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AN INVERSE SPECTRAL PROBLEM FOR A SCHRÖDINGER OPERATOR WITH AN UNBOUNDED POTENTIAL AND A WEIGHT IN $\mathbb{R}^2$

Laure Cardoulis

Abstract. In this paper, we prove a uniqueness theorem for the potential V and the weight M of the following operator $L = (m+M)^{-1}(-\Delta + q + V)$ in $\mathbb{R}^2$. The potential q is a known increasing radial potential which satisfies $\lim_{|x|\to+\infty} q(|x|) = +\infty$ and the potential V is a bounded perturbation of q with compact support. The weight m is a known radial bounded weight which is positive at infinity and the weight M is a bounded perturbation of m with compact support.

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§1. Introduction

In many papers, inverse problems for Schrödinger Operators are studied in the whole space or in half space (see for example [5, 7, 11, 14]). But usually in these papers, the considered potentials decrease towards infinity. There are also many papers on inverse spectral problems for increasing potentials on the line or on the half-line (see [8, 9]). When the potentials are bounded, the methods employed to get uniqueness results use the scattering operator, the scattering amplitude or the Dirichlet to Neumann map.

Our aim here is to study an inverse problem in the whole space for a potential which tends to infinity at infinity. Such unbounded potential occur in the quantum field theory (see [12, XII.3, XII.4, XIII.2] for the Hamiltonian of nonrelativistic quantum mechanics) and we recall (see [12, p. 279]) that the spectral properties of Schrödinger operators are highly dependent on the behaviour of the potential at infinity.

Therefore, because of the unboundedness of our operator, we use a spectral data method developed in [10] for the anharmonic oscillator operator in $\mathbb{R}^2$ and extended in [4] for a Schrödinger operator with a potential which tends to infinity at infinity but without any weight. Note that usually, only small perturbations around a radially symmetric potential are considered (see for example [2]).

In the present paper, we consider the following problem

$$(-\Delta + q + V)u = \lambda(m+M)u \text{ in } \mathbb{R}^2. \tag{1}$$

The variational space, denoted by $V_q(\mathbb{R}^2)$, is the completion of $\mathscr{D}(\mathbb{R}^2)$, the set of C^∞ functions with compact support, for the norm $\|u\|_q = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + qu^2\right)^{1/2}$. We recall that the embedding of $V_q(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$ is compact.

We denote by $\|u\|_{m+M} = \left(\int_{\mathbb{R}^2}(m+M)u^2\right)^{1/2}$ for all $u \in L^2(\mathbb{R}^2)$. According to the hypotheses on m and M (see below (h2)), $\|.\|_{m+M}$ is a norm in $L^2(\mathbb{R}^2)$ equivalent to the usual norm. We denote by $m+M$ the operator of multiplication by $m+M$ in $L^2(\mathbb{R}^2)$. The operator $(-\Delta+q+V)^{-1}(m+M) : (L^2(\mathbb{R}^2), \|.\|_{m+M}) \to (L^2(\mathbb{R}^2), \|.\|_{m+M})$ is positive self-adjoint and compact. So its spectrum is discrete and consists of a positive sequence $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \to 0$ as $n \to +\infty$. We denote by $\lambda_1 = 1/\mu_1$ and by ϕ_1 the corresponding eigenfunction which satisfy $(-\Delta+q+V)\phi_1 = \lambda_1(m+M)\phi_1$ in $\mathbb{R}^2$ and $\|\phi_1\|_{m+M} = 1$. (We recall that λ_1 is simple and $\phi_1 > 0$ (see [1, Th. 2.2]).) We denote by $|x| = r$ and by $\int f$ a primitive of f. We consider the following technical hypotheses:

(h1) $q(x) = q(|x|)$; $q(x) \to +\infty$ as $|x| \to +\infty$; $q \in C^2$; $q' \geq 0$; $V \in L^\infty$; $q(|x|)+V(x) \geq cst > 0$;

(h2) $m(x) = m(|x|)$; $m \in C^1 \cap L^\infty$; $M \in L^\infty$ be such that there exists constants m_1 and m_2, $0 < m_1 \leq m(r)+M(r,\theta) \leq m_2$ for all r,θ;

