Monografías del Seminario Matemático García de Galdeano **33**, 153–159 (2006)

LOCATION OF MISTAKEN DATA USING WAVELETS DEFINED BY A CLASSICAL MEAN

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Abstract. This work may be regarded as a link between classical and modern tools used in Mathematics and other disciplines. We consider the trigonometric de la Vallée Poussin's mean [11], discovered in 1908, to construct wavelet functions [3], really developed eighty years later, and considered the most recent addition to the subject of orthogonal systems. The obtained functions, via the representation of the wavelet coefficients, are successfully applied to the location of mistaken data.

Keywords: De la Vallée Poussin's mean, wavelets, multiresolution analysis.

AMS classification: 42A10, 42C40, 65T99.

§1. Introduction

Wavelets constitute the latest addition to the subject of orthogonal series by their wide applicability. This work provides a construction of wavelets in the framework of a Multiresolution Analysis (MRA) [2, 3, 5] by using the de la Vallée Poussin's mean. Our purpose is to show the usefulness of this classical mean to define, in a *natural* manner, wavelet functions. A MRA scheme deals with a trigonometric polynomial $m_0^{[n]}(\xi)$, which is an approximation of the so called *ideal low pass transfer function* $\chi_{[-\pi/2,\pi/2]}(\xi)$, where ξ represents the variable at frequency domain. In practice, it is not possible to implement an ideal low pass transfer function. It must be approximated (see [5, p. 27]). A measure of this approximation is obtained by the quantity

$$\varepsilon(n) = \int_{-\pi}^{\pi} \left| \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi) - \left| m_0^{[n]}(\xi) \right|^2 \right|^2 d\xi. \tag{1}$$

Our construction involves a trigonometric polynomial $m_0^{[n]}(\xi)$, whose square modulus is defined by using the approximation of the identity associated with the de la Vallée Poussin's sum. In this work we first present some of the standard facts on wavelets. Then we define $m_0^{[n]}(\xi)$ by using the classical mean. We state some basic relations of the introduced functions in order to guarantee the orthogonality of the wavelet systems. We used known theoretical statements to obtain

$$\varepsilon(n) \sim O\left(\frac{1}{n}\right),$$

which may be regarded as a localization property concerning $m_0^{[n]}(\xi)$. Finally, an interesting application is presented. For the detailed proofs, see [6, 7].

§2. Background

A MRA structure of $L^2(\mathbb{R})$ is a nested sequence of closed subspaces V_j, with $j \in \mathbb{Z}$, such that

(i) $\cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots \subset L^2(\mathbb{R})$.

(ii) $\overline{\bigcup_j V_j} = L^2(\mathbb{R}), \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(iii) $f \in V_j \Leftrightarrow f(2^j \cdot) \in V_0$.

(iv) There exists $\varphi \in V_0$, called the scaling function, such that the system $\{\varphi(\cdot - n);\ n \in \mathbb{Z}\}$ is an orthonormal basis for V_0. It is also assumed the condition $|\int_{\mathbb{R}} \varphi(x)\,dx| = 1$.

A wavelet basis is defined as

$$\left\{\psi_{j,k}(x) = 2^{-j/2}\,\psi(2^{-j}x - k)\ ;\ j,k \in \mathbb{Z}\right\},$$

where ψ is a real-valued square-integrable compactly supported function. The basis $\psi_{j,k}(x)$ is derived from a MRA associated to a compactly supported scaling function and a trigonometric polynomial $m_0^{[n]}$. An important property for the scaling function φ is the so-called *two scale relation*:

$$\varphi(x) = \sqrt{2}\sum_n h_n\,\varphi(2x - n), \tag{2}$$

where $(h_n) \in \ell^2(\mathbb{Z})$. By taking the Fourier transform, this relation becomes

$$\hat{\varphi}(\xi) = m_0^{[n]}(\xi/2)\,\hat{\varphi}(\xi/2),$$

where $m_0^{[n]}(\xi)$ is given by

$$m_0^{[n]}(\xi) = \frac{1}{\sqrt{2}}\sum_{k=0}^{2n-1} h_k\,e^{-ik\xi}. \tag{3}$$

