Monografías del Seminario Matemático García de Galdeano **33**, 99–105 (2006)

Optimal basis of a space mixing trigonometric and hyperbolic functions

J. M. Carnicer, E. Mainar and J. M. Peña

Abstract. We provide the basis with optimal shape preserving properties of the space $\bar{H}_1$ generated by 1, t, $\cos t$, $\sin t$, $\cosh t$, $\sinh t$ on $[0,2\pi]$. We illustrate the representation of remarkable curves.

Keywords: Shape preserving representation, trigonometric curve, hyperbolic curve, B-basis.

AMS classification: 65D17, 41A10, 42A10, 41A50.

§1. Introduction

The Bernstein basis is optimal among all other shape preserving bases of the space of polynomials of degree not greater than *n* on a given compact interval [3]. Roughly speaking, a curve designed with the optimal basis imitates the shape of its control polygon more faithfully than using other representations. In [4] it was proved that each space of functions admitting shape preserving representations always has an optimal basis called *the normalized B-basis*. This paper is devoted to the construction of the optimal basis of a space which can be used to design curves of interest in engineering and to show how such curves can be obtained in this space.

§2. Designing with trigonometric and hyperbolic functions

Let us consider the design of curves whose components are functions in the space

$$\bar{H}_1 = \mathrm{span}\langle 1, t, \cos t, \sin t, \cosh t, \sinh t \rangle,$$

on the interval $t \in [0,2\pi]$. First we shall find the optimal basis (normalized B-basis) of $\bar{H}_1$ and later we shall use it for the design of some remarkable curves.

By Theorem 4.1 of [1], there exists a normalized B-basis of the space $\bar{H}_1$ on $[0,2\pi]$ if and only if the space of the derivatives

$$\bar{H}_1' = \langle 1, \cos t, \sin t, \cosh t, \sinh t \rangle$$

is an extended Chebyshev space on $[0,2\pi]$, that is, any nonzero function of the space $\bar{H}_1'$ has at most $\dim \bar{H}_1' - 1 = 4$ zeros (counting multiplicities) on $[0,2\pi]$. In order to show it, we need the following generalization of Rolle's Theorem.

Lemma 1. *Let $f : I \to \mathbb{R}$ be a differentiable function on an interval I. Let $Z_I^*(f)$ be the number of zeros of f (counting multiplicities). Then for any real s, we have*

(a) If $t_1 < t_2$ are two zeros of f in I, then there exists a zero τ of $f' + sf$ such that $t_1 < \tau < t_2$,

(b) $Z_I^*(f' + sf) \geq Z_I^*(f) - 1$.

Proof. The set of zeros and their multiplicities of a function is preserved under multiplication by a positive C^∞ function. Let $g(t) := e^{st} f(t)$. Then the set of zeros of g coincides with the set of zeros of g and $Z_I^*(g) = Z_I^*(f)$. By Rolle's theorem, we have that between any two zeros $t_1 < t_2$ of g there exists a zero τ, $t_1 < \tau < t_2$, of g' and $Z_I^*(g') \geq Z_I^*(g) - 1$. Now let us observe that $f'(t) + sf(t) = e^{-st} g'(t)$. Then (a) follows and, from

$$Z_I^*(f' + sf) = Z_I^*(g') \geq Z_I^*(g) - 1 = Z_I^*(f) - 1,$$

(b) follows. □

We have already discussed in [2] the existence of normalized B-bases in all six dimensional spaces invariant under translations and reflections containing the first degree algebraic and trigonometric polynomials. For the sake of completeness, we give a direct argument to show that $\overline{H}_1$ has a normalized B-basis on $[0, 2\pi]$.

Proposition 2. *The space $\overline{H}_1$ has a normalized B-basis on $[0, 2\pi]$.*

Proof. By Theorem 4.1 of [1], we need to show that $\overline{H}_1' = \langle 1, \cos t, \sin t, \cosh t, \sinh t \rangle$ is an extended Chebyshev space on $[0, 2\pi]$. Let us assume that a function f in $\overline{H}_1'$ has 5 zeros (counting multiplicities) on $[0, 2\pi]$.

