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Fractal splines

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Abstract. Fractal methodology provides a general setting for the understanding of real-world phenomena. In particular, the classical methods of real-data interpolation can be generalized by means of fractal techniques. In this paper we use this kind of procedures to define a family of interpolating mappings associated to a cubic spline. This fact adds a "degree of freedom" to the function, allowing to preserve or modify its properties. In particular, the elements of the class can be defined so that the smoothness of the original be preserved. Under some hypotheses, and using Hermite polynomial techniques, bounds of the interpolation error for function and derivatives are obtained.

Keywords: Fractal interpolation functions, discrete power spectrum, fractal dimension.

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§1. Introduction. Fractal Interpolation Functions

Fractal interpolation techniques provide good deterministic representations of complex phenomena. Barnsley ([1, 2]) was a pioneer in the use of fractal functions to interpolate a set of data. Fractal interpolants of Barnsley can be defined for any continuous function defined on a real compact interval. This method constitutes an advance in the techniques of approximation, since all the classical methods of real-data interpolation can be generalized by means of fractal techniques (see for instance [6, 7]). Fractal interpolation functions are defined as fixed points of maps between spaces of functions using iterated function systems. The theorem of Barnsley and Harrington ([3]) proves the existence of differentiable fractal interpolation functions. In this paper we describe a very general way of constructing smooth fractal functions with the help of Hermite interpolating polynomials. The procedure has a very low computational cost. In the last section of the communication, a particular type of interpolating mappings associated to a cubic spline is defined. Under some hypotheses, bounds of the interpolation error for function and derivatives are obtained. Let $t_0 < t_1 < \cdots < t_N$ be real numbers, and $I = [t_0, t_N] \subset \mathbb{R}$ the closed interval that contains them. Let a set of data points $\{(t_i, x_i) \in I \times \mathbb{R} : \ i = 0, 1, 2, \dots, N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n, \ n \in \{1, 2, \dots, , N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1}, \quad L_n(t_N) = t_n, \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \le l\,|c_1 - c_2|, \quad \forall\, c_1, c_2 \ \in I, \tag{2}$$

for some $0 \le l < 1$. Let $-1 < \alpha_n < 1$, for $n = 1, 2, \dots, N$, and $F = I \times \mathbb{R}$. Let N continuous mappings $F_n : F \to \mathbb{R}$ be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \quad n = 1, 2, \dots, N, \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \le |\alpha_n|\,|x - y|, \quad t \in I, \quad x, y \in \mathbb{R}. \tag{4}$$

Now define functions

$$w_n(t,x) = (L_n(t), F_n(t,x)), \quad \forall n = 1,2,\ldots,N, \tag{5}$$

and consider the following theorem:

Theorem 1 ([1, 2])**.** *The iterated function system (IFS) $\{F,\ w_n : n = 1,2,\ldots,N\}$ defined above admits a unique attractor G that is the graph of a continuous function $f : I \to \mathbb{R}$ which obeys $f(t_i) = x_i$ for $i = 0,1,2,\ldots,N$.*

The previous function f is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t,x))\}_{n=1}^{N}$. The function $f : I \to \mathbb{R}$ is the unique one satisfying the functional equation

$$f(L_n(t)) = F_n(t, f(t)), \quad n = 1,2,\ldots,N, \quad t \in I, \tag{6}$$

or

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)),\ n = 1,2,\ldots,N;\ \ t \in I_n = [t_{n-1}, t_n]. \tag{7}$$

Let $\mathscr{F}$ be the set of continuous functions $f : [t_0, t_N] \to \mathbb{R}$ such that $f(t_0) = x_0$ and $f(t_N) = x_N$. Define a metric on $\mathscr{F}$ by

$$d(f,g) = \|f-g\|_\infty = \max\{|f(t)-g(t)| : t \in [t_0,t_N]\} \quad \forall\ f,g \in \mathscr{F}.$$

