Monografías del Seminario Matemático García de Galdeano **33**, 207–214 (2006)

OPTIMAL INSPECTION OF A SYSTEM WITH TWO TYPES OF FAILURES UNDER AGE DEPENDENT MINIMAL REPAIR

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Abstract. This work describes a maintenance model for a system that presents failures of two types, revealed minor failures (type R) and unrevealed catastrophic failures (type U). The latter are detected by means of an inspection policy at periodic times kT, $k = 1, 2, \dots$ Moreover it is assumed the possibility of imperfect inspections, that is, false alarms of failure as well as undetected failures after an inspection. Type R failures are removed by a minimal repair whereas a perfect repair follows the type U failures. The maintenance procedure is completed with a renewal of the system after the N^{th} type R failure. The objective function is the expected cost in an infinite time span and interest centers on the existence of a finite optimum policy.

Keywords: Maintenance, optimum policy, reliability, unrevealed failures.

AMS classification: 90B25, 60K10.

§1. Introduction

A system failure often means the reduction —partial or even complete— of its ability to fulfill its required function. Most of the times the consequences of a failure can be measured from an economic point of view. Thus, the failure of a component in a manufacturing plant can be responsible of the defective production as well as of the cost incurred due to downtime. The reliability of any system turns out to be a crucial issue and so does the maintenance procedures.

Corrective maintenance is performed after a system failure and can be of different types. The perfect repair brings the system back to an *as-good-as-new* condition by any procedure or even the whole replacement of the system by a new identical one. After an imperfect repair the device is returned to the functioning state but it is no longer *as-good-as-new*. In case that it recovers the state just prior to failure (*as-bad-as-old*), the repair is known as minimal repair. It is important to remark that different probabilistic structures emerge depending on the quality of maintenance actions. In the works of Brown and Proschan [5] as well as Nakagawa and Yasui [7] an imperfect maintenance is achieved with probability p or a perfect one with probability $1-p$. Nakagawa and Yasui [7] study policies which minimize the expected cost rates. Block *et al* [4] present also a perfect repair in combination with a minimal repair. Brown and Proschan [5] as well as Block *et al* [4] analyze the preservation and monotone properties of the model.

In addition, another classification comes out regarding the failure type. Those detected as soon as they occur are known as revealed failures. However many engineering systems may undergo the so-called unrevealed failures, that is, those that are detected only by special tests or inspection. Failures of this sort are typical in systems that are not in continuous operation such as spares or units in stand-by mode. If the failure happens while the mechanism is in an idle period it will remain undiscovered until the following attempt of use unless the system is monitored. The periodic inspection appears to be the right alternative to overcome the prohibitive cost of a continuous monitoring. Badía *et al.* [1] consider the cost optimization by selection of a unique interval for both inspection and maintenance. Zequeira and Bérenger [10] provide an inspection policy along with preventive actions for a system subject to three competing failure types. In a recent work Biswas *et al.* [3] analyze the availability function of a periodically inspected system that experiences a fixed number of imperfect repairs before being perfectly repaired.

Periodic inspections are often applied to detectors of fire, gas, as well as pressure and safety valves installed to prevent special risks. The safety systems of nuclear plants are typical examples of systems under unrevealed failures (Vaurio [9]). In this work we consider a system subject to both failure types, revealed and unrevealed, and design an inspection policy along with a maintenance procedure. The shorter the times between inspections the smaller the downtime. However each inspection involves a cost and the inspection frequency should be weighted against the cost incurred. In this article we aim at minimizing the cost per time unit over an infinite time span and interest centers on the condition under which there exists an optimum inspection interval. The maintenance model along with the cost function as well as the main results concerning the existence of an optimum policy are in the second section where the relevant conclusions are also provided. Some examples illustrating the theoretical results are presented in the last section. This model constitutes an extension of former inspections policies provided in Badía *et al* [1] and [2].

§2. The model and main results

In what follows we consider a system that may undergo two failure types: revealed minor failures (R) and unrevealed catastrophic failures (U). A failure occurring at time t, will be of the type R with probability $p(t)$ and of the type U with probability $q(t) = 1 - p(t)$. Computers serve as practical example of systems of this sort. The existence of a file containing a virus should be checked by means of anti-virus programs and, more often than not, can cause a serious damage in the hard disk. However failures in the power supply are detected as soon as they occur and in general are of less importance.

Periodic inspections are carried out to detect type U failures. Let T denote the elapsed time between two consecutive inspections. We also assume the possibility of imperfect inspections as in Badía *et al* [1]. That is, false alarms of failure as well as undetected failures after an inspection. A minimal repair follows the type R failures meanwhile a perfect repair is carried out every time that a type U failure is discovered. The maintenance procedure is completed with a perfect repair after the Nth type R failure so as to prevent system wearout. Times of inspections are assumed to be negligible but times of both perfect and minimal repairs are taken into account.

