Monografías del Seminario Matemático García de Galdeano **33**, 433–440 (2006)

AN ANTIMAXIMUM PRINCIPLE FOR A DEGENERATE PARABOLIC PROBLEM

Juan Francisco Padial, Peter Takáč and Lourdes Tello

Abstract. We present a mathematical treatment of an important example of a non-Newtonian fluid flow that occurs in a model studying the penitration of water through rocky or sandy dams. This model differs from porous medium models significantly. Here, the nonlinear phenomena are described by the p-Laplacian $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. We treat the following Dirichlet problem for the p-Laplacian with a spectral parameter λ near the first eigenvalue λ_1:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = \lambda\,|u|^{p-2}u + f(x,t), & (x,t) \in \Omega \times (0,T_\infty); \\ \quad u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T_\infty); \\ \quad u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Here, $1 < p < \infty$ and $\lambda \in \mathbf{R}$ is near λ_1. The interval of existence in time, $(0,T_\infty)$, is assumed to be maximal. We show that, if $f(\cdot,t) \geq f_0$ with some $0 \leq f_0 \not\equiv 0$ in $L^\infty(\Omega)$, and $u_0 \in C^1(\overline{\Omega})$ is arbitrary, possibly nonpositive, and $\lambda_1 < \lambda < \lambda_1 + \delta$ where $\delta \equiv \delta(f_0,u_0) > 0$ is small enough, then there is some time $T \in (0,T_\infty)$ such that every solution of problem (1) satisfies $u(x,t) > 0$ for all $x \in \Omega$ and all $t \in (T,T_\infty)$.

Keywords: Non-Newtonian flow, p-Laplacian, parabolic antimaximum principle, first eigenvalue, local and global solution.

AMS classification: 35K65, 35B35, 46E35, 35B33.

§1. Introduction

Beginning with the work of Clément and Peletier [4], various kinds of antimaximum principles have been established for linear and nonlinear *elliptic* operators. In the case of the Dirichlet p-Laplacian Δ_p $(1 < p < \infty)$, $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, which we deal with throughout the present article, the antimaximum principle takes the following form; see Fleckinger et al. [11]:

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with a connected C^2-boundary $\partial\Omega$. Denote by λ_1 the first (smallest) eigenvalue of $-\Delta_p$. Then, given any $f \in L^\infty(\Omega)$ with $0 \leq f \not\equiv 0$ in Ω, there exists a constant $\delta \equiv \delta(f) > 0$ such that, if $\lambda_1 < \lambda < \lambda_1 + \delta$ then every solution $u \in W_0^{1,p}(\Omega)$ of the boundary value problem

$$-\Delta_p u = \lambda\,|u|^{p-2}u + f(x) \ \text{ in } \Omega; \qquad u = 0 \ \text{ on } \partial\Omega, \tag{1}$$

satisfies $u < 0$ in Ω and $\partial u/\partial\nu > 0$ on $\partial\Omega$. In contrast, if $-\infty < \lambda < \lambda_1$ then $u > 0$ in Ω and $\partial u/\partial\nu < 0$ on $\partial\Omega$. As usual, $\partial/\partial\nu$ denotes the outer normal derivative on $\partial\Omega$.

An antimaximum principle for linear *parabolic* operators has been obtained in the recent work of Díaz and Fleckinger [5, Theorem 2.1]. The main result of our present work is an analogue for the nonlinear parabolic operator with Δ_p in the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_p u = \lambda\, |u|^{p-2}u + f(x,t), & (x,t) \in \Omega \times (0,T); \\ \quad u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T); \\ \quad u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2}$$

$T = T_\infty$, where T_∞ ($0 < T_\infty \le \infty$) denotes the maximum time for existence of a weak solution $u\colon \Omega \times (0,T) \to \mathbb{R}$; of course, $T_\infty \equiv T_\infty(f,u_0)$ depends on f and $u_0 \in W_0^{1,p}(\Omega)$. We can state it as follows:

