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A COMPACTNESS RESULT FOR A PSEUDO-PARABOLIC CONSERVATION LAW WITH CONSTRAINT

S. N. Antontsev, G. Gagneux and G. Vallet

Abstract. This work deals with the study of a compactness result for a class of pseudoparabolic problems of type: $\partial_t u - \mathrm{div}\{a(\partial_t u + E)\nabla(u + \tau\partial_t u)\} = 0$. with boundary conditions that takes explicitly into account a nonlinear map of $\partial_t u$.

Keywords: Compactness, pseudoparabolic, differential inclusion.

AMS classification: 35K65, 35K70.

§1. Introduction

In this paper, we are interested in the mathematical analysis of a nonlinear pseudoparabolic problem. This study rises from geological basin formation models initially developed by the Institut Français du Pétrole (IFP). The main feature of these models is characterized by a constraint on the time-derivative of the solution that leads us to consider an original class of conservation laws.

A more precise description of these models have been exposed by S. N. Antontsev *et al.* [2, 3, 1], G. Gagneux *et al.* [7, 6] and G. Vallet [11] for the mathematical aspect of the monolithological case and R. Eymard *et al.* [5] for a numerical approach.

Let us consider a sedimentary basin whose base, denoted by Ω, is a fixed connected open subset of $\mathbb{R}^d$ ($d = 1,2$ in this framework) with a Lipschitzian boundary $\Gamma = \Gamma_s \cup \Gamma_e \cup \partial\Gamma_s$ (with $\Gamma_s \cap \Gamma_e = \emptyset$) and an outward unit normal denoted by $\overrightarrow{n}$.

The sediment height u naturally satisfies the mass balance equation: $\partial_t u + \mathrm{div}\{\vec{q}\} = 0$ in Q, where $\vec{q}$ follows a dynamic extension of the law of Darcy (see C. Cuesta *et al.* [4] for example). Moreover, one introduces a maximum erosion rate E such that: $-\partial_t u \leq E$ in Q, where E takes into account the composition, the structure and the age of the sediment. The coupling of these two constraints is clearly an essential issue since both diffusive sedimentation or erosion and weather limited erosion can occur at the same time in a basin. Then one introduces a new unknown λ satisfying $0 \leq \lambda \leq 1$ and playing the role of a flux limiter, according to the Darcy-Barenblatt's law: $\vec{q} = -\lambda\nabla(u + \tau\partial_t u)$ in Q with $\tau > 0$.

In order to give a mathematical modelling of λ, Th. Gallouët proposes in [9] the following formulation: $1 - \lambda \geq 0$, $\partial_t u + E \geq 0$ with $(1-\lambda)(\partial_t u + E) = 0$ a.e. in Q. Moreover, if H denotes the maximal monotone graph of the Heaviside function, the unilateral global constraint is then implicitly contained in the formulation $\lambda \in H(\partial_t u + E)$ (see G. Vallet [11] and G. Gagneux *et al.* [6] for information about that).

For the boundary conditions: on the boundary Γ_e, one assumes $\overrightarrow{q}.\vec{n}=0$; on the boundary Γ_s, one assumes the unilateral condition: $\overrightarrow{q}.n \in \beta(\partial_t u)$, where β is the real graph of a maximal monotone operator such that (cf. J.-L. Lions [10, p. 422])

$$0 \in \beta(0), \quad \forall x,y \in \mathbb{R},\ \forall(\xi,\theta) \in \beta(x)\times\beta(y),\ (\xi-\theta)(x-y) \geq 0 \quad \text{(monotone)},$$

(in particular, for all $x>0$, $\beta(x) \subset [0,+\infty[$ and $\beta(-x) \subset]-\infty,0]$), that there exists C_1,C_2 in $\mathbb{R}$ such that (growth control),

$$\forall x \in \mathbb{R},\ \forall\theta \in \beta(x),\ C_1\theta^2 \leq x\theta + C_2, \tag{1}$$

there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathscr{C}^1(\mathbb{R})$, compatible with the growth control, such that

$$f_k(0)=0,\ f_k'>0,\ f_k(\pm\infty)=\pm\infty \quad \text{and} \quad f_k^{-1} \to \beta^{-1} \text{ uniformly on any compact set.}$$

