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CONIC STATIONARY SOLUTIONS OF ONE RESTRICTED THREE-BODY PROBLEM

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Abstract. The equations of motion of one three-body problem composed of a dumb-bell (two masses at fixed distance) moving around a central mass under gravitational effects have been stablished. Conic stationary solutions of these equations have been studied and sufficient conditions for stability has been found in term of Lyapunov's stability functions.

Keywords: Three-body problem, dumb-bell problem, stationary solutions.

AMS classification: 70F15, 70F50, 34D20.

§1. Introduction

The interest of the study of the motion of a system composed of three material points M_1, M_2 and M_3 interacting by Newtonian law, in the assumption that the distance between M_2 and M_3 is constant, i.e., the points M_2 and M_3 form a dumb-bell, derives from the fact that it is the simplest problem about traslational-rotary motion of a satellite in a gravitational field and gives the generic conections between the solution of this restricted three body problem and the classical one [2]. Particular cases of this problem can be equivalent to the classical restricted three bodies problem or to the generalized two fixed centres [1]. Not far from this is the problem of the motion of a point in the gravitational field created by a massic segment as an approximation to an elongate body [7, 8], as it is the case in some asteroids. The purpose of this paper is the study of the so call conic stationary solutions of the problem for arbitrary masses of the bodies and arbitrary size of the dumbell. Other particular cases as the linear and isosceles cases have already been studied by the authors [4]. The conic motions is a solutions in which the dumb-bell axis describes a conic circular surface with axis orthogonally disposed with respect to the papallel planes in which the points move around. The constant semiangle θ of the conic surface is the same as the angle between axis of the dumb-bell and the Gz axis, G being the center of mases; the distance z between the planes in which the motion of M_1 and the center of masses of the dumb-bell is performed remains constant. The values of z and θ are not independent and result from the roots of an algebraic equation. These solutions describe the effect of the displacement of the center of masses with respect to the angle θ and other parameters of the problem. We also give sufficient conditions for stability [6].

§2. Formulation of the problem

The system of study is composed of three material points M_1, M_2 and M_3, of masses m_1, m_2 and m_3, mutually attracted by the Newtonian gravitational forces. It will be assumed that

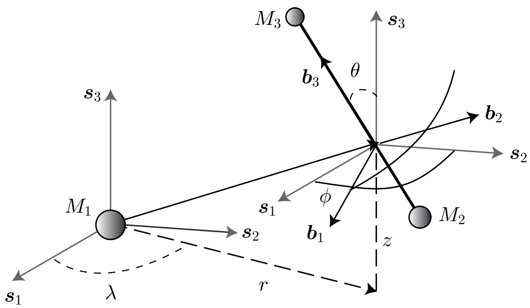


Figure 1: The reference frames

the points M_2 and M_3 are rigidly connected by a segment of constant length l and negligible mass, i.e., they form a dumb-bell.

Let C be the center of masses of the dumb-bell and l_2, l_3 the distances from M_2 and M_3 to C.

The simplest way to study the problem of motion of that system is to consider it referred to an inertial heliocentric frame $\mathscr{S}(M_1, \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ and to use Hamiltonian formulation [4]. The center of masses of the dumb-bell is defined by the cilindrical coordinates (r, z, λ) and the attitude of the dumb-bell in $\mathscr{S}$ is given by two angles, namely nutation θ and precession ϕ (see the figure 1).

We can define an orthonormal rotating frame $(C; \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ (see figure 1) made of the principal axes of inertia, where $\cos\theta = \mathbf{s}_3 \cdot \mathbf{b}_3$ and $\mathbf{b}_1 = \dfrac{\mathbf{s}_3 \times \mathbf{b}_3}{|\mathbf{s}_3 \times \mathbf{b}_3|}$.