Theorem 1. *Let φ_1 denote the eigenfunction associated with λ_1 and normalized by $\varphi_1 > 0$ in Ω and $\int_\Omega \varphi_1^p \,\mathrm{d}x = 1$. Assume that $f \in L^\infty(\Omega \times \mathbb{R}_+)$ satisfies $f(x,t) \ge f_0(x)$ in $\Omega \times \mathbb{R}_+$, where $f_0 \in L^\infty(\Omega)$ is a function with $0 \le f_0 \not\equiv 0$ in Ω, and $u_0 \in W_0^{1,p}(\Omega)$ is such that $u_0 \ge -c\varphi_1$ in Ω, where $c > 0$ is a constant. Then there exist constants $\delta \equiv \delta(f_0,u_0) > 0$ and $T_+ \equiv T_+(f,u_0) \in (0,T_\infty)$ such that, if $\lambda_1 < \lambda < \lambda_1 + \delta$, then every weak solution $u\colon \Omega \times (0,T_\infty) \to \mathbb{R}$ of problem (2) satisfies $u(x,t) > 0$ for all $(x,t) \in \Omega \times [T_+,T_\infty)$ and $(\partial u/\partial \nu)(x,t) < 0$ for all $(x,t) \in \partial\Omega \times [T_+,T_\infty)$.*

This means that even if the initial conditions u_0 are large negative, say, $u_0 = -c\varphi_1$ in Ω with a constant $c > 0$, the solution $u(\,\cdot\,,t)$ eventually becomes positive for all times $t \in [T_+,T_\infty)$. The hypothesis $0 \le f_0 \not\equiv 0$ in Ω can be weakened to $\int_\Omega f_0\varphi_1 \,\mathrm{d}x > 0$ provided the resonant elliptic problem (1) with $\lambda = \lambda_1$ and $f = f_0$ has *no* weak solution. For the elliptic problem (1) this generalization is due to Arcoya and Gámez [3, Theorem 27, p. 1908].

§2. Preliminaries

All Banach and Hilbert spaces used in this article are real. We work with the standard inner product in $L^2(\Omega)$ defined by $\langle u,v\rangle \stackrel{\text{def}}{=} \int_\Omega uv \,\mathrm{d}x$ for $u,v \in L^2(\Omega)$. The orthogonal complement in $L^2(\Omega)$ of a set $\mathscr{M} \subset L^2(\Omega)$ is denoted by $\mathscr{M}^{\perp,L^2}$,

$$\mathscr{M}^{\perp,L^2} \stackrel{\text{def}}{=} \{u \in L^2(\Omega)\colon \langle u,v\rangle = 0 \text{ for all } v \in \mathscr{M}\}.$$

The inner product $\langle\cdot\,,\cdot\rangle$ in $L^2(\Omega)$ induces a duality between the Lebesgue spaces $L^p(\Omega)$ and $L^{p'}(\Omega)$, where $1 \le p,p' \le \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, and between the Sobolev space $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, as well. We keep the same notation also for the duality between the Cartesian products $[L^p(\Omega)]^N$ and $[L^{p'}(\Omega)]^N$. The closure, interior and boundary of a set $S \subset \mathbb{R}^N$ are denoted by $\overline{S}$, $\mathrm{int}(S)$ and ∂S, respectively, and the characteristic function of S by $\chi_S\colon \mathbb{R}^N \to \{0,1\}$. We write $|S|_N \stackrel{\text{def}}{=} \int_{\mathbb{R}^N} \chi_S(x)\,\mathrm{d}x$ if S is also Lebesgue measurable.

We always assume the following

$$\begin{cases} \text{If } N \ge 2 \text{ then } \Omega \text{ is a bounded domain in } \mathbb{R}^N \text{ whose boundary } \partial\Omega \text{ is a} \\ \text{compact manifold of class } C^{1,\alpha} \text{ for some } \alpha \in (0,1), \text{ and } \Omega \text{ satisfies also} \\ \text{the interior sphere condition at every point of } \partial\Omega. \text{ If } N = 1 \text{ then } \Omega \text{ is a} \\ \text{bounded open interval in } \mathbb{R}^1. \end{cases} \tag{H1}$$

For $N \geq 2$, (H1) is satisfied if $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary.

