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ASYMPTOTIC EXPANSIONS OF THE APPELL'S FUNCTION F_2

José Luis López and María Esther García

Abstract. The second Appell's hypergeometric function $F_2(a,b,b',c,c';x,y)$ is considered for large values of its variables x and y. An integral representation of F_2 is obtained in the form of a double integral. This integral is a two-dimensional generalization of the typical one-dimensional integral to which the analytic continuation method of asymptotic expansions of integrals may be applied. We show that the analytic continuation method can also be applied to this kind of two-dimensional integrals. Then, we derive an asymptotic expansion of $F_2(a,b,b',c,c';x,y)$ in decreasing powers of x and y. Coefficients of the expansion are given in terms of the hypergeometric function ${}_3F_2$. As numerical experiments show, the approximation is considerable accurate.

Keywords: Second Appell's hypergeometric function, asymptotic expansions, analytic continuation method.

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§1. Introduction

The Appell's functions F_1, F_2, F_3 and F_4 are generalizations of the Gauss hypergeometric function ${}_2F_1$ [7, p. 224]:

$${}_2F_1(a,b,c;x) \equiv \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k,$$

where $(a)_n$ is the Pochhammer symbol $(a)_n = a(a+1)\ldots(a+n-1)$. In particular, the second Appell's function F_2 is defined by means of the double series [14, p. 789]:

$$F_2(a,b,b',c,c';x,y) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n\, m!\, n!} x^m y^n, \quad |x|+|y|<1. \tag{1}$$

Appell's functions have physical applications in several problems of Quantum Mechanics. For example, they appear in the computation of transition matrices in atomic and molecular physics, such as the transitions that involve Coulombic continuum states [6] or ion-atom collisions [5]. They are also representations of the generalized Slater's and Marvin's integrals [16] and the solution of certain ordinary differential equations and partial differential equations [17]. In fact, there is an extensive mathematical literature devoted to the study of these functions: Sharma has obtained generating functions of the Appell's functions [15]. Some integral representations for F_1 and F_2 have been derived by Manocha [11] and Mittal [12]. The Laplace transforms of these functions have been obtained in [8]. Some reduction formulas for special values of the variables and contiguous relations for Appell's functions have been investigated by Buschman [1, 2]. Carlson has investigated quadratic transformations of Appell's functions [4] and their role on multiple averages [3]. Definition (1) is only valid for $|x|+|y|<1$ and in many of the practical applications cited above it is necessary to consider this function for large values of its variables. Convergent and asymptotic expansions of $F_1(a,b,c,d;x,y)$ for large values of x and/or y have been obtained in [9] by using the distributional approach introduced by Wong [18]. The purpose of this paper is: (i) to define F_2 for large values of x and y and (ii) to obtain an asymptotic expansion of $F_2(a,b,b'c,c';x,y)$ for large values of the variables x and y and any (fixed) value of $a,b,b'c,c'$ using a new technique: the method of analytic continuation introduced in [10]. We face the challenge of obtaining easy algorithms to compute the coefficients of these expansions.

§2. Analytic continuation of F_2 in the variables x and y

2.1. First step

The first step is to find an expression for F_2 valid for $|x|+|y|\geq 1$, in particular when x and y are large. It is given by the following representation of F_2 in the form of a double integral taken from [13], with $x,y\in\mathbb{C}$ no real positives, $c-b>0$, $c'-b'>0$, $b>0$ and $b'>0$:

$$F_2(a,b,b',c,c';x,y)=\frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(c-b)\Gamma(b')\Gamma(c'-b')}I_2(a,b,b',c,c';x,y),$$

with

$$I_2(a,b,b',c,c';x,y)\equiv\int_0^1 ds\int_0^1 dt\, s^{b-1}(1-s)^{c-b-1}t^{b'-1}(1-t)^{c'-b'-1}(1-sx-ty)^{-a}.$$