2.1. Notation and former results

- X time to first failure.
- $r(x)$ failure rate corresponding to X.
- $H(x)$ cumulative failure rate, $H(x) = \int_0^x r(u)du$.
- $H_R(x) = \int_0^x p(t)r(t)dt$ cumulative failure rate of the time to the first type R failure.
- $H_U(x) = \int_0^x q(t)r(t)dt$ cumulative failure rate of the time to the first type U failure.
  It follows that
  $$H(x) = H_R(x) + H_U(x).$$
- Y time to the first type U failure.
- G_i time to the ith type R failure, $i = 1, 2, \ldots, N$.
- $N[0,t]$ number of type R failures in $[0,t]$.

  Next, the different times of repair are defined:

- t_U time of the perfect repair of a type U failure.
- t_R time of the perfect repair after the Nth failure of the type R.
- t_{mr} time of the minimal repair of a type R failure.

  Regarding to costs, the following ones are considered:

- c_i unitary cost of inspection.
- c_f unitary cost of false alarm.
- c_d cost rate per unit of downtime. Downtime occurs while a type U failure remains undiscovered or in case that the system is being repaired.
- $c_{mr,i}(t)$ cost of the minimal repair incurred after the ith type R failure that occurs at time t, $i = 1, 2, \ldots, N-1$.
- c_{r1} cost of the perfect repair after a type U failure.
- $c_{r2}(N,t)$ cost of the perfect repair that follows the Nth type R failure. It depends on both N and the failure time, t.

The objective function is the expected cost per unit of time over an infinite time span. It depends on both time between inspections, T, and the number, N, of type R failures previous to the perfect repair. We provide a necessary and sufficient condition for the existence of an optimal inspection interval, $T^\star$.

The next proposition contains some basic results related to the age dependent minimal policy (see Block *et al* [4]) that will be used through this article.

Proposition 1. *Under the model assumptions, the following results hold:*

(i) The density and reliability functions corresponding to the first type U failure, Y, are

$$f_Y(x) = q(x)r(x)e^{-H_U(x)}, \quad x \geq 0,$$
$$\overline{F}_Y(x) = e^{-H_U(x)}, \quad x \geq 0.$$

(ii) $(N_t)_t$ with $N_t = N[0,t]$ is a non homogeneous Poisson process (NHPP) with $H_R(t)$ being its mean function.

(iii) The density and reliability function corresponding to the time to the ith failure, G_i ($i = 1,2,\ldots,N$) are

$$f_{G_i}(x) = p(x)r(x)\frac{H_R(x)^{i-1}}{(i-1)!}e^{-H_R(x)}, \quad x \geq 0,$$
$$\overline{F}_{G_i}(x) = \sum_{j=0}^{i-1}\frac{H_R(x)^j}{j!}e^{-H_R(x)}, \quad x \geq 0.$$

(iv) Y is independent from both the process $(N_t)_t$ and the variables G_i.

The probability of a false alarm in an inspection is denoted by α meanwhile β represents the probability that a type U failure is not detected.

2.2. Cost function

The model also involves the following random variables:

- K_1 number of inspections previous to a type U failure.
- K_2 number of inspections previous to the Nth type R failure.
- K_3 number of inspections after a type U failure until it is detected.

Therefore

$$K_1 = \left\lfloor \frac{Y}{T} \right\rfloor, \quad K_2 = \left\lfloor \frac{G_N}{T} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of a real number x. In addition K_3 is a geometric random variable with parameter $1-\beta$. Hence its mean value is

$$E(K_3) = \frac{1}{1-\beta} = \delta.$$

A cycle, denoted by τ is the time span between two consecutive renewals of the system.

Let A_1 and A_2 be the following events: $A_1 = \{Y < G_N\}$ y $A_2 = \{Y > G_N\}$. Therefore A_1 represents that the cycle ends after the perfect repair when a type U failure is detected. A_2 corresponds to the the case when the cycle is completed after the perfect repair that follows the Nth failure. Denoting by 1_A the indicator function of the event A and MR the number of minimal repairs in a cycle. The next equalities are obtained by using previous results on the age-dependent minimal repair model (Block *et al* [4]):

$$MR = N[0,Y]1_{A_1} + (N-1)1_{A_2} = N[0,Y]1_{\{N[0,Y]\le N-1\}} + (N-1)1_{\{N[0,Y]\ge N\}},$$
$$E(MR) = \sum_{k=0}^{N-2}\int_0^\infty \frac{H_R(x)^k}{k!}p(x)r(x)e^{-H(x)}dx.$$