We denote $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and always take $1 < p < \infty$. Let λ_1 denote the first (smallest) eigenvalue of the positive Dirichlet p-Laplacian Δ_p, that is,

$$-\Delta_p \varphi_1 = \lambda_1 |\varphi_1|^{p-2}\varphi_1 \ \text{ in } \Omega; \qquad \varphi_1 = 0 \ \text{ on } \partial\Omega, \tag{3}$$

holds with an eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega) \setminus \{0\}$. Eigenvalue λ_1 is simple, by a result due to Anane [1, Théorème 1, p. 727] or Lindqvist [13, Theorem 1.3, p. 157], and it is given by the Rayleigh quotient

$$\lambda_1 \stackrel{\text{def}}{=} \inf\left\{ \int_\Omega |\nabla u|^p \,\mathrm{d}x \colon u \in W_0^{1,p}(\Omega) \ \text{ with } \ \int_\Omega |u|^p \,\mathrm{d}x = 1 \right\}, \tag{4}$$

$\lambda_1 > 0$. Moreover, a minimizer – the corresponding eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega) \setminus \{0\}$ – can be normalized by $\varphi_1 > 0$ in Ω and $\|\varphi_1\|_{L^p(\Omega)} = 1$, owing to the strong maximum principle [17, Prop. 3.2.1 and 3.2.2, p. 801] or [19, Theorem 5, p. 200] (see also [1, Théorème 1, p. 727] or [13, Theorem 1.3, p. 157]). We have $\varphi_1 \in L^\infty(\Omega)$ by [2, Théorème A.1, p. 96]. Consequently, recalling hypothesis (H1), we get even $\varphi_1 \in C^{1+\beta}(\overline{\Omega})$ (where $C^{1+\beta}(\overline{\Omega}) \equiv C^{1,\beta}(\overline{\Omega})$) for some $\beta \in (0, \alpha)$, by a regularity result due to [6, Theorem 2, p. 829] and [18, Theorem 1, p. 127] (interior regularity), and to [8, Theorem 1, p. 1203] (regularity near the boundary). The constant β depends solely on α, N and p. We keep the meaning of the constants α and β throughout the entire article and denote by $\beta' \in (0, \beta)$ an arbitrary, but fixed number. Finally, the Hopf maximum principle [17, Prop. 3.2.1 and 3.2.2, p. 801] or [19, Theorem 5, p. 200] renders

$$\varphi_1 > 0 \ \text{ in } \Omega \qquad \text{and} \qquad \frac{\partial \varphi_1}{\partial \nu} < 0 \ \text{ on } \partial\Omega. \tag{5}$$

We set

$$U \stackrel{\text{def}}{=} \{x \in \Omega \colon \nabla\varphi_1(x) \neq \mathbf{0}\}, \ \text{ hence } \ \Omega \setminus U = \{x \in \Omega \colon \nabla\varphi_1(x) = \mathbf{0}\},$$

and observe that $\Omega \setminus U$ is a compact subset of Ω, by (5).

Often, a function $u \in L^1(\Omega)$ will be decomposed as the orthogonal sum $u = u^{\parallel} \cdot \varphi_1 + u^{\top}$ according to

$$u^{\parallel} \stackrel{\text{def}}{=} \|\varphi_1\|_{L^2(\Omega)}^{-2} \langle u, \varphi_1 \rangle \ \text{ and } \ \langle u^{\top}, \varphi_1 \rangle = 0. \tag{6}$$

Given a linear subspace $\mathscr{M}$ of $L^1(\Omega)$ with $\varphi_1 \in \mathscr{M}$, we write

$$\mathscr{M}^{\top} \stackrel{\text{def}}{=} \{u \in \mathscr{M} \colon \langle u, \varphi_1 \rangle = 0\}.$$

In particular, we will find it convenient to work with the orthogonal sum $L^2(\Omega) = \operatorname{lin}\{\varphi_1\} \oplus L^2(\Omega)^{\top}$.