We perform the following change of variables in this integral:

$$s\longrightarrow s/|x|\equiv x's,\qquad t\longrightarrow t/|y|\equiv y't,\quad \text{with}\quad x'=1/|x|\quad\text{and}\quad y'=1/|y|,$$

which means that x' and/or y' are small when x and/or y are large respectively. We use polar coordinates: $t=r\cos\theta$, $s=r\sin\theta$ and we define $\gamma=y'/x'$. Then,

$$I_2=\gamma x'\int_0^{\frac{\pi}{2}}\sin\theta^{b-1}(\gamma\cos\theta)^{b'-1}F(x',\theta)\,d\theta, \tag{2}$$

where

$$F(x',\theta)\equiv\int_0^\infty h_\theta(x'r)f_\theta(r)\,dr,\qquad f_\theta(r)\equiv(1-r\sin\theta e^{i\alpha}-r\cos\theta e^{i\beta})^{-a}$$

and

$$h_\theta(r)\equiv r^{b+b'-1}(1-r\sin\theta)^{c-b-1}(1-r\gamma\cos\theta)^{c'-b'-1}\chi_{(0,1)}(r\sin\theta)\chi_{(0,1)}(\gamma r\cos\theta).$$

In these formulas, $\chi_{(0,1)}(r)$ represents the characteristic function of the interval $(0,1)$ and α and β the arguments of the respective complex numbers x and y: $x=|x|e^{i\alpha}$ and $y=|y|e^{i\beta}$.

The integral (2) is a two-dimensional generalization of the typical one-dimensional integral to which the analytic continuation method of asymptotic expansions of integrals [10] may be applied. In this paper we extend this method to double integrals and apply it to I_2. To this end we need the following expansions of $f_\theta(r)$ and $h_\theta(r)$:

$$h_\theta(r) = \sum_{k=0}^{m-1} A_k(\theta) r^{k+b+b'-1} + h_m(r), \tag{3}$$

with

$$A_k(\theta) \equiv \sum_{j=0}^{k} \frac{(1+b-c)_j}{j!} (\sin\theta)^j \frac{(1+b'-c')_{k-j}}{(k-j)!} (\gamma\cos\theta)^{k-j}$$

and $h_m(r) = \mathscr{O}(r^{m+b+b'-1})$ when $r \to 0^+$,

$$f_\theta(r) = \sum_{k=0}^{n-1} B_k(\theta) r^{-k-a} + f_n(r), \tag{4}$$

with

$$B_k(\theta) \equiv \frac{(a)_k(-1)^a}{k!(\sin\theta e^{i\alpha} + \cos\theta e^{i\beta})^{a+k}}$$

and $f_n(r) = \mathscr{O}(r^{-a-n})$ when $r \to \infty$. On the other hand, $h_\theta(r) = \mathscr{O}(r^{-N})$ when $r \to \infty$ for any natural number N as large as we wish and $f_\theta(r) = \mathscr{O}(1)$ when $r \to 0^+$.

2.2. Second step

The second step is to check that I_2, in the form given in (2), satisfies the conditions of the method [10]: Condition 1: $1-b-b' < 1 < a+\infty$. Condition 2: $0 < a$ and $1-b-b' < \infty$. The first condition holds because $b+b' > 0$ and the second inequality is trivial. The second condition only restricts the parameter a to positive values because the second inequality holds for any real b and b'.

2.3. Third step

The third step consist of applying the method [10] when $a-b-b'$ is not an integer number. From [10, Theorem 1] we have

$$\begin{aligned} F(x',\theta) = & \sum_{k=0}^{n-1} B_k(\theta) M[h_\theta; 1-k-a] x'^{k+a-1} \\ & + \sum_{k=0}^{m-1} A_k(\theta) M[f_\theta; k+b+b'] x'^{k-1+b+b'} + \int_0^\infty f_n(r) h_m(x'r)\, dr, \end{aligned} \tag{5}$$

where $m = n + [a+1-b-b]$ and the symbol $M[g;z]$ denotes the Mellin transform of a function $g \in L_{loc}(0,\infty)$: $M[g;z] = \int_0^\infty g(r) r^{z-1}\, dr$ when it exists or its analytic continuation as a function of z. The above formula is an asymptotic expansion of $F(x',\theta)$ for small x' as it is proved in [10]. The integral I_2 (and therefore the Appell function F_2) is the θ integral of $F(x',\theta)$ (see (2)). Therefore,