Therefore, the mathematical modelling has to express respectively:

$$\text{the mass balance of the sediment}: \quad \partial_t u - \operatorname{div}(\lambda\nabla[u+\tau\partial_t u]) = 0 \text{ in } Q, \tag{2}$$

$$\text{the boundary conditions on } \partial\Omega = \overline{\Gamma}_e \cup \overline{\Gamma}_s: \quad \begin{cases} \overrightarrow{q}.n = 0 \text{ on }]0,T[\times\Gamma_e, \\ \overrightarrow{q}.n \in \beta(\partial_t u) \text{ on }]0,T[\times\Gamma_s, \end{cases} \tag{3}$$

$$\text{the global unilateral constraint}: \quad \lambda \in H(\partial_t u + E) \text{ in } Q, \tag{4}$$

$$\text{the initial condition}: \quad u|_{t=0} = u_0, \tag{5}$$

where one assumes: $\tau>0$, $u_0 \in H^1(\Omega)$ and $E \in C([0,T];\mathbb{R}^+)$.

This general problem remains an open problem and the aim of this paper is to give some mathematical results when H is replaced by a continuous function a, an approximation Yosida of H for example.

Thus, the pseudoparabolic problem may be written: find u *a priori* in $H^1(0,T;H^1(\Omega))$ such that, for any v in $H^1(\Omega)$ and for a.e. t in $]0,T[$,

$$\int_\Omega \{\partial_t uv + a(\partial_t u+E)\nabla[u+\tau\partial_t u]\nabla v\}\,dx + \int_{\Gamma_s} \beta(\partial_t u)v\,d\sigma \ni 0, \tag{6}$$

with the initial condition $u|_{t=0}=u_0$ a.e. in Ω.

In the sequel a is a continuous function defined on $\mathbb{R}$ that satisfies

$$0 \leq a \leq M, \quad a(x)=0 \text{ if } x \leq 0 \text{ and } a(x)>0 \text{ if } x>0,$$

in order to have implicitly $\partial_t u + E \geq 0$ a.e. in Q and $A(x) = \int_0^x a(s)\,ds$.

Moreover, and for technical reasons one assumes that $a \circ A^{-1}$ is Hölder continuous with exponent $\frac{1}{2}$ in $\mathbb{R}^+$.

Remark 1. Note that

i) the suitable test function $v = -(\partial_t u + E)^-$ leads to:

$$\int_\Omega (\partial_t u+E)^{-2}\,dx - \int_{\Gamma_s} \theta(\partial_t u+E)^-\,d\sigma = -\int_\Omega (\partial_t u+E)^- E\,dx \leq 0 \quad \text{for a.e. } t \text{ in }]0,T[,$$

where $\theta \in \beta(\partial_t u)$. Since $-\theta(\partial_t u+E)^- \geq 0$, one gets the moving obstacle constraint $\partial_t u + E \geq 0$ a.e. in Q.

ii) Since $u_0 \in H^1(\Omega)$, u belongs to $L^\infty(0,T,H^1(\Omega))$ and $\partial_t u$ belongs to $L^\infty(0,T,H^1(\Omega))$. Indeed, by using the test function $v_\varepsilon = \int_0^{\partial_t u} \frac{ds}{a(s+E)+\varepsilon}$ with $\varepsilon > 0$ and passing to limits with ε towards 0^+ lead to

$$\frac{1}{M}\,\|\partial_t u\|^2_{L^2(\Omega)} + \int_\Omega \nabla[u+\tau\partial_t u]\nabla\partial_t u\,dx \leq 0. \tag{7}$$

Thus, on the one hand, for a.e. t in $]0,T[$,

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + \sqrt{t}\,\|\partial_t u\|_{L^2(0,t,L^2(\Omega))};$$

on the other hand,

$$\frac{1}{M}\,\|\partial_t u\|^2_{L^2(\Omega)} + \frac{\tau}{2}\,\|\nabla\partial_t u\|^2_{L^2(\Omega)^N} \leq C\,\|\nabla u_0\|^2_{L^2(\Omega)^N},$$

and one concludes.