We are interested in weak solutions to the evolutionary problem (2) in a cylindrical domain $\Omega \times (0, T)$ with some $0 < T \leq \infty$.

Definition 1. Let $0 < T \leq \infty$. We say that $u \colon \Omega \times (0, T) \to \mathbb{R}$ is a *weak solution* of problem (2) in $\Omega \times (0, T)$ if it satisfies

$$u \in C\left([0, T'] \to L^2(\Omega)\right) \cap L^p\left((0, T') \to W_0^{1,p}(\Omega)\right)$$

for every $T' \in (0,T)$, together with

$$\begin{aligned}
&\int_\Omega u(T')\,\phi(T')\,\mathrm{d}x - \int_0^{T'} \left\langle u, \frac{\partial \phi}{\partial t} \right\rangle_{W_0^{1,p}\times W^{-1,p'}} \mathrm{d}t \\
&\quad + \int_0^{T'}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t - \lambda\int_0^{T'}\int_\Omega |u|^{p-2}u\,\phi\,\mathrm{d}x\,\mathrm{d}t \qquad (7)\\
&= \int_0^{T'}\int_\Omega f(x,t)\,\phi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0(x)\,\phi(x,0)\,\mathrm{d}x
\end{aligned}$$

for all $\phi \in L^p\left((0,T')\to W_0^{1,p}(\Omega)\right)\cap W^{1,p'}\left((0,T')\to W^{-1,p'}(\Omega)\right)$.

For

$$T_\infty \stackrel{\text{def}}{=} \sup\{T>0\colon u \text{ is a weak solution in } \Omega\times(0,T)\}$$

we say that $[0,T_\infty)$ is the *maximal (time) interval of existence* of a weak solution u to problem (2).

For any weak solution, two alternatives are possible: either it exists for all times t, $0\le t<T_\infty=\infty$, or else it blows up in finite time as $t\nearrow T_\infty<\infty$. We will see later that the latter case (blow-up) will be characterized by $\|u(t)\|_{L^p(\Omega)}\to\infty$ as $t\nearrow T_\infty$.

Local (in time) existence of a *weak solution* of problem (2) follows from standard results in Vrabie [20]. Global (in time) existence is guaranteed by a Lyapunov–like functional as long as the norm $\|u(t)\|_{L^p(\Omega)}$ does not blow-up (stays locally bounded in time).

Notice that the solution is unique if $p\ge 2$, by standard arguments, cf. [20], because the nonlinearity on the right hand side is a locally Lipschitz continuous function on $L^p(\Omega)$. Even if the solution might not be unique if $1<p<2$, it is not difficult to construct a "minimal solution" to problem (2) (with respect to the pointwise ordering of functions on $\Omega\times(0,T)$ by "$\le$"). Our hypothesis on the initial conditions u_0, that $u_0\in W_0^{1,p}(\Omega)$ be such that $u_0\ge -c\varphi_1$ in Ω, where $c>0$ is a constant, plays a key role in both, defining and obtaining a minimal solution. Our definition and construction guarantee that a minimal solution is unique.

§3. Main result

We assume that $\Omega\subset\mathbb{R}^N$ satisfies hypothesis (H1). If $2<p<\infty$, we need to impose another technical hypothesis on Ω. To this end, we first introduce a new norm on $W_0^{1,p}(\Omega)$ by

$$\|v\|_{\varphi_1} \stackrel{\text{def}}{=} \left(\int_\Omega |\nabla\varphi_1|^{p-2}|\nabla v|^2\,\mathrm{d}x\right)^{1/2} \qquad \text{for } v\in W_0^{1,p}(\Omega), \qquad (8)$$

and denote by $\mathscr{D}_{\varphi_1}$ the completion of $W_0^{1,p}(\Omega)$ with respect to this norm. That the semi-norm (8) is in fact a norm on $W_0^{1,p}(\Omega)$ follows from an inequality in Takáč [14, ineq. (4.7), p. 200]. The Hilbert space $\mathscr{D}_{\varphi_1}$ coincides with the domain of the closure of the quadratic form

