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Analysis of a conservation law with space-discontinuous advection function

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Abstract. We consider the scalar conservation law in one space dimension:

$$\partial_t u + \partial_x(k(x)g(u)) = 0, \qquad (1)$$

associated with a bounded initial value u_0.

We suppose that the function k is bounded, discontinuous at $x_0 = 0$, and has bounded variations. When k is piecewise-constant, the definition of a weak entropy formulation for the Cauchy problem has been introduced by J. D. Towers in [7]. In [6] the existence and the uniqueness is proved by regularisation of the function k. We generalize the definition of J.D Towers and we adapt the method introduced in [6] to establish an existence and uniqueness property in the case of the homogeneous Dirichlet problem for (1).

Keywords: Conservation law, hyperbolic equation, entropy solution, discontinuous advection.

AMS classification: 35L60, 35B05, 35R05.

§1. Introduction

We are interested in the existence and uniqueness property for a scalar conservation law made of an hyperbolic first-order equation in a one-dimensional bounded domain Ω, for any positive finite T:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}(k(x)g(u)) = 0 & \text{in } Q = \Omega \times]0,T[\,, \\ u(0,x) = u_0(x) & \text{on } \Omega, \\ u = 0 & \text{on a part of }]0,T[\, \times \partial\Omega, \end{cases} \qquad (2)$$

where k is a discontinuous function at a point x_0 of Ω. Such an equation arises in the modelling of continuous sedimentation of solid particles in a liquid or when one considers a two-phase flow in an heterogeneous porous medium without capillarity effects ([3]).

By normalization, we suppose $\Omega =]-1,1[$.

The initial condition u_0 belongs to $L^\infty(\Omega)$ and takes values in $[0,1]$.

The flux function g is Lipschitzian on $[0,1]$ with a constant M_g, $g \geq 0$, $g(0) = g(1) = 0$ and satisfies a nondegeneracy condition:

$$\forall\, \alpha \in \mathbb{R},\ \mathscr{L}\{\lambda \in \mathbb{R},\ g'(\lambda) = \alpha\} = 0.$$

The function k is discontinuous at $x_0 = 0$, $k \in W^{1,+\infty}(]-1,0[)$, $k \in W^{1,+\infty}(]0,1[)$. Thus, we can define:

$$k_L = \lim_{x\to 0^-} k(x) \qquad \text{and} \qquad k_R = \lim_{x\to 0^+} k(x).$$

The mathematical formulation for (2) is given in Section 2 through an entropy inequality on the whole Q using the classical Kruzkov entropy pairs and involving a term that takes into account the jump of k along $\{x_0 = 0\}$. As soon as we are able to transcript the transmission conditions along the interface included in this definition, we are can state, in Section 4, the uniqueness. To do so we need strong traces for u along the interface $\{x_0 = 0\}$.

§2. Definition of an entropy solution

We propose a definition extending that of J. D. Towers ([7]), also used by N. Seguin and J. Vovelle ([6]) or F. Bachmann ([1]), to the case where k depends on the space variable and for the homogeneous Dirichlet problem in a bounded interval of $\mathbb{R}$. So we say that:

Definition 1. A function u of $L^\infty(Q)$ is an entropy solution to (2) if:

(i) $0 \le u(t,x) \le 1 \quad a.e. \text{ on } Q$.

(ii) $\forall \kappa \in [0,1]$, $\forall \varphi \in \mathscr{C}_c^\infty([0,T[\times\Omega)$, $\varphi \ge 0$,

$$\left\{\begin{aligned} &\int_Q \big(|u(t,x)-\kappa|\varphi_t(t,x) + k(x)\Phi(u,\kappa)\varphi_x(t,x)\big)\,dx\,dt \\ &\qquad -\int_Q k'(x)\,\mathrm{sign}(u-\kappa)g(\kappa)\varphi\,dx\,dt + \int_\Omega |u_0-\kappa|\varphi(0,x)\,dx \\ &\qquad + |k_L - k_R|g(\kappa)\int_0^T \varphi(t,0)\,dt \ge 0, \end{aligned}\right. \tag{3}$$

where

$$\Phi(u,\kappa) = \mathrm{sign}(u-\kappa)(g(u)-g(\kappa)).$$

(iii) for $a.e.$ $t \in]0,T[$, $\forall \kappa \in [0,1]$,

$$k(1)\,\mathrm{sign}^+(u_1^\tau(t)-\kappa)(g(u_1^\tau(t)-g(\kappa)) \ge 0, \tag{4}$$
$$k(-1)\,\mathrm{sign}^+(u_{-1}^\tau(t)-\kappa)(g(u_{-1}^\tau(t)-g(\kappa)) \le 0. \tag{5}$$

