

ON CONTROLLABILITY OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

Hassane Bouzahir

Abstract. In this paper, we give sufficient conditions for controllability of a class of partial neutral functional differential equations with infinite delay. We suppose that the linear part is not necessarily densely defined but satisfies the resolvent estimates of the Hille-Yosida theorem. The results are obtained using the integrated semigroups theory. We also announce and avoid a serious problem in the two published papers [9] and [8].

Keywords: Controllability, infinite delay, integrated semigroup, integral solution, neutral functional differential equations.

AMS classification: 34K30, 34K40, 35R10, 45K05.

§1. Introduction

In this paper, we prove a result about controllability to the following partial neutral functional differential equation with infinite delay

$$\begin{cases} \frac{\partial}{\partial t} \mathcal{D}x_t = A\mathcal{D}x_t + Cu(t) + F(t, x_t), t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \quad (1)$$

where the state $x(\cdot)$ takes values in a Banach space $(E, |\cdot|)$, the control $u(\cdot)$ is given in $L^2([0, T], U)$, $T > 0$, the Banach space of admissible control functions with U a Banach space, C is a bounded linear operator from U into E , $A : D(A) \subseteq E \rightarrow E$ is a linear operator on E , \mathcal{B} is the phase space of functions mapping $(-\infty, 0]$ into E , which will be specified later, \mathcal{D} is a bounded linear operator from \mathcal{B} into E defined by

$$\mathcal{D}\varphi = \varphi(0) - \mathcal{D}_0\varphi \text{ for any } \varphi \in \mathcal{B},$$

\mathcal{D}_0 is a bounded linear operator from \mathcal{B} into E and for each $x : (-\infty, T] \rightarrow E$, $T > 0$, and $t \in [0, T]$, x_t represents, as usual, the mapping defined from $(-\infty, 0]$ into E by

$$x_t(\theta) = x(t + \theta) \text{ for } \theta \in (-\infty, 0].$$

F is an E -valued nonlinear continuous mapping on $\mathbb{R}_+ \times \mathcal{B}$.

Treating equations with infinite delay such as Eq. (1) often requires more sophisticated methods and techniques than the finite delay case. For example, to avoid repetitions and understand the interesting properties of the phase space, we suppose that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a

(semi)normed abstract linear space of functions mapping $(-\infty, 0]$ into E , and satisfies the following fundamental axioms which have been first introduced in [18] and widely discussed in [23].

(A) There exist a positive constant H and functions $K(\cdot), M(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $x : (-\infty, \sigma + a] \rightarrow E$, $x_\sigma \in \mathcal{B}$ and $x(\cdot)$ is continuous on $[\sigma, \sigma + a]$, then for every t in $[\sigma, \sigma + a]$ the following conditions hold:

- (i) $x_t \in \mathcal{B}$,
- (ii) $|x(t)| \leq H \|x_t\|_{\mathcal{B}}$, which is equivalent to
- (ii') $|\varphi(0)| \leq H \|\varphi\|_{\mathcal{B}}$, for every $\varphi \in \mathcal{B}$,
- (iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup_{\sigma \leq s \leq t} |x(s)| + M(t - \sigma) \|x_\sigma\|_{\mathcal{B}}$.

(A1) For the function $x(\cdot)$ in (A), $t \mapsto x_t$ is a \mathcal{B} -valued continuous function for t in $[\sigma, \sigma + a]$.

(B) The space \mathcal{B} is complete.

Example 1. Define for a constant γ the following standard space

$$C_\gamma := \left\{ \phi : (-\infty, 0] \rightarrow E \text{ continuous such that } \lim_{\theta \rightarrow -\infty} e^{\gamma\theta} \phi(\theta) \text{ exists in } E \right\}.$$

It is known from [23] that C_γ with the norm $\|\phi\|_\gamma = \sup_{\theta \leq 0} e^{\gamma\theta} |\phi(\theta)|$, $\phi \in C_\gamma$, satisfies the axioms (A), (A1) and (B) with $H = 1$, $K(t) = \max(1, e^{-\gamma t})$ and $M(t) = e^{-\gamma t}$ for all $t \geq 0$.