Then $(\mathscr{F}, d)$ is a complete metric space. Define a mapping $T : \mathscr{F} \to \mathscr{F}$ by

$$(Tf)(t) = F_n\big(L_n^{-1}(t), f \circ L_n^{-1}(t)\big) \quad \forall\, t \in [t_{n-1}, t_n],\ \ n = 1,2,\ldots,N. \tag{8}$$

Using (1)–(4), it can be proved that $(Tf)(t)$ is continuous on the interval $[t_{n-1}, t_n]$ for $n = 1,2,\ldots,N$ and at each of the points $t_1, t_2, \ldots, t_{N-1}$. T is a contraction mapping on the metric space $(\mathscr{F}, d)$

$$\|Tf - Tg\|_\infty \le |\alpha|_\infty \|f-g\|_\infty, \tag{9}$$

where $|\alpha|_\infty = \max\ \{|\alpha_n| : n = 1,2,\ldots,N\}$. Since $|\alpha|_\infty < 1$, T possesses a unique fixed point on $\mathscr{F}$, that is to say, there is $f \in \mathscr{F}$ such that $(Tf)(t) = f(t)$ for $t \in [t_0, t_N]$. This function is the FIF corresponding to w_n. The most widely studied fractal interpolation functions so far are defined by the following IFS:

$$\begin{cases} L_n(t) = a_n t + b_n, \\ F_n(t,x) = \alpha_n x + q_n(t), \end{cases} \tag{10}$$

with

$$a_n = \frac{(t_n - t_{n-1})}{(t_N - t_0)} \quad \text{and} \quad b_n = \frac{(t_N t_{n-1} - t_0 t_n)}{(t_N - t_0)}. \tag{11}$$

α_n is called the vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of the IFS. If $q_n(t)$ is a line ([1, 6]), the FIF is termed affine (AFIF). The cubic FIF ([9, 7]) are constructed using as $q_n(t)$ a cubic polynomial. In many cases the data are evenly sampled, $h = t_n - t_{n-1}$, $t_N - t_0 = Nh$, then $a_n = 1/N$. In the particular case $\alpha_n = 0$, for $n = 1,2,\ldots,N$ then: $F_n(t,x) = q_n(t)$ and $f(t) = q_n \circ L_n^{-1}(t)$ for $t \in I_n$.

§2. Construction of differentiable FIF

In this section we study the construction of differentiable fractal interpolation functions. The theorem of Barnsley and Harrington [3] provides the conditions on the IFS (10) which are sufficient for their existence. We will define IFS satisfying the prescribed hypotheses.

Theorem 2 (Barnsley and Harrington [3]). *Let $t_0 < t_1 < t_2 < \cdots < t_N$ and $L_n(t)$, for $n = 1,2,\dots,N$, the affine function $L_n(t) = a_n t + b_n$ satisfying the expressions (1)–(2). Let $a_n = L_n'(t) = \frac{t_n - t_{n-1}}{t_N - t_0}$ and $F_n(t,x) = \alpha_n x + q_n(t)$, for $n = 1,2,\dots,N$, verifying (3)–(4). Suppose, for some integer $p \geq 0$, $|\alpha_n| < a_n^p$ and $q_n \in C^p[t_0,t_N]$, for $n = 1,2,\dots,N$. Let*

$$F_{nk}(t,x) = \frac{\alpha_n x + q_n^{(k)}(t)}{a_n^k}, \quad k = 1,2,\dots,p, \tag{12}$$

$$x_{0,k} = \frac{q_1^{(k)}(t_0)}{a_1^k - \alpha_1}, \quad x_{N,k} = \frac{q_N^{(k)}(t_N)}{a_N^k - \alpha_N}, \quad k = 1,2,\dots,p.$$