(h3) for all $\lambda > 0$, there exists $r_0 \in \mathbb{R}^+$, such that, for all $r \geq r_0$, $q(r) \geq 1$ and $\sqrt{q(r)} - \frac{\lambda m(r)}{2\sqrt{q(r)}} \geq 0$;

(h4) $\lim_{r\to+\infty} \frac{q'(r)}{q^{3/2}(r)} = 0$;

(h5) there exists $\gamma \in [0,1]$ such that $\frac{q''(r)}{q^{3/2}(r)} = O\left(\frac{1}{r^2 q^{1/2}(r)}, \frac{1}{q^{3/2}(r)}, \frac{q'(r)}{q^2(r)}, \frac{|m'(r)|}{(q(r))^\gamma}\right)$ and $\frac{q'^2(r)}{q^{5/2}(r)} = O\left(\frac{1}{r^2 q^{1/2}(r)}, \frac{1}{q^{3/2}(r)}, \frac{q'(r)}{q^2(r)}, \frac{|m'(r)|}{(q(r))^\gamma}\right)$ at infinity;

(h6) $\frac{1}{r^2 q^{1/2}(r)}$, $\frac{1}{q^{3/2}(r)}$, $\frac{q'(r)}{q^2(r)}$ and $\frac{|m'(r)|}{(q(r))^\gamma} \in L^1([r_0,+\infty[)$;

(h7) $q^{-1/4} e^{-\int (q^{1/2} - \frac{\lambda m}{2} q^{-1/2})} \in L^2([r_0,+\infty[)$;

(h8) $\int_r^{+\infty} o\left(e^{-2\int (q^{1/2} - \frac{\lambda m}{2} q^{-1/2})}\right) = o\left(e^{-2\int (q^{1/2} - \frac{\lambda m}{2} q^{-1/2})}\right)$ for r large enough;

(h9) $q^{-1/4} e^{\int (q^{1/2} - \frac{\lambda m}{2} q^{-1/2})} \notin L^2([r_0,+\infty[)$;

(h10) there exists $\alpha > 1$ and $C_1 > 0$ such that for all $r \geq r_0$ and for all θ, $|V(r,\theta)| \leq \frac{C_1}{r^\alpha} e^{-\int q^{1/2}}$;

(h11) there exists $\beta > 1$ and $C_2 > 0$ such that for all $r \geq r_0$ and for all θ, $|M(r,\theta)| \leq \frac{C_2}{r^\beta} e^{-\int q^{1/2}}$.

Note that (h3) is a consequence of (h1)-(h2) (and permits to define r_0) and that (h2) ensures that if the weight M has a compact support, then the radial weight m is positive outside the support of M. Note also that (h10) (resp. (h11)) is satisfied if the potential V (resp. the weight M) has a compact support. Finally, note that (h7) and (h8) can be replaced by (h12) $e^{-\int (q^{1/2}-\frac{\lambda m}{2}q^{-1/2})} \in L^2([r_0,+\infty[)$.

For example, $q(r)=r^n$ (for $r \geq 1$) satisfies each of the precedent items if and only if $n > \frac{2}{3}$.

We follow here the same method than in [4] where we obtained a uniqueness result for the potential V of the equation (1) and where we assumed that $m+M=1$. The interest of the presence of the weight m is that now, on the contrary of [4], we can take $q(r)=e^r$ with for example $m(r)=\frac{1}{r^2+1}+1$ (indeed (h5) is satisfied since $\frac{q''(r)}{q^{3/2}(r)}=O\left(\frac{|m'(r)|}{q(r)^{1/4}}\right)$, and $\frac{q'^2(r)}{q^{5/2}(r)}=O\left(\frac{|m'(r)|}{q(r)^{1/4}}\right)$ at infinity). But, as in [4] and also as in [6] (for the asymptotic distribution of the eigenvalues), we still do not take $q(r)=\ln(r)$ since $\frac{1}{(\ln(r))^{3/2}} \notin L^1([r_0,+\infty[)$. Denote now for all $p \geq 1$ by

$$C_{\lambda,p}(V,M) := \lim_{r\to+\infty} 2q^{1/4}r^{1/2}e^{\int (q^{1/2}-\frac{\lambda m}{2}q^{-1/2})} \int_0^{2\pi} u(r,\theta)e^{-ip\theta}d\theta, \tag{2}$$

where u is one solution of (1). Our aim is to prove the following theorem.