In these heliocentric coordinates, the Hamiltonian may be expressed as (see [9, 4])

$$\mathscr{H} = \frac{1}{2m}\left(P_r^2 + \frac{(P_\omega - P_\psi)^2}{r^2} + P_z^2\right) + \frac{1}{2A}\left(\frac{P_\psi^2}{\sin^2\theta} + P_\theta^2\right) + U(r, z, \psi, \theta),$$

where the potential function is

$$U = -\mathscr{G} m_1 \left(\frac{m_2}{r_{12}} + \frac{m_3}{r_{13}}\right),$$

the mutual distances r_{1j}, for $j = 2, 3$, are

$$r_{1j}^2 = r^2 + z^2 + l_j^2 - (-1)^j\, 2\, l_j \Big[z\cos\theta + r\sin\theta\sin(\phi - \lambda)\Big]$$

and

$$\begin{aligned} \psi &= \phi - \lambda, & P_\psi &= P_\phi, \\ \omega &= \lambda, & P_\omega &= P_\phi + P_\lambda, \end{aligned}$$

and m and A are the following constants:

$$m = \frac{m_1(m_2 + m_3)}{m_1 + m_2 + m_3} \quad \text{and} \quad A = \frac{m_2 m_3}{m_2 + m_3} l^2.$$

With this election of variables the problem is reduced to four degrees of freedom. Since angle ω is cyclic, its conjugate moment P_ω is an integral of the motion. The Hamiltonian itself is another integral.

Then the equations of motion are

$$\begin{aligned}
\dot r &= \frac{P_r}{m}, & \dot P_r &= \frac{(P_\omega - P_\psi)^2}{mr^3} - \frac{\partial U}{\partial r},\\
\dot z &= \frac{P_z}{m}, & \dot P_z &= -\frac{\partial U}{\partial z},\\
\dot\theta &= \frac{P_\theta}{A}, & \dot P_\theta &= \frac{P_\psi^2 \cos\theta}{A\sin^3\theta} - \frac{\partial U}{\partial \theta},\\
\dot\psi &= -\frac{P_\omega - P_\psi}{mr^2} + \frac{P_\psi}{A\sin^2\theta}, & \dot P_\psi &= -\frac{\partial U}{\partial \psi},
\end{aligned} \tag{1}$$

and equilibria are found by zeroing this system. Thus, there results that

$$P_r = P_z = P_\theta = 0, \qquad P_\omega - P_\psi = \frac{mr^2}{A\sin^2\theta} P_\psi,$$

and

$$\begin{aligned}
\frac{\partial U}{\partial r} &= \frac{mr}{A^2\sin^4\theta} P_\psi^2, & \frac{\partial U}{\partial z} &= 0,\\
\frac{\partial U}{\partial \theta} &= \frac{A\sin\theta\cos\theta}{mr}\frac{\partial U}{\partial r}, & \frac{\partial U}{\partial \psi} &= 0.
\end{aligned}$$

Defining the shorcuts

$$F = \mathscr{G} m_1 \left(\frac{m_3}{r_{13}^3} + \frac{m_2}{r_{12}^3}\right), \qquad G = \mathscr{G} m_1 \left(\frac{m_3 l_3}{r_{13}^3} - \frac{m_2 l_2}{r_{12}^3}\right), \tag{2}$$

the partial derivatives of the potential U may be put as

$$\begin{aligned}
\frac{\partial U}{\partial r} &= Fr + G\sin\theta\sin\psi, & \frac{\partial U}{\partial z} &= Fz + G\cos\theta,\\
\frac{\partial U}{\partial \theta} &= G(-z\sin\theta + r\cos\theta\sin\psi), & \frac{\partial U}{\partial \psi} &= G\sin\theta\cos\psi,
\end{aligned}$$

and equations for equilibria reduce to

$$P_\psi = \frac{A\sin^2\theta}{mr^2 + A\sin^2\theta} P_\omega \tag{3}$$

$$Fr + G\sin\theta\sin\psi = \frac{mr}{A^2\sin^4\theta} P_\psi^2 \tag{4}$$

$$Fz + G\cos\theta = 0, \tag{5}$$

$$A\sin\theta\cos\theta\,(Fr + G\sin\theta\sin\psi) - mrG\,(-z\sin\theta + r\cos\theta\sin\psi) = 0, \tag{6}$$

