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BIHARMONIC PROBLEM IN THE HALF-SPACE

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Abstract. In this paper, we study the biharmonic equation in the half-space $\mathbb{R}^N_+$, with $N \geq 2$. We prove in L^p theory, with $1 < p < \infty$, existence and uniqueness results. We consider data and give solutions which live in weighted Sobolev spaces.

Keywords: Biharmonic problem, half-space, weighted Sobolev spaces.

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§1. Introduction and functional framework

The purpose of this paper is the resolution of the biharmonic problem with nonhomogeneous boundary conditions

$$(P)\quad \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma = \mathbb{R}^{N-1}, \\ \partial_N u = g_1 & \text{on } \Gamma. \end{cases}$$

Since this problem is posed in the half-space, it is important to specify the behaviour at infinity for the data and solutions. We have chosen to impose such conditions by setting our problem in weighted Sobolev spaces, where the growth or decay of functions at infinity are expressed by means of weights. These weighted Sobolev spaces provide a correct functional setting for unbounded domains, in particular because the functions in these spaces satisfy an optimal weighted Poincaré-type inequality. Our analysis is based on the isomorphism properties of the biharmonic operator in the whole space and the resolution of the Dirichlet and Neumann problems for the Laplacian in the half-space. This last one is itself based on the isomorphism properties of the Laplace operator in the whole space and also on the reflection principle inherent in the half-space. Note here the double difficulty arising from the unboundedness of the domain in any direction and from the unboundedness of the boundary itself.

Problem (P) has been investigated by Boulmezaoud (cf. [3]) in weighted Sobolev spaces in L^2 theory for $N \geq 3$ and without the critical cases corresponding to logarithmic factors. The aim of this work is to give results in L^p theory, with $1 < p < \infty$, to reduce critical values and especially to reach weaker solutions from more singular data.

In the sequel, for any integer q, we shall use the following polynomial spaces:

- $\mathscr{P}_q$ is the space of polynomials of degree smaller than or equal to q;
- $\mathscr{P}_q^{\Delta}$ is the subspace of harmonic polynomials of $\mathscr{P}_q$;
- $\mathscr{P}_q^{\Delta^2}$ is the subspace of biharmonic polynomials of $\mathscr{P}_q$;

- $\mathscr{A}_q^\Delta$ is the subspace of polynomials of $\mathscr{P}_q^\Delta$, odd with respect to x_N, or equivalently, which satisfy the condition $\varphi(x',0)=0$;
- $\mathscr{N}_q^\Delta$ is the subspace of polynomials of $\mathscr{P}_q^\Delta$, even with respect to x_N, or equivalently, which satisfy the condition $\partial_N\varphi(x',0)=0$;

with the convention that these spaces are reduced to $\{0\}$ if $q<0$.

Let Ω be an open set of $\mathbb{R}^N$ with $N\geq 2$, $\rho=(1+|x|^2)^{1/2}$ and $\lg\rho=\ln(2+|x|^2)$.

For any $m\in\mathbb{N}$, $p\in\,]1,\infty[$, $(\alpha,\beta)\in\mathbb{R}^2$, we define the following space:

$$W^{m,p}_{\alpha,\beta}(\Omega)=\Big\{u\in\mathscr{D}'(\Omega)\,;\;0\leq|\lambda|\leq k,\;\rho^{\alpha-m+|\lambda|}(\lg\rho)^{\beta-1}\,\partial^\lambda u\in L^p(\Omega);\\ k+1\leq|\lambda|\leq m,\;\rho^{\alpha-m+|\lambda|}(\lg\rho)^{\beta}\,\partial^\lambda u\in L^p(\Omega)\Big\}, \tag{1}$$

where $k=m-N/p-\alpha$ if $N/p+\alpha\in\{1,\dots,m\}$, and $k=-1$ otherwise.