If 2π is not a zero of f, $Z_{2\pi}^*(f) = 0$, then f has 5 zeros on $[0, 2\pi)$. By Lemma 1 (b), the function $g := f' + f$ has at least 4 zeros on $[0, 2\pi)$ and again, by Lemma 1 (b), the function $g' - g = f'' - f$ has at least 3 zeros on $[0, 2\pi)$.

If 2π is a simple zero of f, $Z_{2\pi}^*(f) = 1$, then f has 4 zeros on $[0, 2\pi)$ and a single zero at 2π. By Lemma 1 (a), (b), the function g has at least 4 zeros on $[0, 2\pi)$ and again, by Lemma 1 (b), the function $g' - g = f'' - f$ has at least 3 zeros on $[0, 2\pi)$.

If 2π is a double zero of f, $Z_{2\pi}^*(f) = 2$, then f has 3 zeros on $[0, 2\pi)$ and a double zero at 2π. By Lemma 1 (a), (b), the function g has at least 3 zeros on $[0, 2\pi)$. In addition, 2π is a zero of g. By Lemma 1 (a), (b), the function $g' - g = f'' - f$ has at least 3 zeros on $[0, 2\pi)$.

Summarizing, if the multiplicity of 2π as a zero of f is less than or equal to 2, $Z_{2\pi}^*(f) \leq 2$, then $Z_{[0,2\pi)}^*(g' - g) \geq 3$. Taking into account that $\langle 1, \cos t, \sin t \rangle$ is an extended Chebyshev space on $[0, 2\pi)$, we deduce that $g' - g = 0$ and $g(t) = Ce^t$. Since g vanishes at least once, we deduce that $C = 0$ and g is the zero function. So $f' + f = 0$ and, since f vanishes at least once, we finally deduce that f is the zero function.

If $3 \leq Z_{2\pi}^*(f) \leq 5$, then $h(t) := f(2\pi - t) \in \overline{H}_1'$ has 5 zeros (counting multiplicities) on $[0, 2\pi]$ and $Z_{2\pi}^*(h) \leq 2$. By the above argument, $h = 0$ and then $f = 0$.

So we have shown that $\overline{H}_1'$ is an extended Chebyshev space on $[0, 2\pi]$. □

Now we may proceed to the construction of the normalized B-basis. First we construct a B-basis following the method suggested in Remark 2.3 and Theorem 2.4 of [1] and then, we shall normalize it following Remark 4.1 of [1]. Let us describe the steps of this construction.

First we start with the basis

$$(u_0,\dots,u_5) := \left(1, t, 1-\cos t, t-\sin t, \tfrac{1}{2}(\cosh t+\cos t)-1, \tfrac{1}{2}(\sinh t+\sin t)-t\right),\ t\in[0,2\pi],$$

whose wronskian matrix at $t=0$

$$W(u_0,\dots,u_5)(0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

is lower triangular with positive diagonal entries.

Then we evaluate the wronskian matrix of $(u_5,u_4,\dots,u_0)$ at $t=2\pi$

$$W(u_0,\dots,u_5)(2\pi) = \begin{pmatrix} \sinh(2\pi)/2-2\pi & (\cosh(2\pi)-1)/2 & 2\pi & 0 & 2\pi & 1\\ (\cosh(2\pi)-1)/2 & \sinh(2\pi)/2 & 0 & 0 & 1 & 0\\ \sinh(2\pi)/2 & (\cosh(2\pi)-1)/2 & 0 & 1 & 0 & 0\\ (\cosh(2\pi)-1)/2 & \sinh(2\pi)/2 & 1 & 0 & 0 & 0\\ \sinh(2\pi)/2 & (\cosh(2\pi)+1)/2 & 0 & -1 & 0 & 0\\ (\cosh(2\pi)+1)/2 & \sinh(2\pi)/2 & -1 & 0 & 0 & 0 \end{pmatrix}$$

and compute its LU factorization. We construct the basis $(b_0,\dots,b_5)$ defined by