$\mathscr{Q}_0\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$2\cdot\mathscr{Q}_0(\phi) = \int_\Omega |\nabla\varphi_1|^{p-2}\left\{|\nabla\phi|^2 + (p-2)\left|\frac{\nabla\varphi_1}{|\nabla\varphi_1|}\cdot\nabla\phi\right|^2\right\}\mathrm{d}x \\ -\lambda_1(p-1)\int_\Omega \varphi_1^{p-2}\phi^2\,\mathrm{d}x, \quad \phi\in W_0^{1,p}(\Omega). \tag{9}$$

For $2<p<\infty$ we impose the following additional hypothesis on the domain Ω:

(H2) If $N\geq 2$ and $\partial\Omega$ is not connected, then there is *no* function $v\in\mathscr{D}_{\varphi_1}$, $\mathscr{Q}_0(v)=0$, with the following four properties:

- (i) $v=\varphi_1\cdot\chi_S$ a.e. in Ω, where $S\subset\Omega$ is Lebesgue measurable, $0<|S|_N<|\Omega|_N$;
- (ii) $\overline{S}$ is connected and $\overline{S}\cap\partial\Omega\neq\emptyset$;
- (iii) if V is a connected component of U, then either $V\subset S$ or else $V\subset\Omega\setminus S$;
- (iv) $(\partial S)\cap\Omega\subset\Omega\setminus U$. (Recall $\Omega\setminus U=\{x\in\Omega\colon \nabla\varphi_1(x)=\mathbf{0}\}$.)

It has been *conjectured* in Takáč [14, §2.1] that (H2) always holds true provided (H1) is satisfied. The cases, when Ω is either an interval in $\mathbb{R}^1$ or else $\partial\Omega$ is connected if $N\geq 2$, have been covered within the proof of Proposition 4.4 in [14, pp. 202–205] which claims:

Proposition 2. *Let* $2<p<\infty$ *and assume both hypotheses* (H1) *and* (H2). *Then a function* $u\in\mathscr{D}_{\varphi_1}$ *satisfies* $\mathscr{Q}_0(u)=0$ *if and only if* $u=\kappa\varphi_1$ *for some constant* $\kappa\in\mathbb{R}$.

In particular, there is no function $v\in\mathscr{D}_{\varphi_1}$, $\mathscr{Q}_0(v)=0$, with properties (i)–(iv). This proposition is the only place where (H2) is needed explicitly. All other results in this article depend solely on the conclusion of the proposition which, in turn, implies (H2).

For $1<p<2$ we further require hypothesis (H1), but need to redefine the Hilbert space $\mathscr{D}_{\varphi_1}$ as follows. We define $v\in\mathscr{D}_{\varphi_1}$ if and only if $v\in W_0^{1,2}(\Omega)$, $\nabla v(x)=\mathbf{0}$ for almost every $x\in\Omega\setminus U=\{x\in\Omega\colon\nabla\varphi_1(x)=\mathbf{0}\}$, and

$$\|v\|_{\varphi_1} \stackrel{\text{def}}{=} \left(\int_U |\nabla\varphi_1|^{p-2}|\nabla v|^2\,\mathrm{d}x\right)^{1/2} < \infty. \tag{10}$$

Consequently, $\mathscr{D}_{\varphi_1}$ endowed with the norm $\|\cdot\|_{\varphi_1}$ is continuously embedded into $W_0^{1,2}(\Omega)$. We *conjecture* that $\mathscr{D}_{\varphi_1}$ is dense in $L^2(\Omega)$. This conjecture would immediately follow from $|\Omega\setminus U|_N=0$. The latter holds true if Ω is convex; then also $\Omega\setminus U$ is a convex set in $\mathbb{R}^N$ with empty interior, and hence of zero Lebesgue measure; see [12, Lemma 2.6, p. 55].