In this definition u_1^τ and u_{-1}^τ denote the traces of u respectively in $+1$ and -1 in the sense of A. Vasseur [8] (see also Y. Panov [5]). Indeed it follows from [8]:

Lemma 1. *Let u be an entropy solution to problem (2). If $g \in \mathscr{C}^3(\mathbb{R})$ and if, $\forall(\alpha,\beta) \ne (0,0)$, $\mathscr{L}(\{\lambda \mid \alpha + \beta.g'(\lambda) = 0\}) = 0$, there exists two functions $u_{\pm 1}^\tau$ in $L^\infty(]0,T[)$ such that, for every compact set K of $]0,T[$,*

$$\operatorname*{ess\,lim}_{x\to\pm 1} \int_K \big|u(t,x) - u_{\pm 1}^\tau(t)\big|\,dt = 0.$$

In [5] Panov proved the existence of these strong traces with a flux function only continuous, depending also on the space variable.

Remark 1. Of course, the statement of Lemma 1 also ensures the existence of strong traces for u, γu^+ and γu^-, in $L^\infty(]0,T[)$ along $\{x_0 = 0\}$.

Remark 2. The boundary conditions (4)–(5) can also be written:
for *a.e.* $t \in]0,T[$, $\forall \kappa \in [0,1]$,

$$k(1)(\mathrm{sign}(u_1^\tau(t)-\kappa)+\mathrm{sign}(\kappa))(g(u_1^\tau(t))-g(\kappa)) \geq 0,$$
$$k(-1)(\mathrm{sign}(u_{-1}^\tau(t)-\kappa)+\mathrm{sign}(\kappa))(g(u_{-1}^\tau(t))-g(\kappa)) \leq 0,$$

that are the classical boundary conditions of C. Bardos, A. Y. Leroux and J. C. Nedelec ([2]).

§3. Conditions at the interface $\{x_0 = 0\}$

Let us establish that the previous definition ensures the uniqueness. The proof is based on that proposed in [6] and relies essentially on the transmission condition along $\{x_0 = 0\}$ underlying to entropy inequality (3). Indeed the existence of strong traces for u permits us to state first:

Lemma 2. *Let u be an entropy solution to (2). So, for a.e. $t \in]0,T[$, $\forall \kappa \in [0,1]$,*

$$k_L\Phi(\gamma u^-(t),\kappa) - k_R\Phi(\gamma u^+(t),\kappa) + |k_L - k_R|\, g(\kappa) \geq 0. \tag{6}$$

Proof. Let $\varphi \in \mathscr{C}_c^\infty(Q)$, $\varphi \geq 0$. We refer to the cut-off function ω_ε, $\varepsilon > 0$, introduced in [6]:

$$\omega_\varepsilon(x) = \begin{cases} 0, & \text{if } 2\varepsilon < |x|, \\ \dfrac{-|x|+2\varepsilon}{\varepsilon}, & \text{if } \varepsilon \leq |x| \leq 2\varepsilon, \\ 1, & \text{if } |x| < \varepsilon, \end{cases}$$

such that, when $\varepsilon \to 0^+$, $\omega_\varepsilon(x) \to 0\ \forall x \in \mathbb{R}^*$, and $\omega_\varepsilon(0) = 1$, $\forall \varepsilon \in \mathbb{R}_+^*$. Thanks to a density argument we may choose $\varphi\omega_\varepsilon$ as test-function in (3). We pass to the limit when ε goes to 0^+ by using the Lebesgue dominated convergence Theorem providing that all the terms go to 0 except $|k_L - k_R| g(\kappa) \int_0^T \varphi(t,0)\,dt$ (which does not depend on ε) and

$$I_\varepsilon = \int_Q k(x)\Phi(u,\kappa)\varphi\ {\omega'}_\varepsilon\,dx\,dt.$$

