\mathbf{x}\in\Omega}\min_{\mathbf{a}\in A^r}\langle\mathbf{x}-\mathbf{a}\rangle_2=o\left(\frac{1}{r}\right),\ r\to+\infty,\tag{13}$$

and that the families A^r and $\mathscr{T}_h$ are linked by the relation

$$\exists C>0,\ \forall h\in\mathscr{H},\ \forall r\in\mathbb{N},\ \forall K\in\mathscr{T}_h,\ \frac{\operatorname{card}(A^r\cap K)}{\operatorname{meas}(K)}\leq Cr^2,\tag{14}$$

where $\operatorname{meas}(K)$ is the measure of K. Note that this hypothesis translates a property of "asymptotic regularity" of the density of the points of A^r over the elements K of $\mathscr{T}_h$.

For the first result of convergence we suppose that the number of interpolation points, N, is fixed and that $\varepsilon=\varepsilon(r)$. For any $r\in\mathbb{N}$ and each $h\in\mathscr{H}$, let σ_h^r be the discrete PDE spline associated with L, B^N, $\boldsymbol{\tau}^N g$, A^r, $\boldsymbol{\rho} g$ and ε.

Theorem 3. *Suppose that (4)–(6) and (11)–(14) hold, and that*

$$\varepsilon=o(r^2),\quad r\to+\infty,\tag{15}$$

$$\frac{h^{2(n+1)}r^2}{\varepsilon}=O(1),\quad r\to+\infty.\tag{16}$$

Then,

$$\lim_{r\to+\infty}\|\sigma_h^r-g\|_{n,\Omega}=0.\tag{17}$$

We suppose now that the number of approximation points, $m(r)$, is fixed and that $\varepsilon=\varepsilon(N)$. Likewise, we suppose that $g\in H^{2n}(\Omega)\cap C^{n-1}(\overline{\Omega})$ is the solution of the boundary problem (8). We denote by σ_h^N the discrete PDE spline associated with L, B^N, $\boldsymbol{\tau}^N g$, A^r, $\boldsymbol{\rho} g$ and ε, for all $h\in\mathscr{H}$ and $N\in\mathbb{N}$.

Since the injection of $H^n(\Omega)$ in $C^{n-2}(\overline{\Omega})$ is compact, it is possible to define the operator $\widetilde{\tau}^N : H^n(\Omega)\to\mathbb{R}^{N(n-1)}$ given by, for any $v\in H^n(\Omega)$, $\widetilde{\tau}^N v=(\widetilde{\phi}_i v)_{i=1,\dots,N(n-1)}$, with $\widetilde{\phi}_{(n-1)(l-1)+j+1}=\widetilde{\phi}_{n(l-1)+j+1}$, for all $l=1,\dots,N$ and for all $j=0,\dots,n-2$.

We suppose that

$$\ker \widetilde{\tau}^N \cap \mathbb{P}_{n-1}(\Omega) = \{\mathbf{0}\}, \tag{18}$$

$$\sup_{\mathbf{x}\in\partial\Omega} \min_{\mathbf{b}\in B^N} \langle \mathbf{x}-\mathbf{b}\rangle_2 = O\Big(\frac{1}{N}\Big), \quad N \to +\infty. \tag{19}$$

Theorem 4. *Suppose that (4)–(7), (11)–(13), (18) and (19) hold, with*

$$h = o(1)\,, \;\; N \to +\infty, \qquad \textit{and} \qquad \frac{h^{4n}}{\varepsilon} = o(1)\,, \;\; N \to +\infty.$$

Then we have $\lim_{N\to+\infty} \|\sigma_h^N - g\|_{n,\Omega} = 0$.

§5. Computation

We are now going to obtain the expression of the discrete PDE spline σ_h. Let h be fixed and let us consider a triangulation $\mathscr{T}_h$ of $\overline{\Omega}$ by means of rectangles such that the points of B^N are nodes of its triangulation. We number the basis functions of the finite element space X_h, $\omega_1,\dots,\omega_I$. We can then express σ_h as the following linear combination $\sigma_h(\mathbf{x}) = \sum_{i=1}^{I} \gamma_i \omega_i(\mathbf{x})$, and, if we calculate the unknown coefficients γ_i, we then have the expression of σ_h.