$$(b_5,-b_4,b_3,-b_2,b_1,-b_0) = (u_5,u_4,u_3,u_2,u_1,u_0)U^{-1}.$$

In order to normalize the obtained basis, we solve the linear system

$$L(c_5,c_4,\dots,c_0)^T = (1,0,\dots,0)^T,$$

and then the normalized B-basis is

$$(B_0,\dots,B_5) := (c_0b_0,\dots,c_5b_5).$$

Since the space is invariant under reflections we have for the functions of the normalized B-basis

$$B_i(t) = B_{5-i}(a+b-t),\quad t\in[a,b],\quad i=0,\dots,5.$$

So we only need to compute half of the basis functions B_i, $0\le i\le 2$.

Therefore the normalized B-basis is

$$B_5(t) := \frac{(1-\tanh^2\pi)(2t-\sin t-\sinh t)}{4\pi(1-\tanh^2\pi)-2\tanh\pi},$$

$$B_4(t) := \frac{\tanh^3\pi(2t-\sin t-\sinh t)}{(4\pi(1-\tanh^2\pi)-2\tanh\pi)(\tanh\pi-2\pi)} - \frac{\tanh\pi(2-\cos t-\cosh t)}{2(2\pi-\tanh\pi)},$$

$$B_3(t) := \frac{2t-\sin t-\sinh t}{2(\tanh\pi-2\pi)} - \frac{\tanh\pi(2-\cos t-\cosh t)}{2(\tanh\pi-2\pi)} + \frac{t-\sin t}{2\pi},$$

$$B_2(t) := B_4(2\pi-t),\ B_1(t) := B_4(2\pi-t), B_0(t) := B_5(2\pi-t).$$

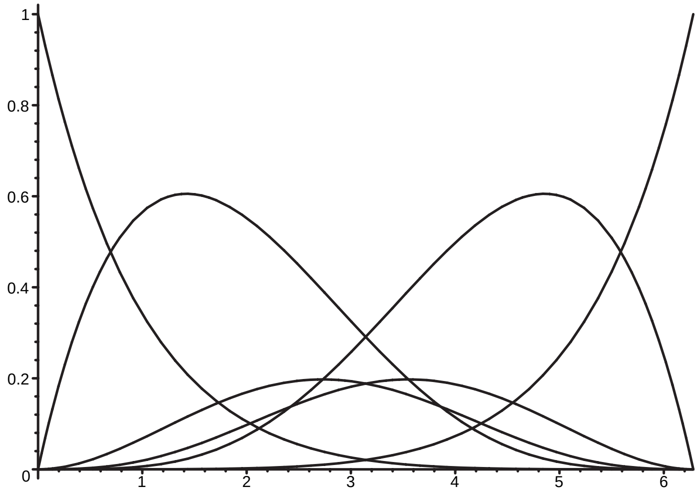

Figure 1: Normalized B-basis of $\bar{H}_1$ on $[0,2\pi]$

Figure 1 shows the graphs of the normalized B-basis of $\bar{H}_1$ on $[0,2\pi]$.

Let us now obtain the control polygons of some curves. For this purpose we need the coefficients of some functions with respect to the normalized B-basis. Let

$$\alpha := \frac{1}{\tanh \pi} - \frac{2\pi}{\sinh^2 \pi} \approx 0.9566321329$$

be the second Greville abscissa, then we can write down the coefficients of the functions $1, t$ (see Table 1). Figure 2 represents a linear segment and its corresponding control polygon $P_0 \cdots P_5$. Note the coincidence of the control points $P_0 = P_3$ and $P_2 = P_5$.

Table 1 also contains the coefficients of the trigonometric functions $1-\cos t, \sin t$, and the hyperbolic functions $\cosh(t-\pi), \sinh t$ with respect to the normalized B-basis.