Indeed, by definition of ω_ε,

$$I_\varepsilon = \int_0^T \frac{1}{\varepsilon}\int_{-2\varepsilon}^{-\varepsilon} k(x)\Phi(u,\kappa)\varphi\,dx\,dt + \int_0^T -\frac{1}{\varepsilon}\int_{\varepsilon}^{2\varepsilon} k(x)\Phi(u,\kappa)\varphi\,dx\,dt.$$

Because $\Phi(\,.\,,\kappa)$ is Lipschitzian on $[0,1]$, and due to the definition of k_L, k_R and $\gamma u^-, \gamma u^+$, we show that $\lim\limits_{\varepsilon\to 0^+} I_\varepsilon = \int_0^T k_L\Phi(\gamma u^-,\kappa)\varphi dt - \int_0^T k_R\Phi(\gamma u^+,\kappa)\varphi\,dt.$ □

Now we highlight the fact that:

Lemma 3. *The inequality (6) is equivalent to two conditions:*

(i) a Rankine-Hugoniot condition: for a.e. $t \in]0,T[$,

$$k_L g(\gamma u^-(t)) = k_R g(\gamma u^+(t)), \tag{7}$$

(ii) an entropy condition: for a.e. $t \in]0,T[$ *such that* $\gamma u^-(t) \neq \gamma u^+(t)$,

- *if* $\operatorname{sign}(\gamma u^-(t) - \gamma u^+(t)) = \operatorname{sign}(k_L - k_R)$:

$$\forall \kappa \in I\,]\gamma u^-(t), \gamma u^+(t)[, \quad k_R \Phi(\gamma u^+(t), \kappa) \leq 0, \tag{8}$$

- *if* $\operatorname{sign}(\gamma u^-(t) - \gamma u^+(t)) = -\operatorname{sign}(k_L - k_R)$:

$$\forall \kappa \in I\,]\gamma u^-(t), \gamma u^+(t)[, \quad k_L \Phi(\gamma u^-(t), \kappa) \geq 0, \tag{9}$$

where $I\,]\alpha,\beta[$ *is the open interval bounded by* α *and* β.

§4. The uniqueness theorem

We are now able to state an uniqueness property for (2) through a Lipschitzian dependence in L^1 of a weak entropy solution with respect to corresponding initial data.

Theorem 4. *Let u and v be two entropy solutions to (2) for initial conditions* $(u_0, v_0) \in \big(L^\infty(]-1,1[;[0,1])\big)^2$. *Then,*

$$\int_0^T \int_{-1}^1 |u(t,x) - v(t,x)| dx dt \leq T \int_{-1}^1 |u_0(x) - v_0(x)|\, dx. \tag{10}$$

Proof. First we state, by using the method of doubling variables (cf. [4]), for any φ in $\mathscr{C}_c^\infty([0,T[\times\Omega)$ vanishing in a neighborhood of $\{x_0 = 0\}$, the following Kruzkov inequality:

$$\begin{aligned}\int_Q (|u(t,x) - v(t,x)|\varphi_t(t,x) + k(x)\Phi(u(t,x),v(t,x))\varphi_x(t,x))\, dx\, dt \\ + \int_\Omega |u_0(x) - v_0(x)|\varphi(0,x)\, dx \ \geq 0.\end{aligned} \tag{11}$$

Then for any φ in $\mathscr{C}_c^\infty([0,T[\times\Omega)$, we can choose in (11) the test-function $\varphi(1-\omega_\varepsilon)$, where ω_ε is defined in the proof of Lemma 2. By taking the ε-limit, it comes

$$\int_Q (|u-v|\varphi_t + k(x)\Phi(u,v)\varphi_x) dx dt + \int_\Omega |u_0 - v_0|\varphi(0,x) dx \ \geq J,$$

with

$$J = \int_0^T (k_L \Phi(\gamma u^-, \gamma v^-) - k_R \Phi(\gamma u^+, \gamma v^+))\varphi(t,0)\, dt.$$