§2. Existence of a strong solution

2.1. A uniqueness lemma

The key for understanding compactness properties in the regularizing procedure is the following assertion:

Lemma 1. *Let us consider κ in $H^1(\Omega)$, E a real number, β a monotone graph (not necessary maximal) and b an essentially bounded nonnegative continuous function such that*

$$\forall x,y \in \mathbb{R},\ |b(x)-b(y)| \leq c|B(x)-B(y)|^{1/2}, \quad \text{where } B(x) = \int_0^x b(s)\,ds.$$

Then, there exists at most one solution w in $H^1(\Omega)$ such that, for any v in $H^1(\Omega)$,

$$0 \in \int_\Omega \big\{wv + b(w+E)\nabla[\kappa+w]\nabla v\big\}\,dx + \int_{\Gamma_s} \beta(w)v\,d\sigma.$$

Proof. One proves this result by using a usual L^1 argument with the following approximation of the positive part $p_\mu : t \mapsto p_\mu(t) = \min(1, \ln(te/\mu)^+)$, $\mu > 0$. □

2.2. The univoque case

Let us assume in this section that $\beta \in \mathscr{C}^1(\mathbb{R})$ with $\beta(0) = 0$ and $\beta' > 0$.

2.2.1. Semi-discretized non degenerated processes

Let us consider $h > 0$, u_0 in $H^1(\Omega)$, $E \geq 0$ and, for a given positive α, $a_\alpha = \max(\alpha, a)$.

Proposition 2. *There exists a unique u_α in $H^1(\Omega)$ such that, for all $v \in H^1(\Omega)$,*

$$\int_\Omega \Big\{\frac{u_\alpha - u_0}{h}v + a_\alpha\Big(\frac{u_\alpha-u_0}{h}+E\Big)\nabla\Big[u_\alpha + \tau\frac{u_\alpha-u_0}{h}\Big]\nabla v\Big\}\,dx + \int_{\Gamma_s}\beta\Big(\frac{u_\alpha-u_0}{h}\Big)v\,d\sigma = 0.$$

Proof. The proof is classical. One uses the fixed point theorem of Schauder-Tychonov for the existence; then Lemma 1 for the uniqueness. □

2.2.2. Semi-discretized degenerated problem

Let us consider $h > 0$, u_0 in $H^1(\Omega)$, $E \geq 0$ and in the sequel, $a(x) = 0$ for any nonnegative real x.

Proposition 3. *There exists a unique u in $H^1(\Omega)$ such that, for all $v \in H^1(\Omega)$,*

$$0 = \int_\Omega \Big\{ \frac{u-u_0}{h} v + a\Big(\frac{u-u_0}{h} + E\Big) \nabla \Big[u + \tau \frac{u-u_0}{h} \Big] \nabla v \Big\}\, dx + \int_{\Gamma_s} \beta\Big(\frac{u-u_0}{h}\Big) v\, d\sigma.$$

Moreover, $(u-u_0)/h + E \geq 0$ a.e. in Ω.

Proof. Note that the last part of the inequality is obvious by using $v = -((u-u_0)/h+E)^-$. By considering $v = \int_0^{(u-u_0)/h} (a_\alpha(s+E))^{-1}\, ds$ in the equation of proposition (2), one gets

$$\begin{aligned} \frac{C_2}{M} \operatorname{meas}(\Gamma_s) \geq \frac{1}{M} \Big\| \frac{u_\alpha - u_0}{h} \Big\|^2_{L^2(\Omega)} + \tau \Big\| \frac{u_\alpha - u_0}{h} \Big\|^2_{H^1(\Omega)} + \frac{C_1}{M} \Big\| \beta\Big(\frac{u_\alpha - u_0}{h}\Big) \Big\|^2_{L^2(\Gamma_s)} \\ + \frac{1}{2h} \big[\|u_\alpha\|^2_{H^1(\Omega)} + \|u_\alpha - u_0\|^2_{H^1(\Omega)} - \|u_0\|^2_{H^1(\Omega)} \big]. \end{aligned} \tag{8}$$

Therefore, $((u_\alpha - u_0)/h)_\alpha$ and $(u_\alpha)_\alpha$ are bounded sequences in $H^1(\Omega)$, and passing to limits is possible since, up to a sub-sequence, $(u_\alpha - u_0)/h$ converges a.e. in Ω and Γ. At last, the uniqueness can be proved by using Lemma 1 with $b = (h+\tau)a$, $\kappa = u_0/(h+\tau)$ and the graph $x \mapsto \{\beta(x)\}$. □

2.2.3. Existence of a solution

Inductively, the following result can be proved: let us consider $N \in \mathbb{N}^*$ with $h = T/N$, u_0 in $H^1(\Omega)$ and $E^k \geq 0$ for any integer k.