Figure 3 shows a complete circle

$$(\sin t, 1-\cos t), \quad t \in [0,2\pi],$$

and its control polygon

$$\begin{pmatrix}0\\0\end{pmatrix}\begin{pmatrix}\alpha\\0\end{pmatrix}\begin{pmatrix}2\pi\\2\pi/\tanh\pi-1\end{pmatrix}\begin{pmatrix}-2\pi\\2\pi/\tanh\pi-1\end{pmatrix}\begin{pmatrix}-\alpha\\0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}.$$

Figure 4 shows the control polygon of the cycloid

$$(t-\sin t, 1-\cos t), \quad t \in [0,2\pi],$$

with respect to the normalized B-basis of $\bar{H}_1$.

Finally, Figure 5 shows the control polygon of the catenary

$$\big(t/\pi - 1, \cosh(t-\pi) - \cosh\pi\big), \quad t \in [0,2\pi].$$

function	c_0	c_1	c_2	c_3	c_4	c_5
1	1	1	1	1	1	1
t	0	α	2π	0	$2\pi-\alpha$	2π
$1-\cos t$	0	0	$\frac{2\pi}{\tanh\pi}-1$	$\frac{2\pi}{\tanh\pi}-1$	0	0
$\sin t$	0	α	2π	-2π	$-\alpha$	0
$\cosh(t-\pi)$	$\cosh\pi$	$\frac{2\pi}{\sinh\pi}$	$\frac{2\pi}{\sinh\pi}$	$\frac{2\pi}{\sinh\pi}$	$\frac{2\pi}{\sinh\pi}$	$\cosh\pi$
$\sinh t$	0	α	2π	2π	$4\pi-\alpha$	$\sinh(2\pi)$

Table 1: Coefficients of linear polynomials, trigonometric and hyperbolic functions