Theorem 1. *We assume that the pairs (q,V) and (q,W) satisfy the hypothesis (h1), the pairs (m,M) and (m,P) satisfy the hypothesis (h2), q and m satisfy the hypotheses (h3) to (h9), V and W are bounded potentials with compact supports, M and P are bounded weights with compact supports. Let $(\phi_l)_l$ be the normalized eigenfunctions associated with $(\lambda_l(V,M))_l$ and let $(\psi_l)_l$ be the normalized eigenfunctions associated with $(\lambda_l(W,P))_l$. If, for all $l \in \mathbb{N}$, $l \geq 1$, and, for all $p \in \mathbb{N}$,*

$$\lambda_l(V,M)=\lambda_l(W,P)=\lambda_l \quad \text{and} \quad C_{\lambda_l,p}(V,M)=C_{\lambda_l,p}(W,P),$$

then

$$\phi_l=\psi_l \text{ (for all } l) \quad \text{and} \quad M=P,\ V=W \text{ in } \Omega=\{x\in\mathbb{R}^2,\ \exists l\geq 2, \phi_l(x)\neq 0\}.$$

We will follow a method used by T. Suzuki in a bounded domain [13]. This method has been generalized by H. Isozaki [10] for the anharmonic operator in $\mathbb{R}^2$ then by L. Cardoulis, M. Cristofol and P. Gaitan [4] for a Schrödinger operator in $\mathbb{R}^2$. This method is the following: (i) first, we study the asymptotic behaviour of the solutions of a second order differential equation which stems from equation (1); (ii) then we prove that, under hypotheses (h1)–(h11), $C_{\lambda,p}(V,M)$ is well defined as a constant; (iii) finally, we prove that all the eigenfunctions associated with (V,M) and (W,P) are the same, and therefore that the perturbations are equal in Ω.

§2. Asymptotic behaviour of a solution

Here we study the asymptotic behaviour of the solutions for the equation (1). Put $|x|=r$. We use polar coordinates to define u_p

$$u_p(r)=r^{1/2}\int_0^{2\pi} u(r,\theta)e^{-ip\theta}d\theta. \tag{3}$$

Using (1) and (3), we get:

$$-u_p''(r)+\Big(\frac{p^2-1/4}{r^2}+q(r)-\lambda m(r)\Big)u_p(r)=f_p(r), \tag{4}$$

where

$$f_p(r)=-r^{1/2}\int_0^{2\pi}V(r,\theta)u(r,\theta)e^{-ip\theta}d\theta+\lambda r^{1/2}\int_0^{2\pi}M(r,\theta)u(r,\theta)e^{-ip\theta}d\theta. \tag{5}$$

We are now able to prove the following theorem.

Theorem 2. *Let q, V and m be potentials and weight defined as before, satisfying (h1) to (h9), a and λ two positive reals. Then the equation*

$$-u''(r)+(\frac{a}{r^2}+q(r)-\lambda m(r))u(r)=0 \tag{6}$$

has a system of fundamental solutions $\{y_1,y_2\}$ with the asymptotic behaviours:

$$y_1\sim\frac{1}{2}q^{-1/4}e^{-\int(q^{1/2}-\frac{\lambda m}{2}q^{-1/2})},\quad y_2\sim q^{-1/4}e^{\int(q^{1/2}-\frac{\lambda m}{2}q^{-1/2})},\quad \text{as } r\to+\infty. \tag{7}$$

Furthermore, the Wronskian $y_1y_2'-y_1'y_2=1$.