$$\begin{aligned} I_2(x') = \gamma x' \int_0^{\frac{\pi}{2}} \sin\theta^{b-1}(\gamma\cos\theta)^{b'-1} \\ \times \left[\sum_{k=0}^{n-1} B_k(\theta) M[h_\theta; 1-k-a] x'^{k+a-1} \right. \\ \left. + \sum_{k=0}^{m-1} A_k(\theta) M[f_\theta; k+b+b'] x'^{k-1+b+b'} + \int_0^\infty f_n(r) h_m(x'r)\, dr \right] d\theta, \end{aligned} \tag{6}$$

that is,

$$I_2(x') = \sum_{k=0}^{n-1} \hat{B}_k x'^{k+a} + \sum_{k=0}^{m-1} \hat{A}_k x'^{k+b+b'} + R_n(x'), \tag{7}$$

where

$$\begin{aligned} \hat{B}_k &\equiv \gamma \int_0^{\frac{\pi}{2}} B_k(\theta) M[h_\theta; 1-k-a] \sin\theta^{b-1} (\gamma\cos\theta)^{b'-1}\, d\theta, \\ \hat{A}_k &\equiv \gamma \int_0^{\frac{\pi}{2}} A_k(\theta) M[f_\theta; k+b+b'] \sin\theta^{b-1} (\gamma\cos\theta)^{b'-1}\, d\theta, \end{aligned} \tag{8}$$

and

$$R_n(x') \equiv \gamma x' \int_0^{\frac{\pi}{2}} \sin\theta^{b-1}(\gamma\cos\theta)^{b'-1}\, d\theta \int_0^\infty f_n(r) h_m(x'r)\, dr.$$

The two first integrals above may be calculated in terms of the hypergeometric function ${}_3F_2$:

$$\begin{aligned} \hat{B}_k = \frac{(-1)^a (a)_k}{k!\, e^{i\alpha(a+k)}} \left[\left(\gamma e^{i(\alpha-\beta)}\right)^{a+k} \frac{\Gamma(c'-b')\,\Gamma(c-b)\,\Gamma(b)\,\Gamma(b'-a-k)}{\Gamma(c)\,\Gamma(-a+c'-k)} U_1 \right. \\ \left. + \left(\gamma e^{i(\alpha-\beta)}\right)^{b'} \frac{\Gamma(b')\,\Gamma(c-b)\,\Gamma(-a+b+b'-k)\,\Gamma(a-b'+k)}{\Gamma(-a+b'+c-k)\,\Gamma(a+k)} U_2 \right] \end{aligned} \tag{9}$$

and

$$\begin{aligned} \hat{A}_k = \sum_{j=0}^{k} \frac{(1+b-c)_j\,(1+b'-c')_{k-j}\,\gamma^{k+b'-j}\,(e^{i\beta})^{j-k-b'}}{j!\,(k-j)!\,(-1)^{b+b'+k}\,(e^{i\alpha})^{j+b}} \\ \times \frac{\Gamma(a-b-b'-k)\,\Gamma(k+b'-j)\,\Gamma(b+j)}{\Gamma(a)} U_3, \end{aligned} \tag{10}$$

where

$$\begin{aligned} U_1 &= {}_3F_2\left(b, a+k, 1+a-c'+k; c, 1+a-b'+k; -\gamma e^{i(\alpha-\beta)}\right), \\ U_2 &= {}_3F_2\left(b', 1+b'-c', -a+b+b'-k; 1-a+b'-k, -a+b'+c-k; -\gamma e^{i(\alpha-\beta)}\right), \\ U_3 &= {}_3F_2\left((b+j)/2, (b+j+1)/2, 0; (1-k-b'+j)/2, 1+(j-k-b')/2; -e^{2i(\beta-\alpha)}\right). \end{aligned}$$

We wonder if the integration in the variable θ performed in (6) doesn't spoil the asymptotic expansion of $F(x',\theta)$ for $x' \to 0$ given in (5). More precisely, we wonder if (7) is a true asymptotic expansion of I_2 for small x' and if it is useful in practice. In order to answer the first question we have to show that the remainder $R_n(x')$ is of the order $\mathcal{O}(x'^{n+a})$ when $x' \to 0$.