Entropy and Rankine-Hugoniot conditions show that J is nonnegative. Indeed let us study, for $a.e.\ t$ of $]0,T[$, the sign of

$$I = k_L \Phi(\gamma u^-, \gamma v^-) - k_R \Phi(\gamma u^+, \gamma v^+).$$

We just focus on the case when $(\gamma u^+ - \gamma v^+)$ and $(\gamma u^- - \gamma v^-)$ have an opposite sign. Otherwise due to (7), $I = 0$. When $\text{sign}(\gamma u^+ - \gamma v^+) = -\text{sign}(\gamma u^- - \gamma v^-) \neq 0$, by using (7), we have

$$I = 2k_L \Phi(\gamma u^-, \gamma v^-) = -2k_R \Phi(\gamma u^+, \gamma v^+).$$

Let's focus on the situation: $\gamma u^- < \gamma v^-$ and $\gamma v^+ < \gamma u^+$ (the other cases being similar).

- If $\gamma u^- < \gamma v^- < \gamma v^+ < \gamma u^+$, then γv^- and γv^+ are in $]\gamma u^-, \gamma u^+[$, so we can use the entropy condition (8)–(9) (according to the sign of $k_L - k_R$) to have $I \geq 0$.
- If $\gamma u^- < \gamma v^+ < \gamma u^+ < \gamma v^-$, then $\gamma u^- - \gamma u^+$ and $\gamma v^- - \gamma v^+$ have an opposite sign. So one of the two have the sign of $k_L - k_R$. As $\gamma v^+ \in]\gamma u^-, \gamma u^+[$ and $\gamma u^+ \in]\gamma v^+, \gamma v^-[$, using (8), we have $I \geq 0$.
- If $\gamma v^+ < \gamma u^- < \gamma u^+ < \gamma v^-$ or $(\gamma v^+ < \gamma u^+ < \gamma u^- < \gamma v^-)$, then γu^+ and γu^- are in $]\gamma v^+, \gamma v^-[$. So as in the first situation, $I \geq 0$.

Now, in order to prove (10), we may choose in (11), for $(t,x) \in [0,T[\times \Omega$, the test-function, for $\varepsilon > 0$,

$$\varphi(t,x) = \theta(t)\alpha_\varepsilon(x),$$

where $\theta \in \mathscr{C}_c^\infty([0,T[)$, $\theta \geq 0$, and α_ε is an element of $\mathscr{C}_c^\infty(\Omega)$ such that $\alpha_\varepsilon \geq 0$, $\alpha_\varepsilon = 1$ on $]-1+\varepsilon, 1-\varepsilon[$ and $|\alpha'_\varepsilon| \leq \frac{2}{\varepsilon}$. So $\alpha_\varepsilon \to 1$ *a.e.* on Ω. We obtain, by taking the limit with respect to ε,

$$\int_Q |u-v|\theta'(t)\,dx\,dt + \int_\Omega |u_0 - v_0|\theta(0)\,dx \\ \geq \int_0^T k(1)\Phi(u_1^\tau, v_1^\tau)\theta(t)\,dt - \int_0^T k(-1)\Phi(u_{-1}^\tau, v_{-1}^\tau)\theta(t)\,dt.$$

By coming back to Definition 1 (iii), we remark that the boundary terms are nonnegative. Indeed, for *a.e.* on $]0,T[$,

- if $u_1^\tau \geq v_1^\tau$, we choose $\kappa = v_1^\tau$ in (4) for u^τ to obtain

$$k(1)\Phi(u_1^\tau(t), v_1^\tau(t))\theta(t) = k(1)(g(u_1^\tau(t)) - g(v_1^\tau(t))\theta(t) \geq 0,$$

- if $u_1^\tau \leq v_1^\tau$, we choose $\kappa = u_1^\tau$ in (4) for v^τ to obtain

$$k(1)\Phi(u_1^\tau(t), v_1^\tau(t))\theta(t) = -k(1)(g(u_1^\tau(t)) - g(v_1^\tau(t))\theta(t) \geq 0.$$

Similarly,

$$k(-1)\Phi(u_{-1}^\tau, v_{-1}^\tau)\theta(t) \leq 0.$$

This way,

$$\int_Q |u-v|\theta_t(t)dxdt + \int_\Omega |u_0 - v_0|\theta(0)dx \geq 0.$$

The conclusion follows with classical arguments which completes the proof of Theorem 4. □

§5. Existence of an entropy solution

The proof relies on a suitable regularization k_ε, $\varepsilon > 0$, of the function k and uses a compactness argument for the sequence $(k_\varepsilon \Phi(u_\varepsilon, \kappa))_{\varepsilon>0}$, where u_ε is the weak entropy solution to the corresponding mollified problem. To do so we need some additional hypotheses:

(H$_1$) There exists $\kappa_0 \in [0,1]$ such that $\Phi(\,.\,,\kappa_0)$ is bijective.