$$Gr\sin\theta\cos\psi = 0. \tag{7}$$

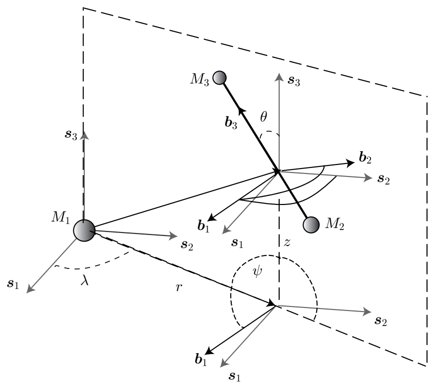


Figure 2: Conic solution: general case

The finding of general solution of this system is rather complicated, hence, we will only look for conic solutions, i.e., verifying $z\cos\theta \neq 0$. Cases $r = 0$ and $\theta = 0$ will be excluded since they correspond to singularities of the problem. Other solutions as linear and isosceles cases have already been studied by the authors [4].

§3. Conic stationary motion

3.1. Existence of conic motions

We will consider here not plane stationary solutions of equations (3)–(7) that satisfy $z\cos\theta \neq 0$.

One particular solution to the equation (7) corresponds to $\psi = \pi/2$ or $3\pi/2$. In this case (see figure 2), the three bodies M_1, M_2 and M_3 lay on the same plane $M_1M_2s_3$, and the axis of the dumb-bell describes a conic surface around the axis M_1s_3 with semiangle at the apex θ; the line passing through M_1 and C also describes a conic surface of semi-angle β, $\cos\beta = z/r$. Hence, we can call it conic solution.

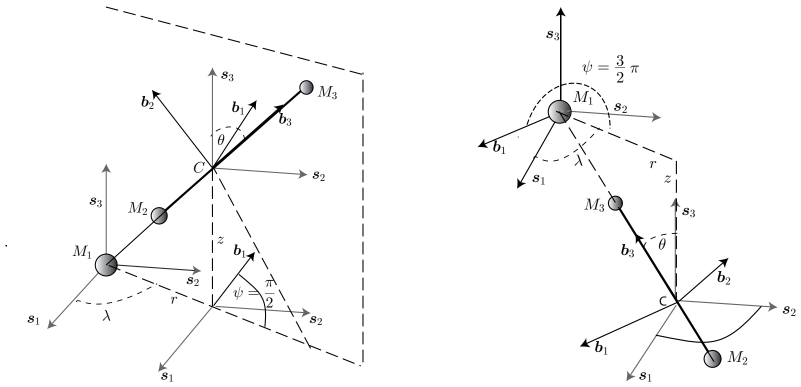
All the bodies must move around Gs_3 axis (G being the center of masses of the whole system) along circles in planes orthogonal to it with frequency of rotation $\dot{\omega}$ and radius given, respectively, by

$$\rho_1 = \frac{m_2+m_3}{m_1+m_2+m_3}\,r, \quad \rho_2 = \frac{m_1}{m_1+m_2+m_3}\,r + l_2\sin\theta, \quad \rho_3 = \frac{m_1}{m_1+m_2+m_3}\,r - l_3\sin\theta.$$

Let us note that for $z = 0$ and $\theta \neq \pi/2$, from (5), it must be $G = 0$, it is to say, $r_{12} = r_{13}$, and again (from (6)) $\theta = \pi/2$, in contradiction with the hypotesis. So, we will study conic stationary solutions with $z \neq 0$ and $\theta \neq \pi/2$. Let us deduce the existence of these equilibria studying the rest of equations. Writing $\varepsilon = \sin\psi = \pm 1$, equations (5) and (6) become:

$$F\,z + G\cos\theta = 0, \tag{8}$$

$$A\sin\theta\cos\theta\,F\,r + \big(\varepsilon A\sin^2\theta\cos\theta - m\,r\,(-z\sin\theta + \varepsilon\,r\cos\theta)\big)G = 0, \tag{9}$$



(a) (b)

Figure 3: Conic solution. (a) Case I, $\varepsilon = +1$, (b) Case II, $\varepsilon = -1$