Proof. **Step 1**: We set $y=\binom{u}{u'}$; using (2.4) we get:

$$y'=\begin{pmatrix}u'\\u''\end{pmatrix}=\begin{pmatrix}0 & 1\\ q(r)-\lambda m(r)+\frac{a}{r^2} & 0\end{pmatrix}\begin{pmatrix}u\\u'\end{pmatrix}=\begin{pmatrix}0 & 1\\ q(r)-\lambda m(r)+\frac{a}{r^2} & 0,\end{pmatrix}y. \tag{8}$$

We denote

$$k=k(r)=\sqrt{q(r)-\lambda m(r)+\frac{a}{r^2}},\quad P=\begin{pmatrix}1 & -1\\ k & k\end{pmatrix},\quad y=P\widetilde{y}. \tag{9}$$

If we combine the two relations (8) and (9), we obtain:

$$\widetilde{y}'=P_1\widetilde{y}\text{ with }P_1=P^{-1}\left[\begin{pmatrix}k & k\\ k^2 & -k^2\end{pmatrix}-P'\right], \tag{10}$$

where P' is the derivative of the matrix P with respect to the variable r. Then, we make a Taylor development for each entry of the matrix P_1. Using the hypotheses (h4) and (h5), we can write (10) under the following form:

$$\widetilde{y}'=\begin{pmatrix}q^{1/2}-\frac{\lambda m}{2q^{1/2}}-\frac{q'}{4q} & -\frac{q'}{4q}\\ -\frac{q'}{4q} & -q^{1/2}+\frac{\lambda m}{2q^{1/2}}-\frac{q'}{4q}\end{pmatrix}\widetilde{y}+R\widetilde{y},$$

with $R:=R\left(\frac{1}{r^2q^{1/2}(r)},\frac{1}{q^{3/2}(r)},\frac{q'(r)}{q^2(r)},\frac{|m'(r)|}{(q(r))^\gamma}\right)$ a 2×2-matrix where all the entries are in the form $O\left(\frac{1}{r^2q^{1/2}(r)},\frac{1}{q^{3/2}(r)},\frac{q'(r)}{q^2(r)},\frac{|m'(r)|}{(q(r))^\gamma}\right)=O(\frac{1}{r^2q^{1/2}(r)})+O(\frac{1}{q^{3/2}(r)})+O(\frac{q'(r)}{q^2(r)})+O(\frac{|m'(r)|}{(q(r))^\gamma})$.

Step 2: Consider $\widetilde{y} = \left(I - \frac{q'}{4q^{3/2}} P_2\right) z$ with $P_2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$. We get:

$$z' = A(q)z + R\left(\frac{1}{r^2 q^{1/2}(r)}, \frac{1}{q^{3/2}(r)}, \frac{q'(r)}{q^2(r)}, \frac{|m'(r)|}{(q(r))^\gamma}\right) z,$$

with

$$A(q) = \begin{pmatrix} q^{1/2} - \frac{\lambda m}{2q^{1/2}} - \frac{q'}{4q} & 0 \\ 0 & -q^{1/2} + \frac{\lambda m}{2q^{1/2}} - \frac{q'}{4q} \end{pmatrix}.$$

Step 3: Now we introduce the new variable v defined by $z = E(q)v$, with

$$E(q) = \begin{pmatrix} q^{-1/4} e^{\int (q^{1/2} - \frac{\lambda m}{2q^{1/2}})} & 0 \\ 0 & \frac{1}{2} q^{-1/4} e^{-\int \left(q^{1/2} - \frac{\lambda m}{2q^{1/2}}\right)} \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}.$$

The matrix $E(q)$ satifies $E(q)' = A(q)E(q)$ and we have $v' = E(q)^{-1} R(q) E(q) v = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} v$. Moreover, using (h6), we deduce that there exists $r_1 \geq r_0$ and $C \in]0,1[$ such that, for all $r \geq r_1$, the integrals $\int_r^{+\infty} \frac{1}{q^{1/2}(t)t^2}\,dt$, $\int_r^{+\infty} \frac{1}{q^{3/2}(t)}\,dt$, $\int_r^{+\infty} \frac{q'(t)}{q^2(t)}\,dt$ and $\int_r^{+\infty} \frac{|m'(t)|}{(q(t))^\gamma}\,dt$ are bounded above by C.