Proposition 4. *For $u^0 = u_0$, there exists a unique sequence $(u^k)_k$ in $H^1(\Omega)$ such that, for all $v \in H^1(\Omega)$,*

$$\begin{aligned} 0 = \int_\Omega \Big\{ \frac{u^{k+1}-u^k}{h} v + a\Big(\frac{u^{k+1}-u^k}{h} + E^k\Big) \nabla \Big[u^{k+1} + \tau \frac{u^{k+1}-u^k}{h} \Big] \nabla v \Big\}\, dx \\ + \int_{\Gamma_s} \beta\Big(\frac{u^{k+1}-u^k}{h}\Big) v\, d\sigma. \end{aligned}$$

Moreover, $\big(u^{k+1}-u^k\big)/h + E^k \geq 0$ a.e. in Ω, and

$$\begin{aligned} 0 \geq \frac{1}{M} \Big\| \frac{u^{k+1}-u^k}{h} \Big\|^2_{L^2(\Omega)} + \tau \Big\| \frac{u^{k+1}-u^k}{h} \Big\|^2_{H^1(\Omega)} \\ + \frac{1}{2h} \Big[\big\|u^{k+1}\big\|^2_{H^1(\Omega)} + \big\|u^{k+1}-u^k\big\|^2_{H^1(\Omega)} \big\|u^k\big\|^2_{H^1(\Omega)} \Big]. \end{aligned} \tag{9}$$

A priori estimations leading to the main result. In order to prove this result, for any sequence $(v_k)_k \subset L^2(\Omega)$, let us note in the sequel

$$v^h = \sum_{k=0}^{N-1} v^{k+1} 1_{[kh,(k+1)h[} \quad \text{and} \quad \tilde{v}^h = \sum_{k=0}^{N-1} \Big[\frac{v^{k+1}-v^k}{h}(t-kh) + v^k \Big] 1_{[kh,(k+1)h[}.$$

Lemma 5. *$(u^h)_h$ and $(\widetilde{u}^h)_h$ are bounded sequences in $L^\infty(0,T,H^1(\Omega))$ and $\partial_t\widetilde{u}^h$ is a bounded sequence in $L^\infty(0,T,H^1(\Omega))$ and then in $L^\infty(0,T,L^2(\Gamma_s))$.*

Proof. This result comes from (9) for the first part and since (9) implies that

$$\frac{1}{M}\Big\|\frac{u^{k+1}-u^k}{h}\Big\|^2_{L^2(\Omega)}+\tau\Big\|\frac{u^{k+1}-u^k}{h}\Big\|^2_{H^1(\Omega)}\leq\frac{1}{2\tau}\|u^{k+1}\|^2_{H^1(\Omega)}. \quad \square$$

Existence of a solution. Following G. Gagneux and G. Vallet [8], our aim is to prove the following result:

Proposition 6. *There exists u in $H^1(Q)$ with $\partial_t u$ in $L^2(0,T,H^1(\Omega))$ such that for any v in $H^1(\Omega)$ and for a.e. t in $]0,T[$,*

$$\int_\Omega\{\partial_t uv+a(\partial_t u+E)\nabla[u+\tau\partial_t u]\nabla v\}\,dx+\int_{\Gamma_s}\beta(\partial_t u)v\,d\sigma=0.$$

Moreover, u belongs to $W^{1,\infty}(0,T,H^1(\Omega))$.

Proof. Since $(\widetilde{u}^h)_h$ is bounded in $H^1(0,T,H^1(\Omega))$, there exists a sub-sequence, still indexed by h, such that for any t, $\widetilde{u}^h(t)\rightharpoonup u(t)$ in $H^1(\Omega)$. Moreover, if $t\in[kh,(k+1)h[$,

$$\big\|u^h(t)-\widetilde{u}^h(t)\big\|_{H^1(\Omega)}=\big\|\widetilde{u}^h(kh)-\widetilde{u}^h(t)\big\|_{H^1(\Omega)}\leq\int_{kh}^{(k+1)h}\big\|\partial_t\widetilde{u}^h(s)\big\|_{H^1(\Omega)}\,ds\leq Ch.$$

Then, for any t, $u^h(t)\rightharpoonup u(t)$ in $H^1(\Omega)$.