These equations compose an indeterminate compatible linear system in the variables F, G (defined by (2)) if its determinant vanishes, it is to say, if

$$(m\,r\,z + \varepsilon A \sin\theta\cos\theta)\,(\varepsilon\,r\cos\theta - z\sin\theta) = 0,$$

what gives us two interesting particular solutions:

$$r_1 = \varepsilon\, z\tan\theta, \tag{10}$$

$$r_2 = -\frac{\varepsilon A}{2mz}\sin 2\theta. \tag{11}$$

3.2. Case I: $r = \varepsilon\, z\tan\theta$

Conditions of equilibria are now written as

$$P_\psi = \frac{A\sin^2\theta}{m\,r^2 + A\sin^2\theta}\,P_\omega, \tag{12}$$

$$F\,r + \varepsilon\,G\sin\theta = \frac{m\,r}{A^2\sin^4\theta}\,P_\psi^2 = \frac{m\,r}{(m\,r^2 + A\sin^2\theta)^2}\,P_\omega^2, \tag{13}$$

$$zF + G\cos\theta = 0, \tag{14}$$

$$\varepsilon\,r\cos\theta - z\sin\theta = 0. \tag{15}$$

The frequency of the motion is given by

$$\dot\omega^2 = \frac{P_\omega^2}{(m\,r^2 + A\sin^2\theta)^2} = \frac{F\,r + \varepsilon\,G\sin\theta}{m\,r} = \varepsilon\tan\theta\,\frac{F\,z + G\cos\theta}{m\,r} = 0.$$

It means that the three bodies are situated at fixed positions on a straight line. It must be $P_\omega = 0$, hence, condition (13) reduces to condition (14).

Now, taking η as the distance M_1 to C and $\nu = m_3/m_2$, condition (14) is transformed into

$$(\eta + l_3)^3 (l_2 - \eta) = \nu\,(l_2 - \eta)^3\,(l_3 + \eta), \quad \text{or} \quad (\eta + \frac{l}{1+\nu})^2 = \nu\,(\frac{\nu\,l}{1+\nu} - \eta)^2,$$

that provides the following solutions:

$$\eta_1 = l\,\frac{-1 + \nu\,\sqrt{\nu}}{(1+\nu)(1+\sqrt{\nu})}, \tag{16}$$

$$\eta_2 = l\,\frac{1 + \nu\,\sqrt{\nu}}{(1+\nu)(-1+\sqrt{\nu})}. \tag{17}$$

3.3. Case II: $r = \dfrac{\varepsilon A \sin 2\theta}{2mz}$

Conditions of equilibria are now written as

$$P_\psi = \frac{A \sin^2\theta}{m r^2 + A \sin^2\theta}\, P_\omega, \tag{18}$$

$$F\,r + \varepsilon\,G \sin\theta = \frac{m\,r}{A^2 \sin^4\theta}\, P_\psi^2 = \frac{m\,r}{(m\,r^2 + A \sin^2\theta)^2}\, P_\omega^2, \tag{19}$$

$$z F + G \cos\theta = 0, \tag{20}$$

$$-m\,r\,z + \varepsilon A \sin\theta \cos\theta = 0. \tag{21}$$

The frequency of the motion is given by

$$\dot\omega^2 = \frac{P_\omega^2}{(m r^2 + A \sin^2\theta)^2} = \frac{F\,r + \varepsilon\,G \sin\theta}{m\,r} \geq 0,$$

hence, taking into account (20),

$$0 \leq F\,r + \varepsilon\,G \sin\theta = F\,(r - \varepsilon\,z \tan\theta) \Longrightarrow r \geq \varepsilon\,z\,\tan\theta \Longleftrightarrow \sin\theta \leq \varepsilon.$$

Introducing the variable $\zeta = z/(l\,\cos\theta)$, condition (20) can be written in the following form:

$$r_{13}^3\left[\nu - \zeta\,(1+\nu)\right] = r_{12}^3\left[\nu + \nu\,(1+\nu)\,\zeta\right],$$

where r_{12} and r_{13} are now written as

$$r_{12}^2 = l^2\,\Big[\frac{\nu^2}{(1+\nu)^2} - \frac{2\,\nu\,\zeta \cos^2\theta}{1+\nu} + \zeta^2 \cos^2\theta\Big] + \frac{A^2 \sin^2\theta}{l^2 m^2 \zeta^2} - \frac{2 A\,\nu \sin^2\theta}{m\,(1+\nu)\,\zeta},$$

$$r_{13}^2 = l^2\,\Big[\frac{1}{(1+\nu)^2} + \frac{2\,\zeta \cos^2\theta}{1+\nu} + \zeta^2 \cos^2\theta\Big] + \frac{A^2 \sin^2\theta}{l^2 m^2 \zeta^2} + \frac{2 A \sin^2\theta}{m\,(1+\nu)\,\zeta}.$$

In this way, the equation (20) is equivalent to a polynomical one of degree thirteen in the variable ζ with coefficients known function of constants of the problem.