Condition (iv) implies

$$|m_0^{[n]}(\xi)|^2 + \left|m_0^{[n]}(\xi+\pi)\right|^2 = 1,$$

which may be regarded as a necessary condition for the orthogonality of the system. Moreover

$$m_0^{[n]}(0) = 1 \quad \text{and} \quad m_0^{[n]}(\pi) = 0. \tag{4}$$

The relations satisfied by the coefficients h_n in (3) (see [2, Chap. 5, Th. 6] and [3, p. 163]) imply that $m_0^{[n]}(\xi)$ approximates the ideal low pass transfer function $\chi_{[-\pi/2,\pi/2]}(\xi)$. The scaling function is used to construct the associated wavelet function, ψ. It must be chosen such that $\{\psi(x-n)\}$ is an orthonormal basis of the space W_0, the orthogonal complement of V_0 in V_{-1}. Then $V_0 \oplus W_0 = V_{-1}$. If such a $\psi(x)$ can be found, then

$$\left\{\psi_{j,k}(x) = 2^{-j/2}\,\psi(2^{-j}x - k)\ ;\ k \in \mathbb{Z}\right\}$$

is an orthonormal basis of W_j, the orthogonal complement of V_j in V_{j-1}, and $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ (see [3, p. 130]). In particular, it follows for $j < j_0$ that

$$V_j = V_{j_0} \bigoplus_{k=0}^{j_0-j-1} W_{j_0-k}. \tag{5}$$

The spaces W_j are called wavelet spaces or detail spaces. A relation between the scaling and the wavelet function is given by

$$\psi(x) = \sqrt{2}\sum_n g_n\,\varphi(2x-n), \tag{6}$$

where $g_n = (-1)^n \overline{h}_{1-n}$.

Remark 1. From the coefficients h_n, a standard recursive procedure on (2) (see [2, p. 67]) is used to find the scaling function and, by (6), the wavelet function is also obtained.

§3. Definition of $\left|m_0^{[n]}(\xi)\right|^2$

We consider the classical de la Vallée Poussin's sum (see [1, p. 1])

$$V_n(t) = 1 + 2\sum_{k=1}^{n} \frac{(n!)^2}{(n-k)!(n+k)!}\cos(kt), \quad t\in(-\pi,\pi). \tag{7}$$

This mean, together with the Fejér's, Poisson's and Jacobi's sums, is considered the last of the important positive approximate identities from the nineteenth century. More precisely, it is well known that $V_n \geq 0$ and

$$\lim_{n\to 0} \|V_n * f - f\|_{L^2(-\pi,\pi)} = O\left(\frac{1}{n^\gamma}\right), \quad f\in L^2(-\pi,\pi)\cap \mathrm{Lip}(\gamma), \tag{8}$$

where $\mathrm{Lip}(\gamma)$ denotes the class of Lipschitz functions of order $\gamma\in(0,1]$ (see [11, vol. I, p. 43]), and $*$ denotes the convolution product (see [4, p. 77]). Property (8) motivates the expression (1) and also the following definition:

Definition 1. The function $\left|m_0^{[n]}(\xi)\right|^2$ is defined by

$$\left|m_0^{[n]}(\xi)\right| = \left(V_n * \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}\right)(\xi). \tag{9}$$

On account of the definition of the convolution product equation (9) takes the form

$$|m_0^{[n]}(\xi)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(u)\,V_n(\xi-u)\,du = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} V_n(\xi-u)\,du. \tag{10}$$

As a consequence, we have that the functions $|m_0^{[n]}(\xi)|^2$ are nonnegative trigonometric cosine polynomials, i.e., they may be expressed as

$$|m_0^{[n]}(\xi)|^2 = a_0 + 2\sum_{k=1}^{n} a_k\cos(k\xi), \quad a_0,a_1,\ldots,a_n\in\mathbb{R}. \tag{11}$$

Remark 2. The coefficients h_n in (3) are computed from a_k by applying the classical Riesz Representation Theorem (see [9, p. 4]) and, in order to obtain an unique solution, the conditions satisfied by h_n in the framework of the MRA scheme, have also to be imposed (see [2, p. 58]).