Since $\partial_t\widetilde{u}^h$ is bounded in $L^\infty(0,T,H^1(\Omega))$, it follows that for any t of Z where $Z\subset[0,T]$ is a measurable set such that $\mathscr{L}([0,T]\backslash Z)=0$, $\partial_t\widetilde{u}^h(t)$ is a bounded sequence in $H^1(\Omega)$. Therefore, up to a sub-sequence indexed by h_t, $\partial_t\widetilde{u}^{h_t}\rightharpoonup\xi(t)$ in $H^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω with $\xi(t)+E(t)\geq 0$ a.e. in Ω, strongly in $L^2(\Gamma)$ and a.e. in Γ by using the compactness of the trace operator from $H^1(\Omega)$ into $L^2(\Gamma)$.

Let us note that, for any t in $[kh,(k+1)h[$ and for all $v\in H^1(\Omega)$,

$$0=\int_\Omega\{\partial_t\widetilde{u}^h v+a(\partial_t\widetilde{u}^h+E^h)\nabla[u^h+\tau\partial_t\widetilde{u}^h]\nabla v\}\,dx+\int_{\Gamma_s}\beta(\partial_t\widetilde{u}^h)v\,d\sigma. \tag{10}$$

Given that, $a(\partial_t\widetilde{u}^{h_t}+E)\nabla v$ converges towards $a(\xi(t)+E)\nabla v$ in $L^2(\Omega)^N$ and that $\nabla[u^{h_t}+\varepsilon\partial_t\widetilde{u}^{h_t}]$ converges weakly towards $\nabla[u(t)+\varepsilon\xi(t)]$ in $L^2(\Omega)^N$, and thanks to the hypothesis on β, $\xi(t)$ is a solution to the problem: at time t, find w in $H^1(\Omega)$ with $w+E(t)\geq 0$ a.e. in Ω, such that for any v in $H^1(\Omega)$,

$$\int_\Omega\{wv+a(w+E(t))\nabla[u(t)+\tau w]\nabla v\}\,dx+\int_{\Gamma_s}\beta(w)v\,d\sigma=0. \tag{11}$$

Thanks to Lemma 1 with $b=\tau a$ in $\mathbb{R}^+$, $\kappa=u(t)/\tau$ and the monotone graph $x\mapsto\{\beta(x)\}$, the solution $\xi(t)$ is unique and all the sequence $\partial_t\widetilde{u}^h(t)$ converges towards $\xi(t)$ weakly in $H^1(\Omega)$. For any f in $H^{-1}(\Omega)$, since $t\mapsto\langle f,\xi(t)\rangle$ is the limit of the sequence of measurable functions

$t \mapsto \langle f, \partial_t \widetilde{u}^h(t)\rangle$, it is a measurable function thanks to Pettis Theorem (K. Yosida [13] p. 131), since $H^1(\Omega)$ is a separable set.

For any v in $L^2(0,T,H^1(\Omega))$, $(\partial_t \widetilde{u}^h(t), v(t))$ converges a.e. in $]0,T[$ towards $(\xi(t), v(t))$. Moreover, $|(\partial_t \widetilde{u}^h(t), v(t))| \leq C\|v(t)\|_{H^1(\Omega)}$ a.e. since the sequence $(\partial_t \widetilde{u}^h)_h$ is bounded in $L^\infty(0,T,H^1(\Omega))$. Thus, the weak convergence in $L^2(0,T,H^1(\Omega))$ of $\partial_t \widetilde{u}^h$ towards ξ can be proved. And one gets that $\xi = \partial_t u$.

At last, for t a.e. in $]0,T[$, passing to limits in (10) leads, for any v in $L^2(0,T,H^1(\Omega))$, to

$$\int_Q \{\partial_t u v + a(\partial_t u + E)\nabla[u + \varepsilon \partial_t u]\nabla v\}\, dxdt + \int_{]0,T[\times\Gamma_s} \beta(\partial_t u) v\, d\sigma dt = 0 \qquad (12)$$

and to the existence of a solution. □

2.3. The multivoque case

Let us prove in this section that if β is the maximal monotone graph presented in the introduction, then

Proposition 7. *There exists u in $H^1(Q)$ with $\partial_t u$ in $L^2(0,T,H^1(\Omega))$ such that for any v in $H^1(\Omega)$ and for a.e. t in $]0,T[$,*

$$0 \in \int_\Omega \{\partial_t u v + a(\partial_t u + E)\nabla[u + \tau \partial_t u]\nabla v\}\, dx + \int_{\Gamma_s} \beta(\partial_t u) v\, d\sigma.$$

Moreover, u belongs to $W^{1,\infty}(0,T,H^1(\Omega))$.