Step 4: We define now the map T and the set $\mathscr{F}$ as follows: $T : v = (v_1, v_2) \mapsto w = (w_1, w_2)$ where $w_1(r) = \int_r^\infty K_1 v(t)\,dt$ and $w_2(r) = \xi + \int_{r_1}^r K_2 v(t)\,dt$ (ξ will be specified later), and $\mathscr{F}$ is the set of $v = (v_1, v_2)$ such that

$$\forall i = 1,2,\ v_i : [r_1, \infty) \to \mathbb{R},\ v_1 = o\Big(e^{-2\int (q^{1/2} - \frac{\lambda m}{2q^{1/2}})}\Big), \quad \sup_{r \geq r_1} |v_2(x)| < +\infty.$$

Note that $(\mathscr{F}, \|.\|_\infty)$ is a Banach space with $\|v\|_\infty = \max(\|v_1\|_\infty, \|v_2\|_\infty)$. Using (h6)–(h8), we can prove that the map $T : \mathscr{F} \to \mathscr{F}$ is a contraction, and so there exists a unique v in $\mathscr{F}$ such that $Tv = v$. For this v in $\mathscr{F}$, we come back up to z then to $\widetilde{y}$, after that to y and finally to u one solution of the equation (6). We obtain that $u \sim E_1 v_1 - E_2 v_2$. Since $v \in \mathscr{F}$, $\frac{E_1}{E_2} v_1 = o(1)$ and we choose ξ such that v_2 has a limit which is not equal to 0 as r tends to infinity. Therefore we deduce that E_2 is an asymptotic behaviour for one solution of (6) and by a standard transformation we get also that E_1 is another asymptotic behaviour for one solution of (6). □

Now we prove that the constant defined by (2) exists.

Lemma 3. *Let u be in $L^2(\mathbb{R}^2)$ such that $(-\Delta + q + V)u = \lambda(m + M)u$ for λ a positive real and q, V, m, M potentials and weights which satisfy the hypotheses (h1) to (h11). Then, each $C_{\lambda,p}(V,M)$ (defined by (2)) exists and is a constant.*

Proof. Let u be one solution of the equation (1). Recall that u_p defined by (3) is one solution of the equation (4) and that f_p is defined by (5). Let y_1 and y_2 be a fundamental system of solutions for (6) whose asymptotics behaviours are given by (7). Then there exists c_1 and c_2, constants, such that

$$u_p(r) = c_1 y_1(r) + c_2 y_2(r) - y_2(r) \int_{r_0}^r y_1(t) f_p(t) dt + y_1(r) \int_{r_0}^r y_2(t) f_p(t) dt.$$

Using (h10) and (h11) we get that $y_i f_p \in L^1([r_0,+\infty[)$ for $i=1,2$. So we can write

$$u_p(r) = \Big(c_1 + \int_{r_0}^{+\infty} y_2(t)\, f_p(t)\, dt\Big) y_1(r) - \Big(\int_r^{+\infty} y_2(t)\, f_p(t)\, dt\Big) y_1(r) \\ + \Big(c_2 - \int_{r_0}^{+\infty} y_1(t)\, f_p(t)\, dt\Big) y_2(r) + \Big(\int_r^{+\infty} y_1(t)\, f_p(t)\, dt\Big) y_2(r).$$

Since $\int_r^{+\infty} y_2(t)\, f_p(t)\, dt = o(1)$, we get $\big(\int_r^{+\infty} y_2(t)\, f_p(t)\, dt\big) y_1(r) \in L^2([r_0,+\infty[)$. We obtain also that $\big(\int_r^{+\infty} y_1(t)\, f_p(t) dt\big)\, y_2(r) \in L^2([r_0,+\infty[)$. Moreover, $y_2 \notin L^2([r_0,+\infty[)$, so we get that $c_2 = \int_{r_0}^{+\infty} y_1(t)\, f_p(t) dt$. Therefore we can prove that: $C_{\lambda,p}(V,M) = c_1 + \int_{r_0}^{+\infty} y_2(t)\, f_p(t)\, dt$. $\square$

§3. Proof of Theorem 1

We assume that $\operatorname{supp} V \subset B := \{x; |x| < R\} = B(0,R)$, $\operatorname{supp} W \subset B(0,R)$, $\operatorname{supp} M \subset B(0,R)$, $\operatorname{supp} P \subset B(0,R)$, with $R > r_0$. We are going to prove that $\varphi_l(x) = \psi_l(x)$, for all x and all $l \geq 1$.