(H$_2$) $\mathscr{L}\{x \in \mathbb{R}^*,\ k(x) = 0\} = 0$.

In this framework we establish that:

Theorem 5. *Under (H$_1$) and (H$_2$), the problem (2) admits at least one entropy solution u.*

Proof. We suppose first that the initial condition u_0 is smooth.

First step: $u_0 \in \mathscr{C}_c^\infty(\Omega)$

We apply the method introduced in [6] (also used in [1]) that is to consider a sequence $(k_\varepsilon)_\varepsilon$ of Lipschitzian functions such that, for every positive ε, $k_\varepsilon = k$ out of $]-\varepsilon, \varepsilon[$ and k_ε is monotone on $[-\varepsilon, \varepsilon]$ (depending on the sign of $k_L - k_R$). That implies:

$$\forall\, x \in \mathbb{R}^*,\ k_\varepsilon(x) \longrightarrow k(x) \ \text{ and } \ |k_\varepsilon|_{BV(\mathbb{R})} \le |k|_{BV(\mathbb{R})}.$$

Then we denote u_ε the unique entropy solution (given by [3]) to the regularized problem:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} + \dfrac{\partial}{\partial x}(k_\varepsilon(x) g(u_\varepsilon)) = 0 & \text{on } Q, \\ u_\varepsilon(0,x) = u_0(x) & \text{on } \Omega, \\ u = 0 & \text{on a part of }]0,T[\, \times \partial\Omega. \end{cases} \tag{12}$$

When we look for some estimates for $(u_\varepsilon)_{\varepsilon>0}$, we are able to state the following lemma coming from [6].

Lemma 6.

(i) For a.e (t,x) *in* Q, $0 \le u_\varepsilon(t,x) \le 1$.

(ii) There exists a constant $C > 0$, *such that, for any* $\kappa \in [0,1]$:

$$|k_\varepsilon(\Phi(u_\varepsilon, \kappa))|_{BV(Q)} \ \le\ C(|u_0|_{BV(\Omega)} + |k|_{BV(Q)}).$$

Lemma 6 implies, by using (H$_1$) and (H$_2$), that $(u_\varepsilon)_{\varepsilon>0}$ tends to a limit denoted u in $L^1(Q)$. Then we introduce the mollified entropy pair, for any κ in $[0,1]$ and any real τ:

$$\Phi_\eta(\tau) = \int_\kappa^\tau \operatorname{sign}_\eta(r-\kappa) g'(r)\, dr \ \text{ and } \ I_\eta(\tau) = \int_\kappa^\tau \operatorname{sign}_\eta(r-\kappa)\, dr,$$

where sign_η denotes the Lipschitzian approximation of the function sign given for any positive η and any nonnegative real x by $\operatorname{sign}_\eta(x) = \min(x/\eta, 1)$ and $\operatorname{sign}_\eta(-x) = -\operatorname{sign}_\eta(x)$.

By coming back to the viscous problem related to (12), we establish that u_ε fulfills the regularized entropy inequality for all φ in $\mathscr{C}_c^\infty([0,T[\times\Omega)$,

$$\int_Q I_\eta(u_\varepsilon)\varphi_t\,dx\,dt + \int_Q k_\varepsilon(x)\Phi_\eta(u_\varepsilon)\varphi_x\,dx\,dt + \int_Q k'_\varepsilon(x)(\Phi_\eta(u_\varepsilon) - I'_\eta(u_\varepsilon)g(u_\varepsilon))\varphi\,dx\,dt + \int_\Omega I_\eta(u_0)\varphi(0,x)\,dx \geq 0. \tag{13}$$