If the conjecture is false, we need to consider also the orthogonal complement

$$\mathscr{D}_{\varphi_1}^{\perp,L^2} = \{v\in L^2(\Omega)\colon \langle v,\phi\rangle = 0 \text{ for all } \phi\in\mathscr{D}_{\varphi_1}\}.$$

Notice that $v \in \mathscr{D}_{\varphi_1}^{\perp,L^2}$ implies $v = 0$ almost everywhere in U. This means that $\mathscr{D}_{\varphi_1}^{\perp,L^2}$ is isometrically isomorphic to a closed linear subspace of $L^2(\Omega \setminus U)$. Moreover, $\chi_{\Omega\setminus U} \notin \mathscr{D}_{\varphi_1}^{\perp,L^2}$ since $\Omega \setminus U$ is a compact subset of Ω; hence, there is a C^1 function $\phi \in \mathscr{D}_{\varphi_1}$, $0 \le \phi \le 1$, with compact support in Ω and such that $\phi = 1$ in an open neighborhood of $\Omega \setminus U$.

Hypothesis (H2) always holds true for $1 < p < 2$; see Takáč [14, Sect. 8, p. 225].

Remark 1. It is not difficult to verify that the conclusion of Proposition 2 remains valid also for $1 < p < 2$, by [14, Remark 8.1, p. 225].

We write $f_0 \equiv \zeta\varphi_1 + f_0^\top$ with $\zeta \in \mathbb{R}$ and $f_0^\top \in L^\infty(\Omega)$.

The main result of our present article is the following *Antimaximum Principle* for problem (2) with any $1 < p < \infty$. This is a more general version of Theorem 1 stated in the Introduction (Section 1); here, function $f(x,t)$ does not need to be nonnegative.

Theorem 3. (Antimaximum Principle). *Let* $1 < p < \infty$ *and assume that* $\Omega \subset \mathbb{R}^N$ *satisfies hypothesis* (H1). *If* $p > 2$, *assume that* Ω *satisfies also hypothesis* (H2). *Let* $f \in L^\infty(\Omega \times \mathbb{R}_+)$ *be such that*

$$f(x,t) \ge f_0(x) \quad \text{in } \Omega \times \mathbb{R}_+, \tag{11}$$

where $f_0 \in L^\infty(\Omega)$ *satisfies* $\int_\Omega f_0\varphi_1\,\mathrm{d}x > 0$ *and the resonant problem*

$$-\Delta_p u = \lambda_1 |u|^{p-2}u + f_0(x) \ \text{ in } \Omega; \qquad u = 0 \ \text{ on } \partial\Omega, \tag{12}$$

has no *weak solution. Finally, assume that* $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *satisfies*

$$u_0(x) \ge -c\varphi_1(x) \quad \text{in } \Omega, \tag{13}$$

where $c > 0$ *is a constant. Then there exist constants* $\delta \equiv \delta(f_0,u_0) > 0$ *and* $T_+ \equiv T_+(f,u_0) \in (0,T_\infty)$ *such that, if* $\lambda_1 < \lambda < \lambda_1 + \delta$, *then every weak solution* $u\colon \Omega \times (0,T_\infty) \to \mathbb{R}$ *of problem* (2) *satisfies* $u(x,t) > 0$ *for all* $(x,t) \in \Omega \times [T_+,T_\infty)$ *and* $(\partial u/\partial\nu)(x,t) < 0$ *for all* $(x,t) \in \partial\Omega \times [T_+,T_\infty)$.