Throughout, we also assume that the operator A satisfies the Hille-Yosida condition:

(H1) there exist $\bar{M} \geq 0$ and $\bar{\omega} \in \mathbb{N}$ such that $]\bar{\omega}, +\infty[\subset \rho(A)$ and

$$\sup \{ (\lambda - \bar{\omega})^n \|(\lambda I - A)^{-n}\| : n \in \mathbb{N}, \lambda > \bar{\omega} \} \leq \bar{M}. \quad (2)$$

Let A_0 be the part of the operator A in $\overline{D(A)}$, which is defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \\ A_0 x = Ax, \text{ for } x \in D(A_0). \end{cases}$$

It is well known that $\overline{D(A_0)} = \overline{D(A)}$ and the operator A_0 generates a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on $D(A)$.

Recall that ([27]) for all $x \in \overline{D(A)}$ and $t \geq 0$, one has $\int_0^t T_0(s)x \in D(A_0)$ and

$$\left(A \int_0^t T_0(s)x ds \right) + x = T_0(t)x. \quad (3)$$

We also recall that $(T_0(t))_{t \geq 0}$ coincides on $\overline{D(A_0)}$ with the derivative of the locally Lipschitz integrated semigroup $(S(t))_{t \geq 0}$ generated by A on E . Which is, according to [5] and [24], a family of bounded linear operators on E , that satisfies

- (i) $S(0) = 0$,
- (ii) for any $y \in E$, $t \rightarrow S(t)y$ is strongly continuous with values in E ,
- (iii) $S(s)S(t) = \int_0^s (S(t+r) - S(r))dr$ for all $t, s \geq 0$, and for any $\tau > 0$ there exists a constant $l(\tau) > 0$ such that

$$\|S(t) - S(s)\| \leq l(\tau) |t - s| \text{ for all } t, s \in [0, \tau].$$

This integrated semigroup is exponentially bounded, that is, there exist two constants \bar{M} and $\bar{\omega}$ such that $\|S(t)\| \leq \bar{M}e^{\bar{\omega}t}$ for all $t \geq 0$.

The theory of neutral functional differential equations with infinite delay in infinite dimension has been recently developed and it is still a field of research (see, for instance, [20], [10] and the references therein). Our aim in this article is to announce and correct a serious problem in two published papers [9] and [8]. That is, the hypotheses (H8 i) in [8] and (H4) in [9] mean that the interval of existence depends on the initial function which contradicts the followed definition of (local) controllability. To avoid the problem, we suppose conditions that assure global existence. We give sufficient conditions for controllability of partial neutral functional differential equations with infinite delay. We suppose that the linear part is not necessarily densely defined but satisfies the resolvent estimates of the Hille-Yosida theorem. The results are obtained using the integrated semigroups theory. We make use of the notion of integral solution and we do not use the analytic semigroups theory.

§2. Main results

We start by introducing the following definition.

Definition 1. Let $T > 0$ and $\varphi \in \mathcal{B}$. We say that a function $x := x(., \varphi) : (-\infty, T) \rightarrow E$, $0 < T \leq +\infty$, is an *integral solution* of Eq. (1) if

- (i) x is continuous on $[0, T)$,
- (ii) $\int_0^t \mathcal{D}x_s ds \in D(A)$ for $t \in [0, T)$,
- (iii) $\mathcal{D}x_t = \mathcal{D}\varphi + A \int_0^t \mathcal{D}x_s ds + \int_0^t Cu(s) + F(s, x_s) ds$ for $t \in [0, T)$,
- (iv) $x(t) = \varphi(t)$, for all $t \in (-\infty, 0]$.

We deduce from [1] and [31] that integral solutions of Eq. (1) are given for $\varphi \in \mathcal{B}$ such that $\mathcal{D}\varphi \in D(A)$ by the following system

$$\begin{cases} \mathcal{D}x_t = S'(t)\mathcal{D}\varphi + \lim_{\lambda \rightarrow +\infty} \int_0^t S'(t-s)B_\lambda (Cu(s) + F(s, x_s)) ds, & t \in [0, T), \\ x(t) = \varphi(t), & t \in (-\infty, 0], \end{cases} \quad (4)$$

where

$$B_\lambda = \lambda(\lambda I - A)^{-1}. \quad (5)$$

To obtain global existence and uniqueness of integral solutions, we have supposed in [1] that

(H2) $K(0) \|\mathcal{D}_0\| < 1$.