In the case $\beta=0$, we simply denote the space by $W^{m,p}_{\alpha}(\Omega)$. Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)}=\Big(\sum_{0\leq|\lambda|\leq k}\big\|\rho^{\alpha-m+|\lambda|}(\lg\rho)^{\beta-1}\,\partial^\lambda u\big\|^p_{L^p(\Omega)}\\ +\sum_{k+1\leq|\lambda|\leq m}\big\|\rho^{\alpha-m+|\lambda|}(\lg\rho)^{\beta}\,\partial^\lambda u\big\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

We also define the semi-norm:

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)}=\Big(\sum_{|\lambda|=m}\big\|\rho^{\alpha}(\lg\rho)^{\beta}\,\partial^\lambda u\big\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

The weights in the definition (1) are chosen so that the corresponding space satisfies two properties. On the one hand, $\mathscr{D}\big(\overline{\mathbb{R}^N_+}\big)$ is dense in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. On the other hand, the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ (cf. [1]):

$$\text{if}\quad \frac{N}{p}+\alpha\notin\{1,\dots,m\}\;\text{ or }\;(\beta-1)p\neq -1, \tag{2}$$

then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/\mathscr{P}_{q'}$ a norm which is equivalent to the quotient norm, with $q'=\inf(q,m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$.

Now, we define the space $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)=\overline{\mathscr{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}}$, whose dual space is denoted by $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$, where p' is the Hölder conjugate of p. Under the assumption (2), the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ which is equivalent to the full norm $\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$.

We shall now recall some properties of the weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings:

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)\hookrightarrow W^{m-1,p}_{\alpha-1,\beta}(\mathbb{R}^N_+)\hookrightarrow\cdots\hookrightarrow W^{0,p}_{\alpha-m,\beta}(\mathbb{R}^N_+)\quad\text{if}\;\;\frac{N}{p}+\alpha\notin\{1,\dots,m\}.$$

When $N/p+\alpha = j \in \{1,\dots,m\}$, then we have:

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \hookrightarrow \dots \hookrightarrow W^{m-j+1,p}_{\alpha-j+1,\beta}(\mathbb{R}^N_+) \hookrightarrow W^{m-j,p}_{\alpha-j,\beta-1}(\mathbb{R}^N_+) \hookrightarrow \dots \hookrightarrow W^{0,p}_{\alpha-m,\beta-1}(\mathbb{R}^N_+).$$

Note that in the first case, for any $\gamma \in \mathbb{R}$ such that $N/p+\alpha-\gamma \notin \{1,\dots,m\}$ and $m \in \mathbb{N}$, the mapping $u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \longmapsto \rho^\gamma u \in W^{m,p}_{\alpha-\gamma,\beta}(\mathbb{R}^N_+)$ is an isomorphism. In both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping $u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \longmapsto \partial^\lambda u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is continuous. Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} m - \left(\dfrac{N}{p}+\alpha\right) - 1, & \text{if } \begin{cases} \frac{N}{p}+\alpha \in \{1,\dots,m\} \text{ and } (\beta-1)p \geq -1 \text{ or} \\ \frac{N}{p}+\alpha \in \{j \in \mathbb{Z}\,;\, j \leq 0\} \text{ and } \beta p \geq -1, \end{cases} \\ \left[m - \left(\dfrac{N}{p}+\alpha\right)\right], & \text{otherwise,} \end{cases} \tag{3}$$

where $[s]$ denotes the integer part of s.

In order to define the traces of functions of $W^{m,p}_\alpha(\mathbb{R}^N_+)$ (here we don't consider the case $\beta \neq 0$), for any $\sigma \in]0,1[$, we introduce the space:

$$\begin{aligned} W^{\sigma,p}_0(\mathbb{R}^N) = \Big\{ u \in \mathscr{D}'(\mathbb{R}^N)\,;\, w^{-\sigma} u \in L^p(\mathbb{R}^N) \;\text{ and }\; \forall i = 1,\dots,N, \\ \int_0^{+\infty} t^{-1-\sigma p}\, dt \int_{\mathbb{R}^N} |u(x+te_i) - u(x)|^p\, dx < \infty \Big\}, \end{aligned} \tag{4}$$