Proof. Following J.-L. Lions idea in [10] p.422, one denotes by $\beta_n = \max(-n, \min(f_n, n))$. Associated with this sequence, one has a sequence (u_n) of solutions to (12).
Of course, this sequence is bounded in $H^1(0,T;H^1(\Omega))$ and the same kind of demonstration can be done in order to prove the existence of a solution to (6).
Let us considering the notations of the previous section. Then, for any t in Z and for the sub-sequence indexed by n_t such that $\partial_t u_{n_t} \rightharpoonup \xi(t)$ in $H^1(\Omega)$, strongly in $L^2(\Gamma)$ and a.e. in Γ, one has for any v in $H^1(\Omega)$,

$$\int_\Omega \{\partial_t u_{n_t} v + a(\partial_t u_{n_t} + E)\nabla[u_{n_t} + \tau \partial_t u_{n_t}]\nabla v\}\, dx + \int_{\Gamma_s} \beta_{n_t}(\partial_t u_{n_t}) v\, d\sigma = 0.$$

Therefore, one gets that $\beta_{n_t}(\partial_t u_{n_t})$ converges weakly in $L^2(\Gamma_s)$ towards an element denoted by ϕ. Therefore, the same kind of demonstration than the one given p. 424 of J.-L. Lions [10] leads to $\phi = \beta^{-1}(\xi(t))$ and proves that $\xi(t)$ is a solution to the problem: at time t, find w in $H^1(\Omega)$ with $w + E(t) \geq 0$ a.e. in Ω, such that for any v in $H^1(\Omega)$,

$$\int_\Omega \{wv + a(w + E(t))\nabla[u(t) + \tau w]\nabla v\}\, dx + \int_{\Gamma_s} \beta(w) v\, d\sigma \ni 0. \qquad (13)$$

Then, one gets a result of existence of a solution in the same way, according to the proposition 6. □

2.4. Conclusion and open problems

A solution to the problem

$$\begin{cases} \partial_t u - \tau \Delta A(\partial_t u + E) - \operatorname{div}(a(\partial_t u + E)\nabla u) = 0 \quad \text{in} \quad Q, \\ \partial_t u + E \geq 0 \quad \text{in} \quad Q, \\ -\tau \partial_n A(\partial_t u + E) - a(\partial_t u + E)\partial_n u = 0 \quad \text{in} \quad]0,T[\times\Gamma_e, \\ -\tau \partial_n A(\partial_t u + E) - a(\partial_t u + E)\partial_n u \in \beta(\partial_t u) \quad \text{in} \quad]0,T[\times\Gamma_s, \\ u|_{t=0} = u_0 \quad \text{in} \quad \Omega, \end{cases}$$

has been found in $\mathrm{Lip}([0,T],H^1(\Omega))$.

In order to conclude that this problem is well-posed in the sense of Hadamard, one still has to prove that such a solution is unique. This is still an open problem, mainly due to a behaviour of hysteresis type of the equation.

Let us cite a recent paper of Z. Wang *et al.* [12] where the uniqueness of the solution to a similar equation has been proved. The equation is posed in the one-dimension space case, and the method is based on a Holmgren approach.

The above solution is a solution to a perturbation of the real problem since a has to be the graph of the Heaviside function in order to satisfy the condition (4). This problem is open too.

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Stanislav N. Antontsev
Departamento de Matemática,
Universidade da Beira Interior
Covilhã 6201-001 (Portugal)
`anton@ubi.pt.`

Gérard Gagneux
Laboratoire de Mathématiques Appliquées
CNRS UMR 5142 - Université de Pau et des Pays de l'Adour
IPRA, B.P. 1155, 64013 Pau Cedex, France
`gerard.gagneux@univ-pau.fr`

Guy Vallet
Laboratoire de Mathématiques Appliquées
CNRS UMR 5142 - Université de Pau et des Pays de l'Adour
IPRA, B.P. 1155, 64013 Pau Cedex, France
`guy.vallet@univ-pau.fr`