The following remark is about the function f_0 which appears in the resonant problem in the statement of the main theorem:

Remark 2. We notice that, given any $f_0(x) = f^\top(x) + \zeta\cdot\varphi_1(x)$ with $\zeta \in \mathbb{R}$ and $f^\top \in L^\infty(\Omega)$ satisfying $f^\top \not\equiv 0$ in Ω and $\int_\Omega f^\top\varphi_1\,\mathrm{d}x = 0$, it follows from [15, Theorems 3.1 and 3.5] that there exist two constants $\zeta_*, \zeta^* \in \mathbb{R}$, $-\infty < \zeta_* < 0 < \zeta^* < \infty$, such that the elliptic problem (12) has a weak solution $u \in W_0^{1,p}(\Omega)$ if and only if $\zeta_* \le \zeta \le \zeta^*$.

Theorem 3 will be proved in a number of steps in a separate work. In fact, we obtain a much more precise result if the time T_+ in this theorem is chosen large enough:

Corollary 4 (“Large” Positive Solutions)**.** *In the situation of Theorem 3 above, we can choose* $T_+ \equiv T_+(f,u_0) \in (0,T_\infty)$ *such that, if* $\lambda_1 < \lambda < \lambda_1 + \delta$, *then every weak solution* $u\colon \Omega \times (0,T_\infty) \to \mathbb{R}$ *of problem* (2) *satisfies*

$$u(x,t) = \tau(t)\big(\varphi_1(x) + v^\top(x,t)\big) \quad \text{for all } (x,t) \in \Omega \times (T_+,T_\infty), \tag{14}$$

where functions τ *and* $v^\top$ *have the following properties:*

(a) $\tau\colon [T_+, T_\infty) \to (0, \infty)$ *is locally absolutely continuous with* $\tau \in W^{1,p'}(T_+, T')$ *for every* $T' \in (T_+, T_\infty)$*, and* $\tau(t) \to +\infty$ *as* $t \nearrow T_\infty$*; and*

(b) $v^\top \in C^{1+\beta,(1+\beta)/2}\big(\overline{\Omega} \times [T_+, T']\big)$ *for every* $T' \in (T_+, T_\infty)$*, with* $\int_\Omega v^\top(x,t)\,\varphi_1\,\mathrm{d}x = 0$ *and* $|v^\top(x,t)| \leq \frac{1}{2}\varphi_1(x)$ *for all* $x \in \Omega$ *and* $T_+ \leq t < T_\infty$*, and* $\big\|v^\top(\,\cdot\,,t)\big\|_{C^{1+\beta'}(\overline{\Omega})} \to 0$ *as* $t \nearrow T_\infty$*, whenever* $0 < \beta' < \beta$.

This means that the solution $u(\,\cdot\,,t)$ eventually becomes positive and behaves like $\tau(t)\varphi_1$ for all times $t \in [T_+, T_\infty)$. The asymptotic behavior of $\tau(t)$ as $t \nearrow T_\infty$ is determined by the (positive) solution $z\colon [T_+, T_\infty) \to (0, \infty)$ of the ordinary differential equation

$$\|\varphi_1\|^2_{L^2(\Omega)} \cdot \frac{\mathrm{d}}{\mathrm{d}t} z(t) = (\lambda - \lambda_1) z(t)^{p-1} + \langle f(\,\cdot\,,t), \varphi_1 \rangle, \quad T_+ \leq t < T_\infty, \tag{15}$$

with a suitable initial condition at $t = T_+$. Notice that $\langle f(\,\cdot\,,t), \varphi_1 \rangle \geq \langle f_0, \varphi_1 \rangle > 0$ holds by (11).

Acknowledgements

The work all three authors was supported in part by Ministerio de Ciencia y Tecnología (Spain) and the German Academic Exchange Service (DAAD, Germany) within the exchange program “Acciones Integradas”.

References

[1] ANANE, A. Simplicité et isolation de la première valeur propre du p-laplacien avec poids. *Comptes Rendus Acad. Sc. Paris, Série I, 305* (1987), 725–728.

[2] ANANE, A. *Etude des Valeurs Propres et de la Résonance pour l'Opérateur p-Laplacien*. Thèse de Doctorat, Université Libre de Bruxelles, Brussels, 1988.