The length of a cycle turns out to be

$$\tau = T(K_1+K_3)1_{A_1} + G_N 1_{A_2} + t_U 1_{A_1} + t_R 1_{A_2} + t_{mr}MR.$$

The probabilities of A_1 and A_2 are derived from the results in Proposition 1:

$$P(A_1) = \int_0^\infty \overline{F}_{G_N}(x)f_Y(x)dx = \sum_{k=0}^{N-1}\int_0^\infty q(x)r(x)\frac{H_R(x)^k}{k!}e^{-H(x)}dx,$$
$$P(A_2) = \int_0^\infty \overline{F}_Y(x)f_{G_N}(x)dx = \int_0^\infty p(x)r(x)\frac{H_R(x)^{N-1}}{(N-1)!}e^{-H(x)}dx.$$

X_N and $X_N^\star$ denote two auxiliary random variables with the following density functions:

$$f_{X_N}(x) = \frac{p(x)r(x)\frac{H_R(x)^{N-1}}{(N-1)!}e^{-H(x)}}{P(A_2)}, \quad f_{X_N^\star}(x) = \frac{\sum_{k=0}^{N-1}q(x)r(x)\frac{H_R(x)^k}{k!}e^{-H(x)}}{P(A_1)},$$

and $S_N^\star(T)$ the next expectation

$$S_N^\star(T) = E\left(\left\lfloor\frac{X_N^\star}{T}\right\rfloor\right).$$

Then

$$E(K_1 1_{A_1}) = P(A_1)S_N^\star(T), \quad E(G_N 1_{A_2}) = P(A_2)E(X_N).$$

The mean number of minimal repairs in a cycle is obtained by means of well known results on minimal repair. Hence the mean length of a cycle turns out to be

$$E(\tau) = P(A_1)T(S_N^\star(T)+\delta) + P(A_2)E(X_N) + P(A_1)t_U + P(A_2)t_R + t_{mr}E(MR).$$

The cost of a cycle, $C(\tau)$, is expressed in terms of the random variables below.

- I number of inspections in a cycle, $\quad I = (K_1+K_3)1_{A_1} + K_2 1_{A_2}$.
- I_0 number of inspections in a cycle previous to a type U failure, $\quad I_0 = K_1 1_{A_1} + K_2 1_{A_2}$.
- F number of false alarms in a cycle.
- CMR cost incurred in a cycle due to the minimal repairs.
- D downtime of the system in a cycle, $\quad D = \tau - (YA_1 + G_N 1_{A_2})$.

p	N^*	$T^{\star}_{N^{\star}}$	$Q(T^{\star}_{N^{\star}},N^{\star})$
0.1	2	3.246	11.677
0.2	2	2.75	11.494
0.3	3	3.57	11.304
0.4	3	3.092	11.085
0.5	4	3.75	10.869
0.6	5	4.215	10.638

Table 1: Optimum policy, $N^{\star}$, $T^{*}_{N^{\star}}$ and optimum cost, $Q(T^{\star}_{N^{\star}},N^{\star})$.

The distribution of F when I_0 is known is a binomial random variable with parameters $n = I_0$ and $p = \alpha$. Therefore

$$E(F) = \alpha E(I_0).$$

In addition

$$\begin{aligned} CMR &= \left(\sum_{i=1}^{N[0,Y]} c_{mr,i}(G_i)\right) 1_{A_1} + \left(\sum_{i=1}^{N-1} c_{mr,i}(G_i)\right) 1_{A_2} \\ &= \left(\sum_{i=1}^{N[0,Y]} c_{mr,i}(G_i)\right) 1_{\{N[0,Y]\le N-1\}} + \left(\sum_{i=1}^{N-1} c_{mr,i}(G_i)\right) 1_{\{N[0,Y]\ge N\}}. \end{aligned}$$

The cost of a cycle is

$$C(\tau) = c_i I + c_f F + c_{r1} 1_{A_1} + c_{r2}(N, G_N) + CMR + c_d D.$$

The next result is obtained from well-known results related to the age-dependent minimal repair model (Block *et al* [4]):

$$E(CMR) = \int_0^{\infty} \sum_{j=0}^{N-2} c_{mr,j+1}(x) \frac{H_R(x)^j}{j!} e^{-H_R(x)} p(x) r(x) e^{-H_U(x)} dx.$$