where $w = \rho$ if $N/p \neq \sigma$ and $w = \rho\,(\lg \rho)^{1/\sigma}$ if $N/p = \sigma$, and $e_1,\dots,e_N$ is the canonical basis of $\mathbb{R}^N$. It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{\sigma,p}_0(\mathbb{R}^N)} = \left(\left\| \frac{u}{w^\sigma} \right\|^p_{L^p(\mathbb{R}^N)} + \sum_{i=1}^N \int_0^{+\infty} t^{-1-\sigma p}\, dt \int_{\mathbb{R}^N} |u(x+te_i) - u(x)|^p\, dx \right)^{1/p},$$

which is equivalent to the norm

$$\left(\left\| \frac{u}{w^\sigma} \right\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\sigma p}}\, dx\, dy \right)^{1/p}.$$

Similarly, for any real number $\alpha \in \mathbb{R}$, we define the space:

$$\begin{aligned} W^{\sigma,p}_\alpha(\mathbb{R}^N) = \Big\{ u \in \mathscr{D}'(\mathbb{R}^N)\,;\, w^{\alpha-\sigma} u \in L^p(\mathbb{R}^N), \\ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\rho^\alpha(x)\,u(x) - \rho^\alpha(y)\,u(y)|^p}{|x-y|^{N+\sigma p}}\, dx\, dy < \infty \Big\}, \end{aligned}$$

where $w = \rho$ if $N/p+\alpha \neq \sigma$ and $w = \rho\,(\lg \rho)^{1/(\sigma-\alpha)}$ if $N/p+\alpha = \sigma$. For any $s \in \mathbb{R}^+$, we set

$$\begin{aligned} W^{s,p}_\alpha(\mathbb{R}^N) = \Big\{ u \in \mathscr{D}'(\mathbb{R}^N)\,;\, 0 \leq |\lambda| \leq k,\; \rho^{\alpha-s+|\lambda|}\,(\lg \rho)^{-1}\,\partial^\lambda u \in L^p(\mathbb{R}^N); \\ k+1 \leq |\lambda| \leq [s]-1,\; \rho^{\alpha-s+|\lambda|}\,\partial^\lambda u \in L^p(\mathbb{R}^N);\; |\lambda| = [s],\; \partial^\lambda u \in W^{\sigma,p}_\alpha(\mathbb{R}^N) \Big\}, \end{aligned}$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \dots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. It is a reflexive Banach space equipped with the norm:

$$\|u\|_{W^{s,p}_{\alpha}(\mathbb{R}^N)} = \Big(\sum_{0 \le |\lambda| \le k} \big\|\rho^{\alpha - s + |\lambda|} \, (\lg \rho)^{-1} \, \partial^{\lambda} u\big\|^p_{L^p(\mathbb{R}^N)} + \sum_{k+1 \le |\lambda| \le [s]-1} \big\|\rho^{\alpha - s + |\lambda|} \, \partial^{\lambda} u\big\|^p_{L^p(\mathbb{R}^N)} \Big)^{1/p} + \sum_{|\lambda| = [s]} \|\partial^{\lambda} u\|_{W^{\sigma,p}_{\alpha}(\mathbb{R}^N)}.$$

We can similarly define, for any real number β, the space:

$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^N) = \Big\{ v \in \mathscr{D}'(\mathbb{R}^N) \, ; \, (\lg \rho)^{\beta} \, v \in W^{s,p}_{\alpha}(\mathbb{R}^N) \Big\}.$$

We can prove some properties of the weighted Sobolev spaces $W^{s,p}_{\alpha,\beta}(\mathbb{R}^N)$. We have the algebraic and topological imbeddings in the case where $N/p + \alpha \notin \{\sigma, \dots, \sigma + [s]\}$:

$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^N) \hookrightarrow W^{s-1,p}_{\alpha-1,\beta}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta}(\mathbb{R}^N),$$
$$W^{s,p}_{\alpha,\beta}(\mathbb{R}^N) \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta}(\mathbb{R}^N) \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-s,\beta}(\mathbb{R}^N).$$

When $N/p + \alpha = j \in \{\sigma, \dots, \sigma + [s]\}$, then we have:

$$W^{s,p}_{\alpha,\beta} \hookrightarrow \cdots \hookrightarrow W^{s-j+1,p}_{\alpha-j+1,\beta} \hookrightarrow W^{s-j,p}_{\alpha-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{\sigma,p}_{\alpha-[s],\beta-1},$$
$$W^{s,p}_{\alpha,\beta} \hookrightarrow W^{[s],p}_{\alpha+[s]-s,\beta} \hookrightarrow \cdots \hookrightarrow W^{[s]-j+1,p}_{\alpha-\sigma-j+1,\beta} \hookrightarrow W^{[s]-j,p}_{\alpha-\sigma-j,\beta-1} \hookrightarrow \cdots \hookrightarrow W^{0,p}_{\alpha-s,\beta-1}.$$

If u is a function on $\mathbb{R}^N_+$, we denote its trace of order j on the hyperplane Γ by:

$$\forall j \in \mathbb{N}, \quad \gamma_j u : x' \in \mathbb{R}^{N-1} \longmapsto \partial^j_N u(x', 0).$$

Let's recall the following trace lemma due to Hanouzet (cf. [5]) and extended by Amrouche-Nečasová (cf. [1]) to this class of weighted Sobolev spaces:

Lemma 1. *For any integer $m \ge 1$ and real number α, the mapping*

$$\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{m-1}) : \; \mathscr{D}\big(\overline{\mathbb{R}^N_+}\big) \longrightarrow \prod_{j=0}^{m-1} \mathscr{D}(\mathbb{R}^{N-1}),$$

can be extended to a linear continuous mapping, still denoted by γ,

$$\gamma : \; W^{m,p}_{\alpha}(\mathbb{R}^N_+) \longrightarrow \prod_{j=0}^{m-1} W^{m-j-1/p,p}_{\alpha}(\mathbb{R}^{N-1}).$$

Moreover γ is surjective and $\ker \gamma = \overset{\circ}{W}{}^{m,p}_{\alpha}(\mathbb{R}^N_+)$.

§2. On the Laplace equation in $\mathbb{R}^N_+$

We shall now recall the fundamental results of the Laplace equation in the half-space, with nonhomogeneous Dirichlet or Neumann boundary conditions. Let's first recall the results of the Dirichlet problem

$$(P_D)\ \begin{cases} \Delta u = f & \text{in } \mathbb{R}^N_+, \\ u = g & \text{on } \Gamma. \end{cases}$$

Theorem 2 (Amrouche-Nečasová). *Let $\ell \in \mathbb{Z}$ and assume that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell\}. \tag{5}$$

For any $f \in W^{-1,p}_{\ell}(\mathbb{R}^N_+)$ and $g \in W^{1-1/p,p}_{\ell}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in \mathscr{A}^{\Delta}_{[1+\ell-N/p']},\ \langle f, \varphi \rangle_{W^{-1,p}_{\ell}(\mathbb{R}^N_+)\times \mathring{W}^{1,p'}_{-\ell}(\mathbb{R}^N_+)} = \langle g, \partial_N \varphi \rangle_{W^{1/p',p}_{\ell}(\Gamma)\times W^{-1/p',p'}_{-\ell}(\Gamma)}, \tag{6}$$

problem (P_D) has a solution $u \in W^{1,p}_{\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{A}^{\Delta}_{[1-\ell-N/p]}$.

Theorem 3 (Amrouche-Nečasová). *Let $\ell \in \mathbb{Z}$ and $m \geq 1$ be two integers and assume that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell+1\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell-m\}. \tag{7}$$

For any $f \in W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)$ and $g \in W^{m+1-1/p,p}_{m+\ell}(\Gamma)$, satifying the compatibility condition (6), problem (P_D) has a solution $u \in W^{m+1,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{A}^{\Delta}_{[1-\ell-N/p]}$.