Step 1: We prove in this step that for all l, $\varphi_l(x) = \psi_l(x)$, if $|x| > R$. We have

$$(-\Delta + q)\varphi_l = \lambda_l m \varphi_l \quad \text{and} \quad (-\Delta + q)\psi_l = \lambda_l m \psi_l \text{ in } \mathbb{R}^2 \setminus B.$$

We decompose φ_l and ψ_l relative to the trigonometric functions basis $\{e^{-ik\theta}\}_k$, and we want to prove that all the coefficients are equal. Let

$$b_{l,p}(r) = r^{\frac{1}{2}} \int_0^{2\pi} \varphi_l(r,\theta)\, e^{-ip\theta}\, d\theta \quad \text{and} \quad \tilde{b}_{l,p}(r) = r^{\frac{1}{2}} \int_0^{2\pi} \psi_l(r,\theta)\, e^{-ip\theta}\, d\theta.$$

If $r > R$ then (see (3) and (4)) the function $b_{l,p}$ is solution of

$$-b''_{l,p}(r) + \Big(\frac{p^2 - \frac{1}{4}}{r^2} + q(r) - \lambda_l m(r)\Big) b_{l,p}(r) = 0.$$

This equation has the following fundamental system of solutions (see Theorem 2)

$$b_{l,p}(r) = C_1 y_1(r) + C_2 y_2(r) \quad \text{with} \quad \begin{cases} y_1 \sim \frac{1}{2} q^{-1/4}\, e^{\int \left(-q^{1/2} + \frac{\lambda_l m}{2} q^{-1/2}\right)}, \text{ as } r \to +\infty, \\ y_2 \sim q^{-1/4}\, e^{\int \left(q^{1/2} - \frac{\lambda_l m}{2} q^{-1/2}\right)}, \text{ as } r \to +\infty. \end{cases}$$

Since $b_{l,p} \in L^2$, $y_1 \in L^2$, and $y_2 \notin L^2$, we deduce that $C_2 = 0$, so we get that $b_{l,p}(r) = C_1 y_1(r)$. By the same way, we obtain that $\tilde{b}_{l,p}(r) = \tilde{C}_1 y_1(r)$. Moreover, due to the hypothesis $C_{\lambda_l,p}(V,M) = C_{\lambda_l,p}(W,P)$, we obtain $C_1 = \tilde{C}_1$. Therefore $b_{l,p}(r) = \tilde{b}_{l,p}(r)$ for all l and for all $r > R$. So, for all l, $\phi_l(r) = \psi_l(r)$ if $r > R$.

Step 2: We prove here that $\varphi_1(x) = \psi_1(x)$ if $|x| < R$. We consider

$$K(x,y) = \sum_{p \geq 1} \varphi_p(x) \{\psi_p(y) - \varphi_p(y)\} \tag{11}$$

and we have $K(x,y) = \sum_{p\geq 1} \psi_p(y)\{\varphi_p(x) - \psi_p(x)\}$. Let x be such that $|x| < R$. By the first step, if $|y| > R$, (11) ensures that $K(x,y) = 0$. If we denote by $V_y = -\triangle_y + q(|y|) + V(y)$, for $y \in \mathbb{R}^2$, then $V_y K(x,y) = 0$ if $|y| > R$. We put for all $t > 0$

$$F_t(x,y) = \sum_{p\geq 1} e^{-t\lambda_p}\, \psi_p(y)\{\varphi_p(x) - \psi_p(x)\}. \tag{12}$$

Since $e^{-t\lambda_p} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}(-\lambda_p)^n$, we obtain

$$F_t(x,y) = \sum_{p\geq 1}\sum_{n=0}^{+\infty} \frac{t^n}{n!}(-\lambda_p)^n\, \psi_p(y)\{\varphi_p(x) - \psi_p(x)\},$$

and then

$$F_t(x,y) = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\sum_{p\geq 1}(-\lambda_p)^n\, \psi_p(y)\{\varphi_p(x) - \psi_p(x)\} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}(-V_y)^n K(x,y).$$