If $F_{n-1,k}(t_N, x_{N,k}) = F_{nk}(t_0, x_{0,k})$, with $n = 2,3,\dots,N$ and $k = 1,2,\dots,p$, then

$$\{(L_n(t), F_n(t,x))\}_{n=1}^N$$

determines a FIF $f \in C^p[t_0,t_N]$ and $f^{(k)}$ is the FIF determined by

$$\{(L_n(t), F_{nk}(t,x))\}_{n=1}^N, \quad k = 1,2,\dots,p.$$

From here on, we consider a uniform partition in order to simplify the calculus,

$$a_n = \frac{1}{N}. \tag{13}$$

If we consider a generic polynomial q_n (for instance) the equality proposed in the theorem implies the resolution of systems of equations. We will proceed in a different way. In order to define an IFS satisfying Theorem 2, we consider the mappings (10) where

$$q_n(t) = g \circ L_n(t) - \alpha_n\, b(t), \tag{14}$$

g is a continuous function satisfying $g(t_i) = x_i$, $i = 0,1,\dots,N$, and $b(t)$ is a real continuous function, $b \neq g$, such that $b(t_0) = x_0$ and $b(t_N) = x_N$. In the reference [8] we proved some properties about this IFS.

Definition 1. Let $g \in \mathscr{C}(I)$, $\Delta : t_0 < t_1 < \cdots < t_N$ a partition of the closed interval $I = [t_0,t_N]$. Let b be as in the previous paragraph and $\alpha = (\alpha_1, \dots \alpha_N)$ the scaling vector of the IFS defined by (10) and (14). The corresponding FIF $g_{\Delta b}^{\alpha}$, or simply g^α, is termed α-fractal function of g with respect to the partition Δ and the function b.

Theorem 3. *[8] The α-fractal function g^α of g with respect to Δ and b satisfies the inequality*

$$\|g^\alpha - g\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\, \|g - b\|_\infty, \tag{15}$$

with $|\alpha|_\infty = \max_{1 \leq n \leq N}\{|\alpha_n|\}$. Besides, g^α interpolates to g, that is to say,

$$g^\alpha(t_n) = g(t_n) \quad \forall n = 0,1,\dots,N.$$

Remark 1. By the equation (7), for $t \in I_n,\ n = 1,2,\ldots,N$,

$$g^\alpha(t) = g(t) + \alpha_n (g^\alpha - b) \circ L_n^{-1}(t). \tag{16}$$

The first step is to see which conditions we should impose to $b(t)$ so that the hypotheses of Barnsley & Harrington Theorem are satisfied, assuring the existence of differentiable FIF. Let us consider $p \geq 0$, $|\alpha_n| < 1/N^p$ and $q_n(t) \in C^p[t_0,t_N]$, for $n = 1,2,\ldots,N$. The prescribed conditions, for $n = 2,3,\ldots,N$ and $k = 1,2,\ldots,p$, are:

$$F_{n-1,k}(t_N, x_{N,k}) = F_{nk}(t_0, x_{0,k}). \tag{17}$$

The theorem considers $F_{nk}(t,x) = \left(\alpha_n x + q_n^{(k)}(t)\right)/a_n^k$. In this particular case, as $L_n(t) = t/N + b_n$ and $L_n'(t) = 1/N = a_n$, we have by (14)

$$q_n^{(k)}(t) = \frac{1}{N^k}\, g^{(k)}(L_n(t)) - \alpha_n\, b^{(k)}(t), \quad \forall k = 0,1,\ldots,p. \tag{18}$$

So that (17) becomes:

$$N^k \alpha_{n-1}\, \frac{g^{(k)}(t_N) - N^k \alpha_N\, b^{(k)}(t_N)}{1 - N^k \alpha_N} - \alpha_{n-1}\, N^k\, b^{(k)}(t_N) = N^k \alpha_n\, \frac{g^{(k)}(t_0) - N^k \alpha_1\, b^{(k)}(t_0)}{1 - N^k \alpha_1} - \alpha_n\, N^k\, b^{(k)}(t_0).$$

If we consider constant scale factors $\alpha_n = \alpha$, for all $n = 1,\ldots,N$:

$$g^{(k)}(t_N) - b^{(k)}(t_N) = g^{(k)}(t_0) - b^{(k)}(t_0). \tag{19}$$

A sufficient condition in order to satisfy this equality is

$$\begin{cases} b^{(k)}(t_0) = g^{(k)}(t_0), \\ b^{(k)}(t_N) = g^{(k)}(t_N), \quad k = 0,1,2,\ldots,p. \end{cases} \tag{20}$$