Concerning the Neumann problem

$$(P_N)\ \begin{cases} \Delta u = f & \text{in } \mathbb{R}^N_+, \\ \partial_N u = g & \text{on } \Gamma, \end{cases}$$

we can give the following result

Theorem 4 (Amrouche). *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$ and assume that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \ldots, -\ell-m\}. \tag{8}$$

For any $f \in W^{m,p}_{m+\ell}(\mathbb{R}^N_+)$ and $g \in W^{m+1-1/p,p}_{m+\ell}(\Gamma)$ satifying the compatibility condition

$$\forall \varphi \in \mathscr{N}^{\Delta}_{[\ell-N/p']},\ \langle f, \varphi \rangle_{W^{0,p}_{\ell}(\mathbb{R}^N_+)\times W^{0,p'}_{-\ell}(\mathbb{R}^N_+)} + \langle g, \varphi \rangle_{W^{1-1/p,p}_{\ell}(\Gamma)\times W^{-1/p',p'}_{-\ell}(\Gamma)} = 0, \tag{9}$$

problem (P_N) has a solution $u \in W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{N}^{\Delta}_{[2-\ell-N/p]}$.

Remark 1. Note that for these three theorems, the solutions continuously depend on the data with respect to the quotient norm.

§3. Generalized solutions of Δ^2 in $\mathbb{R}^N_+$

Firstly, we establish a global result for the biharmonic operator in the whole space:

Theorem 5. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$ and assume that*

$$\frac{N}{p'} \notin \{1, \dots, \ell + \min\{m, 2\}\} \quad \textit{and} \quad \frac{N}{p} \notin \{1, \dots, -\ell - m\}, \tag{10}$$

then the biharmonic operator

$$\Delta^2 : W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N)/\mathscr{P}^{\Delta^2}_{[2-\ell-N/p]} \longrightarrow W^{m-2,\,p}_{m+\ell}(\mathbb{R}^N) \perp \mathscr{P}^{\Delta^2}_{[2+\ell-N/p']}$$

is an isomorphism.

Secondly, we characterize the kernel $\mathscr{K}^m$ of the operator $(\Delta^2, \gamma_0, \gamma_1)$ in $W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)$. For any $q \in \mathbb{Z}$, we introduce the space $\mathscr{B}_q$ as a subspace of $\mathscr{P}^{\Delta^2}_q$:

$$\mathscr{B}_q = \left\{ u \in \mathscr{P}^{\Delta^2}_q \,;\, u = \partial_N u = 0 \text{ on } \Gamma \right\}.$$

Then we define the two operators Π_D and Π_N by:

$$\forall r \in \mathscr{A}^{\Delta}_k,\ \Pi_D r = \frac{1}{2}\int_0^{x_N} t\, r(x', t)\, dt \ \text{ and } \ \forall s \in \mathscr{N}^{\Delta}_k,\ \Pi_N s = \frac{1}{2} x_N \int_0^{x_N} s(x', t)\, dt.$$

Lemma 6. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$ and assume that $\dfrac{N}{p} \notin \{1, \dots, -\ell - m\}$, then*

$$\mathscr{K}^m = \mathscr{B}_{[2-\ell-N/p]} = \Pi_D \mathscr{A}^{\Delta}_{[-\ell-N/p]} \oplus \Pi_N \mathscr{N}^{\Delta}_{[-\ell-N/p]}. \tag{11}$$

Thirdly, we establish a global result for the homogeneous problem in the half-space:

$$(P^0) \ \begin{cases} \Delta^2 u = 0 & \text{in } \mathbb{R}^N_+, \\ u = g_0 & \text{on } \Gamma, \\ \partial_N u = g_1 & \text{on } \Gamma. \end{cases}$$