By substituting in (10), we obtain, for all $v \in H^{Nh}$,

$$\sum_{i=1}^{I} \gamma_i \left(\langle \boldsymbol{\rho}\omega_i, \boldsymbol{\rho} v\rangle_m + \varepsilon(\omega_i, v)_L\right) + \langle \boldsymbol{\lambda}, \boldsymbol{\tau}^N v\rangle_{Nn} = \langle \boldsymbol{\beta}, \boldsymbol{\rho} v\rangle_m + \varepsilon\,(f,v)_{0,\Omega}\,,$$

subject to the restrictions $\boldsymbol{\tau}^N\left(\sum_{i=1}^{I}\gamma_i\omega_i\right) = \mathbf{y}$, which are equivalent to

$$\begin{cases} \displaystyle\sum_{i=1}^{I} \gamma_i \left(\langle \boldsymbol{\rho}\omega_i, \boldsymbol{\rho}\omega_j\rangle_m + \varepsilon(\omega_i,\omega_j)_L\right) + \langle \boldsymbol{\lambda}, \boldsymbol{\tau}^N \omega_j\rangle_{Nn} = \langle \boldsymbol{\beta}, \boldsymbol{\rho}\omega_j\rangle_m + \varepsilon(f,\omega_j)_{0,\Omega}, \quad 1 \le j \le I, \\ \displaystyle\sum_{i=1}^{I} \gamma_i \boldsymbol{\tau}_j^N(\omega_i) = \mathbf{y}_j, \quad 0 \le j \le Nn, \end{cases}$$

that is a linear system with $I + Nn$ equations and the unknowns $\gamma_1,\dots,\gamma_I,\lambda_1,\dots,\lambda_{Nn}$. Its matricial form is

$$\begin{pmatrix} C & D \\ D^t & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \widehat{\mathbf{f}} \\ \mathbf{y} \end{pmatrix},$$

where $C = \left(\langle \boldsymbol{\rho}\omega_i, \boldsymbol{\rho}\omega_j\rangle_m + \varepsilon(\omega_i,\omega_j)_L\right)_{1\le i,j\le I}$, $D = \left(\phi_j(\omega_i)\right)_{1\le i\le I, 1\le j\le Nn}$, $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_I)^t$, $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_{Nn})^t$, $\widehat{\mathbf{f}} = \left(\langle \boldsymbol{\beta}, \boldsymbol{\rho}\omega_i\rangle_m + \varepsilon(f,\omega_i)_{0,\Omega}\right)^t_{1\le i\le I}$, and $\mathbf{y} = (\mathbf{y}_1,\dots,\mathbf{y}_{Nn})^t$.

If we write $A = \left(\omega_j(\mathbf{a}_i)\right)_{1\le i\le m, 1\le j\le I}$ and $R = \left((\omega_i,\omega_j)_L\right)_{1\le i,j\le I}$, then $C = A^t A + \varepsilon R$ and $\widehat{\mathbf{f}} = A^t\boldsymbol{\beta} + \varepsilon\widetilde{\mathbf{f}}$, with $\widetilde{\mathbf{f}} = \left((\omega_1,f)_{0,\Omega},\dots,(\omega_I,f)_{0,\Omega}\right)^t$.

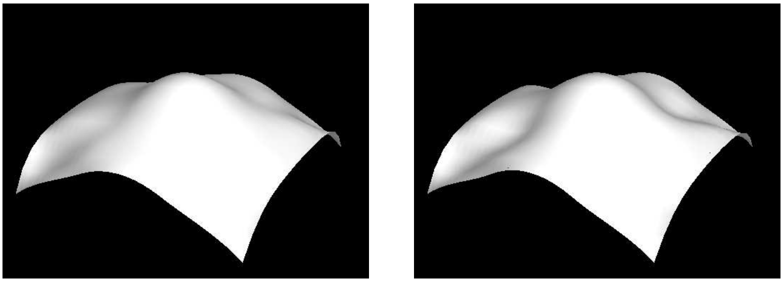

Figure 1: Discrete PDE splines with 225 points of approximation, $d=5$, $N=16$ and $\varepsilon=10^{-5}$ (left) and $m=900$, $d=5$, $N=16$, $\varepsilon=10^{-9}$ (right).

§6. Numerical and graphical examples

We present two examples in order to test the validation of the smoothing method that we have constructed. In both, we have taken $\Omega=(0,1)\times(0,1)$. Likewise, we have taken a triangulation $\mathscr{T}_h$ of $\overline{\Omega}$, made of $d\times d$ equal squares, such that the points of B^N are nodes of this triangulation in $\partial\Omega$, except the geometric vertices of Ω, as explained in the introduction. The finite element space X_h is constructed on $\mathscr{T}_h$ from the Bogner-Fox-Schmit rectangle of class C^1. For point set A^r, we have considered a randomly distributed set on $D=\{(x,y)\in\mathbb{R}^2 : \langle(x-0.5,y-0.5)\rangle_2\le 0.2\}\subset\Omega$. In addition, we have considered

$$Lu=\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)^2 u(x,y),$$

and $f(x,y)=0$. The boundary conditions are particular for each example.