[3] ARCOYA, D., AND GÁMEZ, J. L. Bifurcation theory and related problems: anti-maximum principle and resonance. *Comm. P.D.E. 26*, 9–10 (2001), 1879–1911.

[4] CLÉMENT, PH., AND PELETIER, L. A. An anti-maximum principle for second order elliptic operators. *J. Differential Equations 34* (1979), 218–229.

[5] DÍAZ, J. I., AND FLECKINGER, J. Positivity for large time of solutions of the heat equation: the parabolic antimaximum principle. *Discrete and Continuous Dynamical Systems 10*, 1–2 (2003), 193–200.

[6] DIBENEDETTO, E. $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal. 7*, 8 (1983), 827–850.

[7] DIBENEDETTO, E. *Degenerate Parabolic Equations*. Universitext, Springer-Verlag, New York-Berlin-Heidelberg, 1993.

[8] LIEBERMAN, G. M. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal. 12*, 11 (1988), 1203–1219.

[9] DRÁBEK, P., GIRG, P., TAKÁČ, P., AND ULM, M. The Fredholm alternative for the p-Laplacian: bifurcation from infinity, existence and multiplicity of solutions. *Indiana Univ. Math. J. 53*, 2 (2004), 433–482.

[10] DRÁBEK, P., GIRG, P., AND TAKÁČ, P. Nonlinear perturbations of homogeneous quasilinear operators: bifurcation from infinity, existence and multiplicity. *J. Differential Equations 204*, 2 (2004), 265–291.

[11] FLECKINGER, J., GOSSEZ, J.-P., TAKÁČ, P., AND DE THÉLIN, F. Existence, nonexistence et principe de l'antimaximum pour le p-laplacien. *Comptes Rendus Acad. Sc. Paris, Série I, 321* (1995), 731–734.

[12] FLECKINGER, J., GOSSEZ, J.-P., TAKÁČ, P., AND DE THÉLIN, F. Nonexistence of solutions and an anti-maximum principle for cooperative systems with the p-Laplacian. *Mathematische Nachrichten 194* (1998), 49–78.

[13] LINDQVIST, P. On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$. *Proc. Amer. Math. Soc. 109*, 1 (1990), 157–164.

[14] TAKÁČ, P. On the Fredholm alternative for the p-Laplacian at the first eigenvalue. *Indiana Univ. Math. J. 51*, 1 (2002), 187–237.

[15] TAKÁČ, P. On the number and structure of solutions for a Fredholm alternative with the p-Laplacian. *J. Differential Equations 185* (2002), 306–347.

[16] TAKÁČ, P. L^∞-Bounds for Weak Solutions of an Evolutionary Equation with the p-Laplacian. In *Proceedings of the 2004 International Conference on Function Spaces, Differential Operators and Nonlinear Analysis (FSDONA) in honor of Alois Kufner*, May 28 – June 2, 2004, Brno-Milovy, Czech Republic, P. Drábek and J. Rákosník, Eds. Math. Inst. of the Academy of Sciences of the Czech Republic (MÚ AV ČR), Prague, 2005, pp. 327–354.

[17] TOLKSDORF, P. On the Dirichlet problem for quasilinear equations in domains with conical boundary points. *Comm. P.D.E. 8*, 7 (1983), 773–817.

[18] TOLKSDORF, P. Regularity for a more general class of quasilinear elliptic equations. *J. Differential Equations 51* (1984), 126–150.

[19] VÁZQUEZ, J. L. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim. 12* (1984), 191–202.

[20] VRABIE, I. I. *Compactness Methods for Nonlinear Evolutions*. Longman Scientific and Technical, Essex, England, 1987.

Juan Francisco Padial and Lourdes Tello
E.T.S. Arquitectura,
U.P.M. de Madrid,
Madrid, Spain
`jfpadial@aq.upm.es` and
`ltello@aq.upm.es`

Peter Takáč
Institute for Mathematics,
Universität Rostock
Universitätsplatz 1, D–18055 Rostock,
Germany
`peter.takac@uni-rostock.de`