Proposition 1. *Under the previous notation, we have*

$$\int_{-\pi}^{\pi} V_n(t)\,dt = 2\pi.$$

If the integral is evaluated on intervals $[a,b] \subset \mathbb{R}$ with $b-a=2\pi$, the same conclusion can be taken.

Proof. The detailed proof can be found in [6, 7] □

Proposition 2. *The functions $|m_0^{[n]}(\xi)|^2$ defined in (9) satisfy*

$$|m_0^{[n]}(\xi)|^2 + |m_0^{[n]}(\xi+\pi)|^2 = 1. \tag{12}$$

Proof. From (10), a change of variable in both summands of (12) yields

$$\begin{aligned} |m_0^{[n]}(\xi)|^2 + |m_0^{[n]}(\xi+\pi)|^2 &= \frac{1}{2\pi}\int_{\xi-\pi/2}^{\xi+\pi/2} V_n(s)\,ds + \frac{1}{2\pi}\int_{\xi+\pi/2}^{\xi+3\pi/2} V_n(s)\,ds \\ &= \frac{1}{2\pi}\int_{\xi-\pi/2}^{\xi+3\pi/2} V_n(s)\,ds = 1, \end{aligned}$$

where, for the last equality, we have used Proposition 1. □

The functions $|m_0^{[n]}(\xi)|^2$ does not satisfy, at first, (4). In order to get this relation, it can be consider, without loss of generality, an affine transformation.

§4. Localization-best approximation property

In this section we derive that $m_0^{[n]}(\xi)$ is well localized at frequency.

Proposition 3. *The functions $|m_0^{[n]}(\xi)|^2$ satisfy*

$$\big\| |m_0^{[n]}(\xi)|^2 - \chi_{[-\frac{\pi}{2},\frac{\pi}{2}]}(\xi) \big\|_{L^2(-\pi,\pi)} \sim O\Big(\frac{1}{\sqrt{n}}\Big).$$

Proof. The detailed proof can be found in [6, 7]. □

$h_0 = 0.4380$	$h_1 = 0.8248$	$h_2 = 0.2914$	$h_3 = -0.1294$
$h_4 = -0.0228$	$h_5 = 0.0119$	$h_6 = 0.0004$	$h_7 = -0.0002$

Table 1: Coefficients for the two scale relation

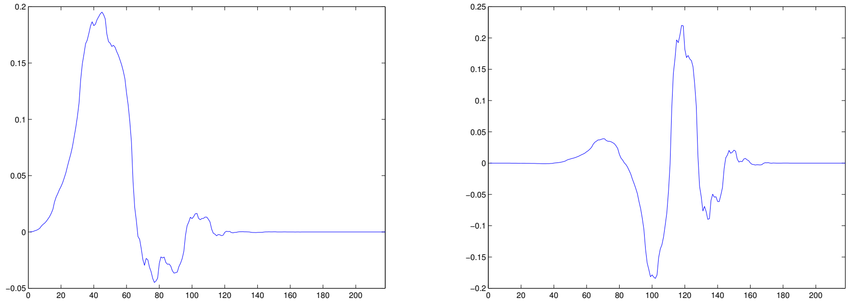

Figure 1: Scaling function (left) and wavelet function (right)

§5. Applications

We have considered $n = 8$ to obtain the functions given by (9):

$$|m_0^{[8]}(\xi)|^2 = \frac{1}{2208}\cos^2(\xi/2) \times \big(-1666 - 1084\cos(\xi) + 495\cos(2\xi) + 94\cos(3\xi) - 46\cos(4\xi) - 2\cos(5\xi) + \cos(6\xi)\big).$$

We have obtained (see Remark 2) the coefficients in Table 1.