We pass to the limit in (13) with respect to ε to obtain

$$\int_Q (I_\eta(u)\varphi_t + k(x)\Phi_\eta(u)\varphi_x)\,dx\,dt + \int_Q k'(x)(\Phi_\eta(u) - I'_\eta(u)g(u))\varphi\,dx\,dt + \int_\Omega I_\eta(u_0)\varphi(0,x)\,dx + (g(\kappa)+C_g\eta)|k_R-k_L|\int_0^T \varphi(t,0)\,dt \geq 0, \tag{14}$$

where $C_g = 2M_g$. This way, the limit with respect to η provides (3)

To establish that u satisfies (4)-(5), we refer to the viscous problem associated with (12) and we make sure that for any positive η and any positive ε,

$$\int_Q \{k_\varepsilon\Phi_\eta^+(u_\varepsilon)\varphi_x + I_\eta^+(u_\varepsilon)\varphi_t\}\,dx\,dt + \int_Q \{\Phi_\eta^+(u_\varepsilon) - g(u_\varepsilon)\,\mathrm{sign}_\eta^+(u_\varepsilon-\kappa)\}k'_\varepsilon\varphi\,dx\,dt \geq 0, \tag{15}$$

where

$$I_\eta^+(\lambda) = \int_\kappa^\lambda \mathrm{sign}_\eta^+(r-\kappa)\,dr \text{ and } \Phi_\eta^+(\lambda) = \int_\kappa^\lambda g'(r)\,\mathrm{sign}_\eta^+(r-\kappa)\,dr.$$

Now we take the ε-limit in (15) with the same arguments as those used to obtain (14) from (13). It comes

$$|k_L-k_R|(g(\kappa)+\eta\,C_g)\int_0^T \varphi(t,0)\,dt + \int_Q \{I_\eta^+(u)\varphi_t + k\Phi_\eta^+(u)\varphi_x\}\,dx\,dt + \int_Q \{\Phi_\eta^+(u) - g(u)\,\mathrm{sign}_\eta^+(u-\kappa)\}k'\varphi\,dx\,dt \geq 0. \tag{16}$$

Then choosing appropriate test-functions in (16) and using the definition of u_1^τ and u_{-1}^τ yield to (4)-(5).

Second step: $u_0 \in L^\infty(\Omega)$

We use a mollification process to come back to the first step. Indeed we consider a sequence $(u_0^j)_{j\in\mathbb{N}^*}$ such that $u_0^j \in \mathscr{C}_c^\infty(\Omega)$ and (u_0^j) tends to u_0 in $L^1(\Omega)$. We denote u^j the entropy solution to (2) associated with the initial condition u_0^j so that, for any j, u^j fulfills (14) and (16). The comparison result (10) ensures that $(u_j)_j$ is a Cauchy sequence in $L^1(Q)$ so tends to a limit, denoted u in $L^1(Q)$. Then, the j-limit in (14) and (16) ensures that u is an entropy solution to (2). □

References

[1] BACHMANN, F. Analysis of a scalar conservation law with a flux function with discontinuous coefficients. *Advances in Differential Equations 9*, 11–12 (2004), 1317–1338.

[2] BARDOS, C., LEROUX, A. Y, AND NEDELEC, J. C. First order quasilinear equations with boundary conditions. *Comm. in Partial Differential Equations 4* (1979), 1017–1034.

[3] GAGNEUX, G., AND MADAUNE-TORT, M. *Analyse Mathématique de Modèles Nonlinéaires de l'Ingénierie Pétrolière*. Springer-Verlag, Berlin, 1996.

[4] KRUZKOV, S. N. First-order quasilinear equations with several independent variables. *Mat. Sb. 81* (1970), 228–255.

[5] PANOV, E. Y. Existence of strong traces for generalized solutions of multidimensional scalar conservation laws. *Preprint* (2005).

[6] SEGUIN, N., AND VOVELLE, J. Analysis and approximation of a scalar conservation law with a flux fonction with discontinuous coefficients. *Math. Models Methods Appl. Sci. 13*, 2 (2003), 221–257.

[7] TOWERS, J. D. Convergence of a difference scheme for conservation laws a with discontinuous flux. *SIAM J. Numer. Anal. 38* (2000), 681–698.

[8] VASSEUR, A. Strong traces for solutions of multidimensional scalar conservation laws. *Arch. Rational Mech. Anal. 160* (2001), 181–193.

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