§3. Bounds for the remainder

The four step of our analysis consist of finding bounds for the remainder $R_n(x')$ and show that $R_n(x') = \mathcal{O}(x'^{n+a})$ when $x' \to 0$. After the change of variables ($rx' \to r$) we have

$$R_n(x') = \gamma \int_0^{\frac{\pi}{2}} \sin\theta^{b-1} (\gamma \cos\theta)^{b'-1}\, d\theta \int_0^\infty f_n\left(\frac{r}{x'}\right) h_m(r)\, dr.$$

Using the Laurent formula for the Taylor remainder $f_n(r)$ in the expansion (4) of $f(r)$ we obtain

$$\left| f_n\left(\frac{r}{x'}\right)\right| \le \hat{C}_n \frac{|x'|^{n+a}}{|r|^{n+a}} \Longrightarrow |R_n(x')| \le \hat{C}_n M_n |x'|^{n+a},$$

where $\hat{C}_n$ is a positive constant independent of x',

$$M_n \equiv \int_0^{\frac{\pi}{2}} \sin\theta^{b-1} (\gamma \cos\theta)^{b'-1}\, d\theta \int_0^\infty r^{-n-a} |h_m(r)|\, dr$$

and $h_m(r) = h_\theta(r) - \sum_{k=0}^{m-1} A_k(\theta) r^{k-1+b+b'}$. Then, the job will be done when we prove that $M_n < \infty$. To show this, we take a $w \in (0,1)$ and write

$$\begin{aligned} M_n \le \int_0^{\frac{\pi}{2}} & \sin\theta^{b-1} (\gamma\cos\theta)^{b'-1}\, d\theta \\ & \times \left[\int_0^w r^{b+b'-1-n-a} |\tilde{h}_m(r)|\, dr + \int_w^\infty r^{b+b'-1-n-a} |\tilde{h}(r)|\, dr \right. \\ & \left. + \sum_{k=0}^{m-1} |A_k(\theta)| \int_w^\infty r^{k-1+b+b'-a-n}\, dr \right], \end{aligned} \tag{11}$$

with $\tilde{h}(r) = r^{1-b-b'} h_\theta(r)$. The constant M_n above is given by the sum of three terms (one for every one of the three terms inside the braces). We will find bounds for each one of them.

1) The singularities of $\tilde{h}(r)$ are away from $(0,w)$. Then, using the Cauchy formula for the derivative of $\tilde{h}$, we find $|\tilde{h}_m(r)| \le C_m(w) r^m$ with $C_m(w) < \infty$ and independent of r. Therefore,

$$\int_0^w r^{b+b'-1-n-a} |\tilde{h}_m(r)|\, dr \le C_m(w) \frac{w^{m+b+b'-a-n}}{m+b+b'-a-n}. \tag{12}$$

2) Using Cartesian variables we find that

$$
\begin{aligned}
&\int_w^\infty r^{-a-n}\,dr\int_0^{\frac{\pi}{2}}\sin\theta^{b-1}(\gamma\cos\theta)^{b'-1}(1-r\sin\theta)^{c-b-1}(1-\gamma r\cos\theta)^{c'-b'-1}\\
&\qquad\qquad\times\chi_{(0,1)}(r\sin\theta)\chi_{(0,1)}(\gamma r\cos\theta)\,d\theta\\
&=\int_0^{\frac{\pi}{2}}\sin\theta^{b-1}(\gamma\cos\theta^{b'-1})\,d\theta \qquad\qquad (13)\\
&\qquad\times\int_w^{\min\{1/\sin\theta,1/(\gamma\cos\theta)\}} r^{-a-n}(1-r\sin\theta)^{c-b-1}(1-\gamma r\cos\theta)^{c'-b'-1}\,dr\\
&\equiv C_n(w).
\end{aligned}
$$

The constant $C_m(w)<\infty$ because $c-b>0$ and $c'-b'>0$.

3) The integrals in third term inside the braces of (11) are given by

$$
\int_w^\infty r^{k-1+b+b'-a-n}dr=\frac{w^{k+b+b'-a-n}}{k+b+b'-a-n}. \tag{14}
$$