§4. Sufficient conditions for stability of the conic solutions.

The stationary solutions are defined by the following found values:

$$P_r^{(0)} = P_z^{(0)} = P_\theta^{(0)} = 0, \quad P_\psi^{(0)}, \quad r^{(0)}, \quad z^{(0)}, \quad \psi^{(0)}, \quad \theta^{(0)}.$$

Introducing the vector $\mathbf{v} = (y_1, y_2, y_3, y_4, x_1, x_2, x_3, x_4)$ of variations of the coordinates and momenta

$$\begin{aligned} &y_1 = P_r, && y_2 = P_z, && y_3 = P_\psi - P_\psi^{(0)}, && y_4 = P_\theta, \\ &x_1 = r - r^{(0)}, && x_2 = z - z^{(0)}, && x_3 = \psi - \psi^{(0)}, && x_4 = \theta - \theta^{(0)}, \end{aligned}$$

the Hamiltonian of the linearized perturbed problem [2] is, formaly, the same as the non-linearized, but with coefficients evaluated at the equilibrium solution. Consequently, the quadratic part of the Hamiltonian of the linearized perturbed problem is the sum of a positive defined part, the kinetic energy, and the Hessian of the potential energy. This last part is

$$\mathscr{V}_2 = \frac{1}{2} \sum_{i,j=1}^{4} V_{ij}\, x_i\, x_j, \tag{22}$$

where V_{ij} are the following second derivatives of the potential evaluated at the equilibrium solution. We will use this function as a Lyapunov function for our analysis of the statibility. In this way, the Lyapunov's stability of the stationary solutions follows (taking into account the Dirichlet theorem) from the fact that the quadratic form (22) be positively defined, i.e., in agreement with the Jacobi's criterium, if all the principal minors of the matrix which elements are (V_{ij}) have positive value. In the case of conic solutions, the matrix (V_{ij}) has $V_{13} = V_{23} = V_{34} = 0$, hence, the conditions of Jacobi's criterium become:

$$V_{11} > 0, \quad V_{11}V_{22} - V_{12}^2 > 0, \quad V_{33} > 0, \tag{23}$$

$$\det \begin{bmatrix} V_{11} & V_{12} & V_{14} \\ V_{12} & V_{22} & V_{24} \\ V_{14} & V_{24} & V_{44} \end{bmatrix} > 0. \tag{24}$$

These conditions, in the case I, reduce to

$$\begin{aligned} V_{11} &= F - 3\mathscr{G} m_1 \left[\frac{m_2}{r_{12}^5} (\eta - l_2)^2 + \frac{m_3}{r_{13}^5} (\eta^2 - l_3^2) \right] \sin^2\theta \geq 0, \\ V_{11}V_{22} - V_{12}^2 &= F\left(F - 3\mathscr{G} m_1 \left[\frac{m_2}{r_{12}^5} (\eta - l_2)^2 + \frac{m_3}{r_{13}^5} (\eta^2 - l_3^2) \right] \right) \geq 0, \\ V_{33} &= \varepsilon\, \eta\, G \sin^2\theta \geq 0. \end{aligned}$$

Analogously, we should procede in the case II.

§5. Conclusions

The equations of motion of one three-body problem composed of a dumb-bell (two masses at fixed distance) moving around a central mass have been stablished. Conic stationary solutions of these equations have studied and sufficient conditions for stability has been found in term of Lyapunov's stability functions. It seems that this could be a good approximation for the study of the motion of a body around to a massic segment and, one more general situation, the motion of two solid bodies under a gravitational field.

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