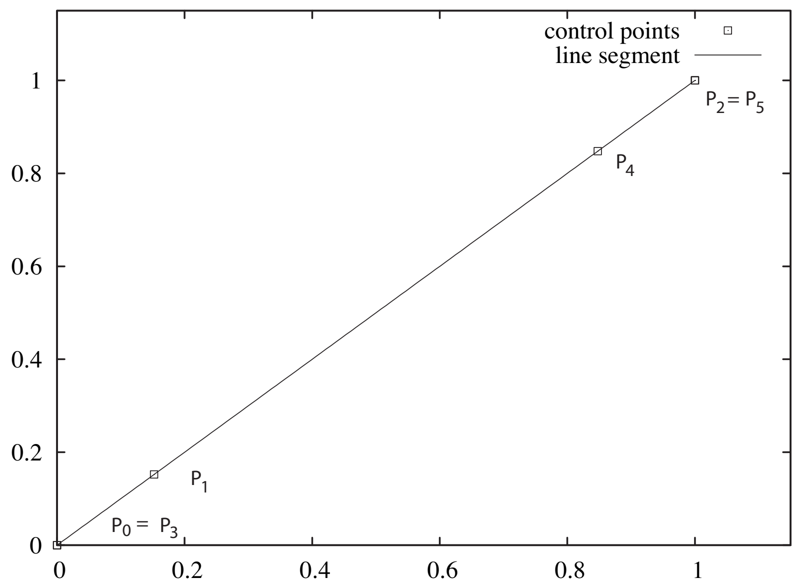

Figure 2: Control polygon of a linear segment in $\overline{H}_1$

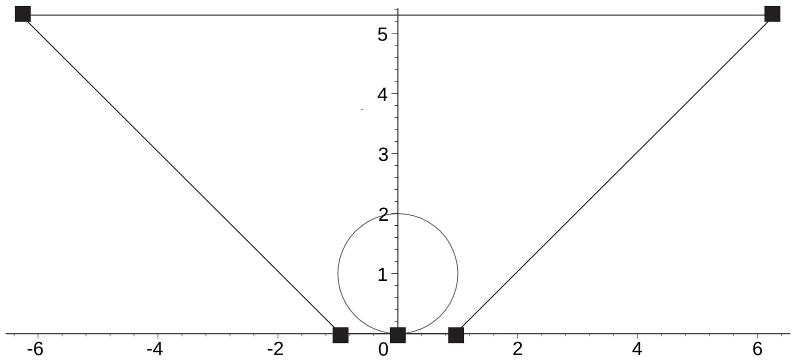

Figure 3: Control polygon of a circle

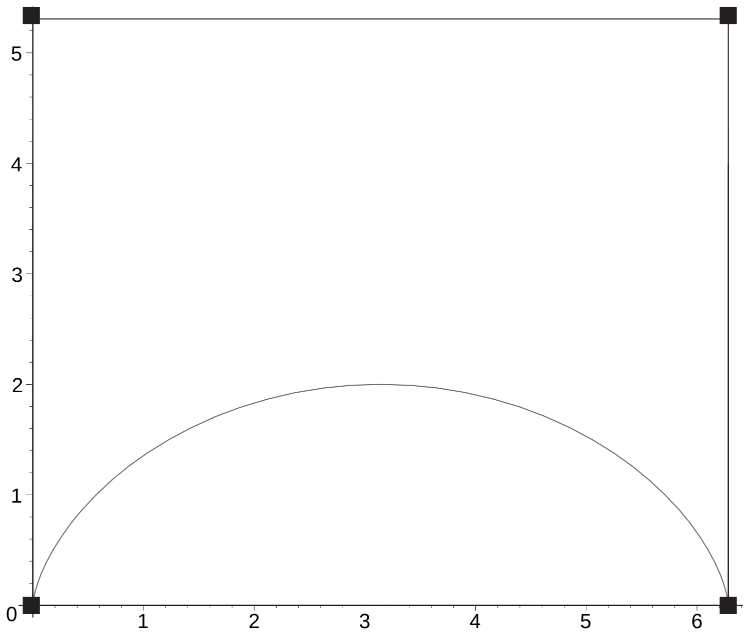

Figure 4: Control polygon of a cycloid

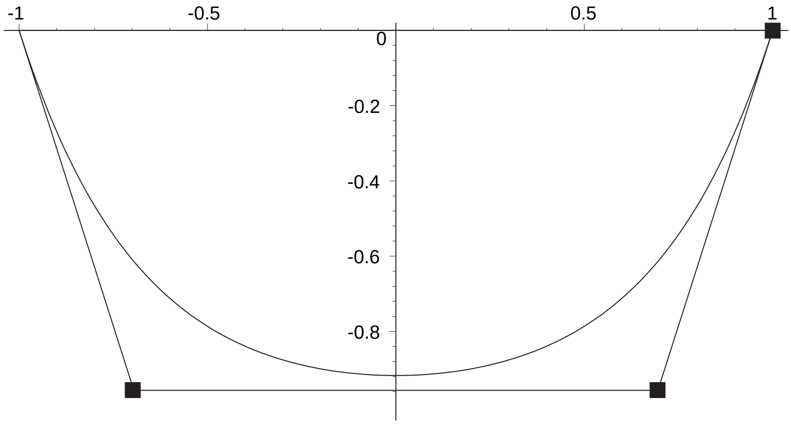

Figure 5: Control polygon of a catenary in $\overline{H}_1$

Acknowledgements

Partially supported by the Spanish Research Grant BFM2003-03510, by Gobierno de Aragón and Fondo Social Europeo.

References

[1] Carnicer, J. M., Mainar, E., and Peña, J. M. Critical length for design purposes and extended Chebyshev spaces. *Const. Approx. 20* (2004), 55–71.

[2] Carnicer, J. M., Mainar, E., and Peña, J. M. Shape preservation regions for six-dimensional spaces. To appear in *Adv. in Com. Math.* (2007).

[3] Carnicer, J. M., and Peña, J. M. Shape preserving representations and optimality of the Bernstein basis. *Adv. Comput. Math. 1* (1993), 173–196.

[4] Carnicer, J.M., and Peña, J. M. Totally positive bases for shape preserving curve design and optimality of B-splines. *Comput. Aided Geom. Design 11* (1994), 635–656.

J. M. Carnicer and J. M. Peña
Departamento de Matemática Aplicada
Universidad de Zaragoza
50009 Zaragoza, Spain
`carnicer@unizar.es` and `jmpena@unizar.es`

E. Mainar
Departamento de Matemáticas, Estadística y Computación
Universidad de Cantabria
39005 Santander, Spain
`mainare@unican.es`