So $F_t(x,y) = 0$ if $|y| > R$. Now, multiplying $F_t(x,y)$ by $e^{t\lambda_1}$, we obtain, for $|y| > R$,

$$e^{t\lambda_1} F_t(x,y) = 0 = \psi_1(y)\{\varphi_1(x) - \psi_1(x)\} + \sum_{p\geq 2} e^{-t(\lambda_p-\lambda_1)}\, \psi_p(y)\{\varphi_p(x) - \psi_p(x)\}. \tag{13}$$

Multiplying by a test function, we can prove that the limit, as t tends to infinity, of the second term of the previous sum in (13) is equal to zero. Then (13) implies that $\psi_1(y)\{\varphi_1(x) - \psi_1(x)\} = 0$ for $|x| < R$ and $|y| > R$. So, since ψ_1 has no zero, we have $\varphi_1(x) - \psi_1(x) = 0$ for $|x| < R$. Thus, $\varphi_1(x) = \psi_1(x)$ for $x \in \mathbb{R}^2$.
Step 3: Recall that $x \in \mathbb{R}^2$ be such that $|x| < R$. Since $\phi_1 = \psi_1$, the function F_t defined by (12) becomes: for all $t > 0$, $F_t(x,y) = \sum_{p\geq 2} e^{-t\lambda_p}\psi_p(y)(\phi_p(x) - \psi_p(x))$. We can choose y such that $|y| > R$ and $\psi_2(y) \neq 0$ (since ψ_2 has non compact support). Multiplying F_t by $e^{t\lambda_2}$, we obtain also

$$0 = \psi_2(y)\{\varphi_2(x) - \psi_2(x)\} + \sum_{p\geq 3} e^{-t(\lambda_p-\lambda_2)}\, \psi_p(y)\{\varphi_p(x) - \psi_p(x)\}.$$

As in the second step, we can prove that the limit, as $t \to +\infty$, of the second term is equal to 0, so we get that $\phi_2(x) = \psi_2(x)$ for $|x| < R$. Therefore, for all $x \in \mathbb{R}^2$, $\phi_2(x) = \psi_2(x)$. Step by step, we obtain that, for all $l \geq 1$, and for all $x \in \mathbb{R}^2$, $\phi_l(x) = \psi_l(x)$. Therefore we get: for all $l \geq 1$,

$$(-\Delta + q + V)\phi_l = \lambda_l(m+M)\phi_l \quad \text{and} \quad (-\Delta + q + W)\phi_l = \lambda_l(m+P)\phi_l \text{ in } \mathbb{R}^2.$$

So, for all $l \geq 1$, $(V - W)\phi_l = \lambda_l(M-P)\phi_l$. Since $\phi_1 > 0$, we have $V - W = \lambda_1(M-P)$ and so, for all $l \geq 2$, $\lambda_1(M-P)\phi_l = \lambda_l(M-P)\phi_l$. Since $\lambda_1 < \lambda_l$ for all $l \geq 2$, we obtain: for all $l \geq 2$, $(M-P)\phi_l = 0$. Therefore $M = P$ in $\Omega = \{x \in \mathbb{R}^2,\ \exists l \geq 2,\ \phi_l(x) \neq 0\}$ and so $V = W$ in Ω.

Remark 1. The set $\mathbb{R}^2 \setminus \Omega = \{x \in \mathbb{R}^2,\ \forall l \geq 2,\ \phi_l(x) = 0\}$ does not contain any bounded open subset. Indeed, using a unique continuation theorem (see [3], [12, Th. XIII.63, p. 240]) for the operator $-\Delta + q + V$ in any ball B (note that $|(-\Delta + q + V)\phi_l|^2 = |\lambda_l(m+M)\phi_l|^2 \leq cst\,|\phi_l|^2$ in B), since ϕ_l is an eigenfunction, ϕ_l cannot vanish in B.

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Laure Cardoulis
Université de Toulouse 1, Ceremath/UMR MIP
Place Anatole France, 31000 Toulouse, France
laure.cardoulis@univ-tlse1.fr