Thus we look for a function b such that b agrees with g at the extremes of the interval until the p-th derivative. The conditions (20) will be satisfied if a Hermite interpolating polynomial b is considered, with nodes t_0, t_N and p derivatives at the extremes. Briefly let us remember some concepts on Hermite polynomial interpolation. Consider the real numbers ξ_i, $y_i^{(k)}$, for $k = 0,1,\ldots n_i - 1$ and $i = 0,1,\ldots,m$, with $\xi_0 < \xi_1 < \cdots < \xi_m$. The Hermite interpolation problem for these data consists of determining a polynomial P whose degree does not exceed n, where $n + 1 = \sum_{i=0}^m n_i$, and which satisfies the following interpolation conditions:

$$P^{(k)}(\xi_i) = y_i^{(k)}, \quad k = 0,1,\ldots n_i - 1, \quad i = 0,1,\ldots,m. \tag{21}$$

The existence and uniqueness of the polynomial P verifying the previous conditions (21) it is assured ([10]). Hermite interpolating polynomials can be given explicitly by :

$$P(x) = \sum_{i=0}^{m} \sum_{k=0}^{n_i - 1} y_i^{(k)} L_{ik}(x),$$

where the polynomials $L_{ik} \in \prod_n$ are generalized Lagrange polynomials ([10]). In the case in study, the sought polynomial P is the function $b(t)$. Let us assume $m = 1$, the interpolation points are the extremes of the interval $[t_0, t_N]$, $\xi_0 = t_0$, $\xi_1 = t_N$, and for $r = 0, 1, \ldots, p$:

$$b^{(r)}(t_0) = g^{(r)}(t_0), \quad b^{(r)}(t_N) = g^{(r)}(t_N),$$

where $b, g \in \mathscr{C}^p$ and $n_i = p+1$, for $i = 0, 1$, then $n + 1 = \sum_{i=0}^{1} n_i = 2p + 2$ so that $n = 2p + 1$. As consequence $b(t)$ is a Hermite interpolating polynomial of degree $2p + 1$. The function g can be arbitrarily chosen satisfying $g \in C^p$.

We consider a theorem of Ciarlet, Schultz & Varga concerning Hermite interpolation.

Theorem 4 (Ciarlet, Schultz & Varga [4])**.** *Let $g \in C^r[t_0, t_N]$ with $r \geq 2p + 2$, let Δ be any partition of $[t_0, t_N]$, $\Delta : t_0 < t_1 < \cdots < t_N$, and let $\varphi(t)$ be the unique interpolation of $g(t)$ in H_Δ^{p+1}, i.e., $g^{(l)}(t_n) = \varphi^{(l)}(t_n)$, for all $0 \leq n \leq N$, $0 \leq l \leq p$. Then,*

$$\|g^{(k)} - \varphi^{(k)}\|_\infty \leq \frac{\|\Delta\|^{2p+2-k}}{2^{2p+2-2k}\, k!\, (2p+2-2k)!}\, \|g^{(2p+2)}\|_\infty, \tag{22}$$

for all $k = 0, 1, \ldots, p + 1$.

In the case in study we need a single subinterval of length $T = b - a$. Using (15) and (22) for $k = 0$ and $\varphi = b$,

$$\|g^\alpha - g\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\, \|g - b\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\, \frac{T^{2p+2}}{2^{2p+2}\, (2p+2)!}\, \|g^{(2p+2)}\|_\infty. \tag{23}$$

§3. Interpolation Error Bounds

We consider again a uniform partition and constant scaling factors α. According to theorem of Barnsley & Harrington the IFS associated with the k-th derivative of a FIF can be expressed as:

$$\begin{cases} L_n(t) = \frac{1}{N}t + b_n, \\ F_{nk}(t, x) = N^k \alpha x + N^k q_n^{(k)}(t), \quad k = 0, 1, 2, \ldots, p. \end{cases} \tag{24}$$