Bounds (12), (13) and (14) show that

$$
\begin{aligned}
M_n\le&\int_0^{\frac{\pi}{2}}\sin\theta^{b-1}(\gamma\cos\theta)^{b'-1}\,d\theta\\
&\times\left[C_m(w)\frac{w^{m+b+b'-a-n}}{m+b+b'-a-n}+\sum_{k=0}^{\infty}A_k(\theta)\frac{w^{k+b+b'-a-n}}{k+b+b'-a-n}\right]\\
&+C_n(w)<\infty
\end{aligned}
$$

and then $R_n(x')=\mathscr{O}(x'^{n+a})$ when $x'\to0$. We have shown that the asymptotic expansion for I_2 given above is a true asymptotic expansion of F_2 when $x'\to0$. To resume, we have shown that, for fixed a, b, c, b', c' and γ, with $c-b>0$, $c'-b'>0$, $a>0$, $b>0$, $b'>0$, the variables $x,y\in\mathbb{C}\setminus\mathbb{R}^+$ and $\gamma\equiv|x|/|y|$:

$$
F_2(a,b,b',c,c';x,y)=\frac{\Gamma(c)\Gamma(c')/(\Gamma(b)\Gamma(b'))}{\Gamma(c-b)\Gamma(c'-b')}\left[\sum_{k=0}^{n-1}\frac{\hat{B}_k}{|x|^{k+a}}+\sum_{k=0}^{m-1}\frac{\hat{A}_k}{|x|^{k+b+b'}}+R_n(x)\right], \tag{15}
$$

with $\hat{B}_k$ and $\hat{A}_k$ given in (9) and (10) respectively. The remainder term $R_n(x)$ verifies $R_n(x)=\mathscr{O}(x^{-n-a})$ when $x\to\infty$. The numerical experiments detailed in Tables 1 and 2 show the accuracy of expansion (15). The relative error decreases when x and y increase. It decreases also when n, the number of terms of the expansion, increases up to a critical value.

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$x = y$	I_2	I_2 exp		Relative error	
		$n = 1$	$n = 2$	$n = 1$	$n = 2$
-10	0.0191841	0.0453056	0.00530564	1.361622385	0.723435553
-100	0.00108557	0.00143269	0.00103269	0.319758283	0.048711736
-1000	0.0000414835	0.0000453056	0.0000413056	0.092135427	0.0042884520
$-10-10i$	$0.00844655-0.01078211i$	$0.0103091-0.0248882i$	$0.0103091-0.00488824i$	1.03883	0.45129
$-100-100i$	$0.000317933-0.000608233i$	$0.000326001-0.000787035i$	$0.000326001-0.000587035i$	0.26079	0.0330481
$-1000-1000i$	$0.0000102809-0.0000229581i$	$0.0000103091-0.0000248882i$	$0.0000103091-0.0000228882i$	0.0767366	0.00299639

Table 1: Parameter values: $a = 3/2$, $b = b' = 1$, $c = c' = 3$.

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x, y	I_2	I_2 exp $n=1$	I_2 exp $n=2$	Relative error $n=1$	Relative error $n=2$
-10, -10	0.0985077	0.111491	0.0888381	0.131799849	0.098160854
-100, -100	0.03466	0.0352565	0.0345402	0.01721004	0.003456433
-1000, -1000	0.0111277	0.0111491	0.0111264	0.001923128	0.000116825
-10, -12	0.0948268	0.116554	0.090613	0.229125099	0.044436804
-100, -120	0.0331225	0.0368575	0.0360372	0.112763227	0.087997584
-1000, -1200	0.0106212	0.0116554	0.0116294	0.097371295	0.09492336
$-10-10i$, $-10-10i$	$0.0819734 - 0.0285693i$	$0.0866159 - 0.0358775i$	$0.0814614 - 0.0234334i$	0.099737	0.0843932
$-100-100i$, $-100-100i$	$0.0272294 - 0.0110132i$	$0.0273904 - 0.0113455i$	$0.0272274 - 0.0109519i$	0.0125713	0.00208811
$-1000-1000i$, $-1000-1000i$	$0.00865643 - 0.00357595i$	$0.00866159 - 0.00358775i$	$0.00865643 - 0.0035753i$	0.00137507	0.0000694003

Table 2: Parameter values: $a = 1/2$, $b = b' = 1$, $c = c' = 3$.

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José Luis López
Departamento de Matemática e Informática
Universidad Pública de Navarra
Campus de Arrosadía s/n
31006-Pamplona
jl.lopez@unavarra.es

María Esther García
Departamento de Matemática Aplicada
Facultad de Veterinaria
Universidad de Zaragoza
50009-Zaragoza
estlabec@unizar.es