(H3) $F : [0, +\infty) \times \mathcal{B} \rightarrow E$ is continuous and there exists $\beta_0 > 0$ such that

$$|F(t, \varphi_1) - F(t, \varphi_2)| \leq \beta_0 \|\varphi_1 - \varphi_2\|_{\mathcal{B}} \text{ for } \varphi_1, \varphi_2 \in \mathcal{B} \text{ and } t \geq 0.$$

Using Theorem 7 in [1], we get the following result.

Theorem 1. Assume that (H1), (H2) and (H3) hold. Let $\varphi \in \mathcal{B}$ such that $\mathcal{D}\varphi \in \overline{D(A)}$. Then, there exists a unique integral solution $x(\cdot, \varphi)$ of Eq. (1), defined on $(-\infty, +\infty)$.

Definition 2. Under the above conditions, Eq. (1) is said to be *controllable* on an interval $J = [0, \delta]$, $\delta > 0$, if for every initial function $\varphi \in \mathcal{B}$ with $\varphi(0) - \mathcal{D}_0\varphi \in \overline{D(A)}$ and $e_1 \in \overline{D(A)}$, there exists a control $u \in L^2(J, U)$ such that the solution $x(\cdot)$ of Eq. (1) satisfies $x(\delta) = e_1$.

Theorem 2. Suppose that (H1), (H2) and (H3) hold. Let $x(\cdot)$ be the integral solution of Eq. (1) on $(-\infty, \delta)$, $\delta > 0$, and assume that (see [28]) the linear operator W from U into $D(A)$ defined by

$$Wu = \lim_{\lambda \rightarrow +\infty} \int_0^\delta S'(\delta - s) B_\lambda C u(s) ds,$$

induces an invertible operator \tilde{W} on $L^2(J, U)/\text{Ker } W$ and there exist positive constants N_1 and N_2 satisfying $\|C\| \leq N_1$ and $\|\tilde{W}^{-1}\| \leq N_2$, then Eq. (1) is controllable on $J = [0, \delta]$, provided that

$$\left(\|\mathcal{D}_0\| + \beta_0 \bar{M} e^{\bar{\omega}\delta} \delta + \beta_0 N_1 N_2 \bar{M}^2 e^{\bar{\omega}\delta} \delta^2 \right) K_\delta < 1,$$

where $K_\delta := \max_{0 \leq t \leq \delta} K(t)$.

Proof. Let $x(\cdot)$ be the integral solution of Eq. (1) on $(-\infty, \delta)$, $\delta > 0$. It suffices to take for $t \in J$

$$u(t) = \tilde{W}^{-1} \left\{ \lim_{\lambda \rightarrow +\infty} \int_0^\delta S'(\delta - s) B_\lambda C u(s) ds \right\} (t).$$

That is

$$u(t) = \tilde{W}^{-1} \left\{ x(\delta) - \mathcal{D}_0 x_\delta - S'(\delta) \mathcal{D}\varphi - \lim_{\lambda \rightarrow +\infty} \int_0^\delta S'(\delta - s) B_\lambda F(s, x_s) ds \right\} (t).$$

We can show that by a fixed point argument, with this control the integral solution $x(\cdot)$ of Eq. (1) exists and satisfies $x(\delta) = e_1$. (see [1] for details about this method). \square

Remark 1. Supposing that all linear operators W from U into $\overline{D(A)}$ defined by

$$Wu = \lim_{\lambda \rightarrow +\infty} \int_a^b S'(b - s) B_\lambda C u(s) ds, \quad (6)$$

$0 \leq a < b \leq T$, $T > 0$, induce invertible operators \tilde{W} on $L^2([a, b], U)/\text{Ker } W$ such that there exist positive constants N_1 and N_2 satisfying $\|C\| \leq N_1$ and $\|\tilde{W}^{-1}\| \leq N_2$, taking $\delta = T/N$, N large enough and following [1], a similar argument as the above proof can be used inductively in $[n\delta, (n+1)\delta]$, $1 \leq n \leq N-1$, to see that Eq. (1) is controllable on $[0, T]$ for all $T > 0$.

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Hassane BOUZAHIR
Université Ibn Zohr
Ecole Nationale des Sciences Appliquées
BP 1136, 80000 Agadir-MOROCCO
hbouzahir@yahoo.fr