Example 1. For the first example, we have constructed a discrete PDE spline. For this purpose, we take the boundary conditions from the function given by

$$h(x,y)=-((y-0.5)^2+1)((x-0.5)^2+1),$$

i.e., $h_0(x,y)=h(x,y)$ and $h_1(x,y)=\frac{\partial h}{\partial \mathbf{n}}(x,y)$, for all $(x,y)\in\partial\Omega$. Finally, we take $\boldsymbol{\beta}=\boldsymbol{\rho} g$ where

$$g(x,y)=3\sqrt{0.2-(x-0.5)^2-(y-0.5)^2}-2.3, \quad \forall (x,y)\in D.$$

Figure 1 shows two graphs of discrete PDE splines in X_h associated with L, B^N, $\boldsymbol{\tau}^N h$, A^r, $\boldsymbol{\rho} g$ and ε. We can observe the effect of the parameter ε. When this parameter decreases the discrete PDE spline comes closer to the approximation points.

Example 2. We have used our method to approximate a test function. We have chosen Nielson's function given by $N(x,y)=\frac{y}{2}\cos^4(4(x^2+y-1))$.

To get a quantitative measure of the degree of approximation provided by each discrete PDE spline in X_h associated with L, B^N, $\boldsymbol{\tau}^N N$, A^r, $\boldsymbol{\rho} N$ and ε, we have computed an estimation

$d \times d$	N	m	ε	E_{rel}	$d \times d$	N	m	ε	E_{rel}
3×3	8	0	10^{-5}	5.63×10^{-1}	5×5	16	0	10^{-5}	4.11×10^{-2}
		225	10^{-6}	3.55×10^{-1}			225	10^{-6}	1.71×10^{-2}
		625	10^{-9}	1.53×10^{-1}			625	10^{-9}	1.55×10^{-2}
7×7	24	0	10^{-5}	2.17×10^{-2}	10×10	36	0	10^{-5}	8.61×10^{-3}
		225	10^{-6}	1.12×10^{-2}			225	10^{-6}	8.04×10^{-3}
		625	10^{-9}	0.21×10^{-2}			625	10^{-9}	7.72×10^{-3}

Table 1: Error for some discrete PDE splines approximating Nielson's function.

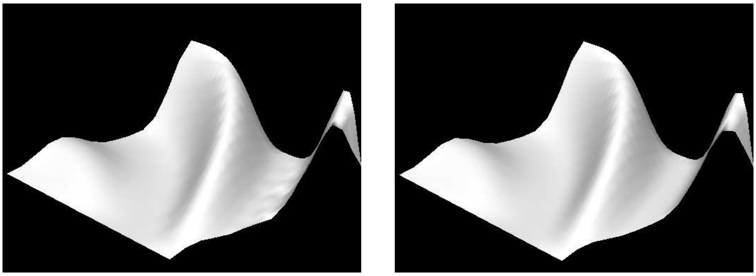

Figure 2: Discrete PDE splines corresponding to $m = 225$, $d = 5$, $N = 16$ and $\varepsilon = 10^{-9}$ with $E_{rel} = 0.0155002$ (left), and to $m = 625$, $d = 10$, $N = 36$ and $\varepsilon = 10^{-9}$, with $E_{rel} = 0.007725$ (right), both of which approximate Nielson's function.

of the relative error E_{rel} in the L^2 norm, given by

$$E_{rel} = \left(\sum_{i=1}^{1600} |\sigma_h(x_i) - N(x_i)|^2 \Big/ \sum_{i=1}^{1600} |N(x_i)|^2 \right)^{1/2},$$

where N is the approximating function and $\{x_i\}_{i=1,\dots,1600}$ is a fixed set of points regularly distributed on $\overline{\Omega}$. Table 1 shows the relative error computed for discrete PDE splines with various parameter values. As can be observed, the quality of fitting is influenced by all the parameters but is not random. It is necessary to change all of them in terms of Theorem 3 for a good fitting. Moreover, as the relative error is not monotone with respect to ε, we can surmise the existence of an optimal value of ε, when the other parameters are fixed, according to the GCV method (see Wahba [9]). Figure 2 shows the graphs of two discrete PDE spline which approximate Nielson's function to different values of the problem parameters. Finally, when the data come from the boundary function we get the convergence even if we do not take approximation points in Ω. In this case, the method is in reality the Galerkin method.

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