Some additional calculations allow to derive the expected cost of a cycle:

$$E(C(\tau)) = c_d E(\tau) + \Psi(N) + (c_i + c_f \alpha)(P(A_1) S^{\star}_N(T) + P(A_2) S_N(T)),$$

where

$$S_N(T) = E\left(\left\lfloor \frac{X_N}{T} \right\rfloor\right),$$

$$\Psi(N) = P(A_1)(c_i \delta + c_{r1} - c_d E(X^{\star}_N)) + P(A_2)(E(c_{r2}(N, X_N)) - E(X_N)) + E(CMR).$$

The cost function, denoted by $Q(T,N)$, is the expected cost per unit of time over an infinite time span. The key result of the renewal-reward processes states that $Q(T,N)$ converges, as time goes by, to the ratio between the expected cost of a cycle and its expected length (see Ross [8]). Therefore, the forthcoming result holds.

Proposition 2. *$Q(T,N)$ turns out to be*

$$Q(T,N) = c_d + \frac{\Psi(N) + (c_i + c_f\alpha)[P(A_1)S_N^\star(T) + P(A_2)S_N(T)]}{E[\tau]}. \tag{1}$$

Theorem 3. *If $P(A_1) > 0$, then there exists $T_N^\star$ ($0 < T_N^\star < \infty$) that minimizes $Q(T,N)$ in 1 if and only if*

$$\Psi(N) < 0. \tag{2}$$

Proof. Using some properties of $S_N^\star(T)$ and $S_N(T)$ already derived (Badía *et al* [1]), if $0 < P(A_1)$, it follows that

$$\lim_{T\to\infty} E(\tau) = \infty \quad \text{and} \quad \lim_{T\to\infty}(P(A_1)S_N^\star(T) + P(A_2)S_N(T)) = 0,$$
$$\lim_{T\to 0} E(\tau) < \infty \quad \text{and} \quad \lim_{T\to 0}(P(A_1)S_N^\star(T) + P(A_2)S_N(T)) = \infty.$$

Therefore

$$\lim_{T\to 0} Q(T,N) = \infty \quad \text{and} \quad \lim_{T\to\infty} Q(T,N) = Q(\infty,N) = c_d.$$

If $\Psi(N) < 0$, there exists $T_0 \in (0,\infty)$ such that $P(A_1)S_N^\star(T_0) + P(A_2)S_N(T_0) = -\Psi(N)$. $Q(T,N) > c_d$ for $T < T_0$ and $Q(T,N) < c_d$ for $T > T_0$ and Theorem 3 holds.

Whenever $\Psi(N) \geq 0$, $Q(T,N) \geq c_d = Q(\infty,N)$ and $T_N^\star = \infty$. □

Remark 1. Condition (2) is equivalent to the next one involving the expected uptime, $E[A]$:

$$P(A_1)E(X_N^\star) + P(A_2)E(X_N) = E(A) > \frac{P(A_1)(c_i\delta + c_{r1}) + P(A_2)E(c_{r2}(N,X_N))}{c_d}.$$

The foregoing inequality implies that an inspection policy is rewarding whenever the system uptime compensates the costs incurred.

§3. Examples

Time to failure is assumed to be an exponential random variable with mean equal to one.The probabilities of false alarm and undetected failure after inspection are, respectively, $\alpha = \beta = 0.1$. The rest of the parameters in the model are given as follows: $c_i = 1$, $c_f = 4$, $c_d = 12$, $c_{r1} = 10$, $c_{mr,i}(t) = it$, $,i = 1,\ldots,N-1$, $c_{r2}(N,t) = Nt$, $t_U = t_R = 1$, $t_{mr} = 0.5$.

With respect to the the optimum number of type R failures previous to the perfect repair, $N^\star$, we follow the procedure proposed by Nakagawa [6]. First, the optimum $T_N^\star$ is derived, for a given N, afterwards the optimum, $N^\star$, is obtained. Then

$$Q(T_{N^\star}^\star, N^\star) = \min_N Q(T_N^\star, N).$$

Table 1 shows both the optimum inspection interval, $T_{N^\star}^\star$, and the optimum number of type R failures, $N^\star$, previous to the perfect repair as well as the corresponding optimum cost, $Q(T_{N^\star}^\star, N^\star)$. Table 1 reveals that the optimum inspection interval, $T_{N^\star}^\star$ is non-monotonic with p. Nevertheless, the greater the probability of a type R failure, p, the greater $N^\star$ and the lower the optimum cost. Note that when p increases, the probability of a catastrophic failure, which causes higher costs than a revealed one, is reduced and so is the optimum cost.

Acknowledgements

This work has been supported by the University of Zaragoza-Ibercaja under project IBE2004-CIE-02.

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