By Remark 1, we have obtained the scaling and wavelet functions plotted in Figure 1. We have implemented (see [2, p. 258]) these functions in order to locate some collected mistaken data in connection with the electrical consumption's analysis represented in Figure 2.

By using the scaling and the wavelet functions, and taking into account (5), we have obtained the projections of f onto the approximation space V_4 and the detail spaces W_4, W_3, W_2 and W_1. The largest wavelet coefficients in the projections onto the detail spaces detect the domain where the function f is not regular or, which is the same thing, where the mistaken data occur (see [5, p. 171]). Taking equal to zero the largest wavelet coefficients, it is possible to define a new detail projections. From (5), the sum with the new detail projections and the projection represented in Figure 3 (left) leads to the reconstruction of the original electrical signal without the wrong data represented in Figure 4.

§6. Concluding remark

We have used the classical de la Vallée Poussin's mean to define certain trigonometric polynomial required to construct wavelet functions in the framework of a standard MRA scheme. We have used the positivity of the mean and also an asymptotic formula in order to compute the coefficients h_n of (3) and to guarantee the localization-best approximation property. We

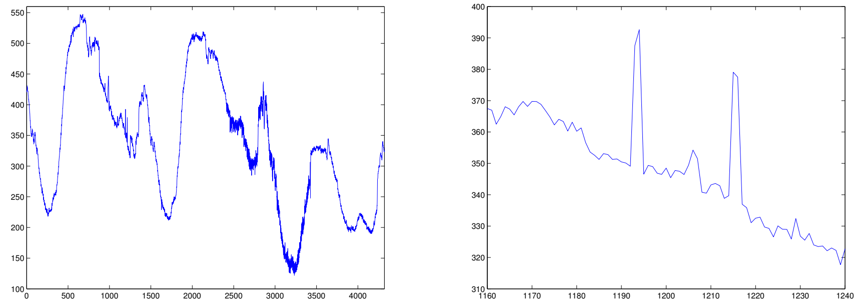

Figure 2: Electrical consumption: Global representation (left) and the partial zoom f (right)

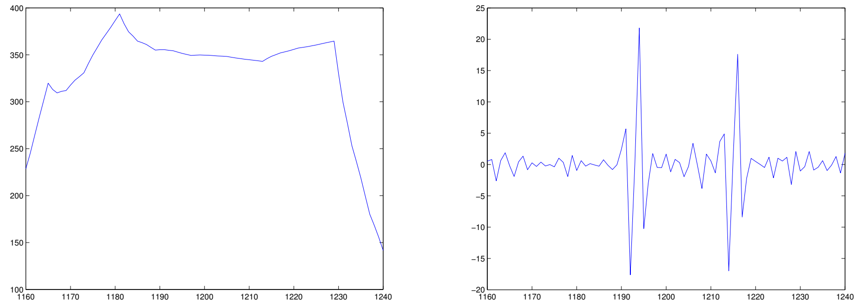

Figure 3: Projection of f onto V_4 (left) and onto W_1 (right)

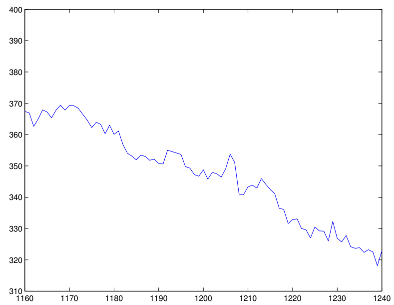

Figure 4: Reconstruction of f without the wrong data

present an explicit application to locate and remove the mistaken data of a given electrical signal. The detailed proofs and some similar constructions by using another classical means can be found in [6, 7]. We expect to promote the use of the classical approximation theory in connection with the subject of wavelets.

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