In our case,

$$q_n(t) = g \circ L_n(t) - \alpha_n b(t),$$

where $b(t)$ is a Hermite interpolating polynomial of degree $2p + 1$ of g. The derivatives of $q_n(t)$ are:

$$q_n^{(k)}(t) = \frac{1}{N^k}\, g^{(k)}(L_n(t)) - \alpha\, b^{(k)}(t), \quad k = 0, 1, 2, \ldots, p, \tag{25}$$

so that the IFS defining the k-th derivative of the FIF g_b^α is expressed as:

$$\begin{cases} L_n(t) = \frac{1}{N}t + b_n, \\ F_{nk}(t, x) = N^k \alpha x + g^{(k)} \circ L_n(t) - N^k \alpha\, b^{(k)}(t), \quad k = 0, 1, 2, \ldots, p, \end{cases} \tag{26}$$

that is to say, $q_{nk}(t) = g^{(k)} \circ L_n(t) - N^k \alpha b^{(k)}(t)$, $k = 0,1,2,\ldots,p$, so that the $k-$th derivative of the function α-fractal of g respect to α and b, g_b^α, agrees with the α-fractal function of $g^{(k)}$ respect to $N^k\alpha$ and $b^{(k)}$, that is to say:

$$(g_b^\alpha)^{(k)} = (g^{(k)})_{b^{(k)}}^{N^k\alpha}, \quad k = 0,1,2,\ldots,p.$$

To bound the difference between the $k-$th derivative of g and the $k-$th derivative of g_b^α we can use Theorem 3:

$$\|(g_b^\alpha)^{(k)} - g^{(k)}\|_\infty = \|(g^{(k)})_{b^{(k)}}^{N^k\alpha} - g^{(k)}\|_\infty \le \frac{N^k|\alpha|}{1-N^k|\alpha|}\|g^{(k)} - b^{(k)}\|_\infty.$$

Considering that $b(t)$ is the Hermite interpolating polynomial of degree $2p+1$ of g, the theorem of Ciarlet, Schultz & Varga can be used in order to bound $\|g^{(k)} - b^{(k)}\|_\infty$, so that applying (22) for $\varphi = b$ and considering $g \in C^{(2p+2)}$

$$\begin{aligned}\|(g_b^\alpha)^{(k)} - g^{(k)}\|_\infty &\le \frac{N^k|\alpha|}{1-N^k|\alpha|}\|g^{(k)} - b^{(k)}\|_\infty \\ &\le \frac{N^k|\alpha|}{1-N^k|\alpha|}\frac{T^{2p+2-k}}{2^{2p+2-2k}\,k!\,(2p+2-2k)!}\,\|g^{(2p+2)}\|_\infty, \quad k = 0,1,\ldots,p.\end{aligned}$$

§4. Fractal Cubic Splines

In this section we study a particular case, considering the IFS (10) with $a_n = 1/N$, $\alpha_n = \alpha$, for all $n = 1,2,\ldots,N$, and $q_n(t) = g \circ L_n(t) - \alpha_n b(t)$, where g is a cubic spline with respect to a uniform partition Δ and b is a Hermite interpolating polynomial satisfying the described conditions (20) where $p = 2$ ($b(t)$ is a polynomial of degree $n = 5$). In order to bound the interpolation error we consider the following theorem:

Theorem 5 (Hall & Meyer [5]). *Let $f \in C^4[a,b]$ and $|f^{(4)}(t)| \le L$ for all $t \in [a,b]$. Let $\Delta = \{a = t_0 < t_1 < \cdots < t_N = b\}$ be a partition of the interval $[a,b]$, with constant distances between nodes $h = t_n - t_{n-1}$. Let S_Δ be the spline function that interpolates the values of the function f at the points $t_0,t_1,\ldots,t_N \in \Delta$, being S_Δ type I or II. Then,*

$$\|f^{(r)} - S_\Delta^{(r)}\|_\infty \le C_r\,L\,h^{4-r} \quad (r = 0,1,2), \tag{27}$$

with $C_0 = 5/384$, $C_1 = 1/24$, $C_2 = 3/8$. The constants C_0 and C_1 are optimum.

Remark 2. A spline is type I if its first derivatives at a and b are known. A spline is type II if it can be explicitly represented by its second derivatives at a and b.

To estimate $\|x - g_b^\alpha\|_\infty$, it is easy to observe that

$$\|x - g_b^\alpha\|_\infty \le \|x - g\|_\infty + \|g - g_b^\alpha\|_\infty.$$

The first adding can be bounded applying the theorem of Hall and Meyer since $g = S_\Delta(t)$ is a cubic spline. Thus

$$\|x - g\|_\infty \le C_0 L h^4. \tag{28}$$

In the second term, Theorem 3 can be used; from (15)

$$\|g - g_b^\alpha\|_\infty \le \frac{|\alpha|}{1-|\alpha|}\ \|g - b\|_\infty. \tag{29}$$

From (28)–(29)

$$\|x - g_b^\alpha\|_\infty \le C_0 L h^4 + \frac{|\alpha|}{1-|\alpha|}\ \|g - b\|_\infty. \tag{30}$$

The inequality above can be transformed considering the following result:

Lemma 6. *If $|\alpha| < 1/N^2$, there exists $s = s(N)$ such that $0 < s < 1$ and $|\alpha| \le 1/N^{2+s}$.*

Proof. By hypothesis $|\alpha| < 1/N^2$. Since $1/N^{2+x} \to 1/N^2$ as $x \to 0^+$, there exists $s = s(N)$ such that $0 < s < 1$ and $|\alpha| \le 1/N^{2+s}$. □

As a consequence, if $|\alpha| < 1/N^2$, there exists s such that

$$\frac{|\alpha|}{1-|\alpha|} \le \frac{1}{N^{2+s}-1},$$

and we obtain the following result.

Theorem 7. *Let $x(t)$ be a function verifying $x(t) \in C^4[t_0,t_N]$ and $|x^{(4)}(t)| \le L$ for all $t \in [t_0,t_N]$. Let $s = s(N) > 0$ such that $0 < s < 1$ and $|\alpha| \le \frac{1}{N^{2+s}}$. Then,*

$$\|x - g_b^\alpha\|_\infty \le K_0 h^4 + \frac{1}{N^{2+s}-1}\ \|g - b\|_\infty,$$

or

$$\|x - g_b^\alpha\|_\infty \le K_0 h^4 + \frac{h^{2+s}}{T^{2+s} - h^{2+s}}\ \|g - b\|_\infty, \tag{31}$$

where $K_0 = LC_0$ is the constant of Hall and Meyer Theorem and $T = t_N - t_0 = Nh$.

We proceed in a similar way for the derivatives of the spline.

Theorem 8. *Let $x(t)$ be a function verifying $x(t) \in C^4[t_0,t_N]$ and $|x^{(4)}(t)| \le L$ for all $t \in [t_0,t_N]$. Let $g(t)$ be a cubic spline and $b(t)$ be a Hermite interpolating polynomial of degree 5 of g. Let $s = s(N) > 0$ be, such that $0 < s < 1$ and $|\alpha| \le \frac{1}{N^{2+s}}$. Then,*

$$\|x' - (g_b^\alpha)'\|_\infty \le K_1 h^3 + \frac{1}{N^{1+s}-1}\ \|g' - b'\|_\infty,$$

and

$$\|x'' - (g_b^\alpha)''\|_\infty \le K_2 h^2 + \frac{1}{N^{s}-1}\ \|g'' - b''\|_\infty, \tag{32}$$

where $K_1 = LC_1$, $K_2 = LC_2$ are the constant of Hall and Meyer Theorem and $T = t_N - t_0 = Nh$.

The differences $\|g^{(k)} - b^{(k)}\|_\infty$ can be bounded considering that b is the interpolating Hermite polynomial of g and using a theorem of interpolation error for this kind of approximants.

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