Lemma 7. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$. Under hypothesis (8), for any $g_0 \in W^{m+2-1/p,\,p}_{m+\ell}(\Gamma)$ and $g_1 \in W^{m+1-1/p,\,p}_{m+\ell}(\Gamma)$, satisfying the compatibility condition*

$$\forall \varphi \in \mathscr{B}_{[2+\ell-N/p']}\,, \quad \langle g_1, \Delta\varphi \rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi \rangle_\Gamma = 0, \tag{12}$$

problem (P^0) admits a solution $u \in W^{m+2,\,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-\ell-N/p]}$, and which continuously depends on the data with respect to the quotient norm.

Finaly, we can give the main result for the biharmonic operator in the half-space:

Theorem 8. *Let $\ell \in \mathbb{Z}$. Under hypothesis (5), for any $f \in W_\ell^{-2,p}(\mathbb{R}_+^N)$, $g_0 \in W_\ell^{2-1/p,p}(\Gamma)$ and $g_1 \in W_\ell^{1-1/p,p}(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathscr{B}_{[2+\ell-N/p']},\ \langle f, \varphi\rangle_{W_\ell^{-2,p}(\mathbb{R}_+^N)\times \mathring{W}_{-\ell}^{2,p'}(\mathbb{R}_+^N)} + \langle g_1, \Delta\varphi\rangle_\Gamma - \langle g_0, \partial_N \Delta\varphi\rangle_\Gamma = 0, \tag{13}$$

problem (P) admits a solution $u \in W_\ell^{2,p}(\mathbb{R}_+^N)$, unique up to an element of $\mathscr{B}_{[2-\ell-N/p]}$, and which continuously depends on the data with respect to the quotient norm.

Proof. We can readily check the necessity of condition (13), where $\langle g_1, \Delta\varphi\rangle_\Gamma$ denotes the duality bracket $\langle g_1, \Delta\varphi\rangle_{W_\ell^{1-1/p,p}(\Gamma)\times W_{-\ell}^{-1/p',p'}(\Gamma)}$, and $\langle g_0, \partial_N\Delta\varphi\rangle_\Gamma$ the duality bracket

$$\langle g_0, \partial_N\Delta\varphi\rangle_{W_\ell^{2-1/p,p}(\Gamma)\times W_{-\ell}^{-1-1/p',p'}(\Gamma)}.$$

Then, by Lemma 1, we can consider the lifted problem

$$(P^\star)\ \begin{cases} \Delta^2 u = f & \text{in } \mathbb{R}_+^N,\\ u = 0 & \text{on } \Gamma,\\ \partial_N u = 0 & \text{on } \Gamma,\end{cases}$$

where $f \in W_\ell^{-2,p}(\mathbb{R}_+^N)$ and $f \perp \mathscr{B}_{[2+\ell-N/p']}$, which corresponds to (13).

We shall give now a characterization of $W_\ell^{-2,p}(\mathbb{R}_+^N)$:

Lemma 9. *For any $f \in W_\ell^{-2,p}(\mathbb{R}_+^N)$, there exists $F = (F_{ij})_{1\le i,j\le N} \in W_\ell^{0,p}(\mathbb{R}_+^N)^{N^2}$ such that $f = \operatorname{div}\operatorname{div} F = \sum_{i,j=1}^N \partial_{ij}^2 F_{ij}$, with $\sum_{i,j=1}^N \|F_{ij}\|_{W_\ell^{0,p}(\mathbb{R}_+^N)} \le C\|f\|_{W_\ell^{-2,p}(\mathbb{R}_+^N)}$.*

Finally, we can give the outline of the proof of the existence:

Step 1. Assume that $2+\ell-N/p' < 0$. Let $f \in W_\ell^{-2,p}(\mathbb{R}_+^N)$. Then by Lemma 9, we can write $f = \partial_{ij}^2 F_{ij}$. Let $\tilde{F}_{ij}$ the extension of F_{ij} to $\mathbb{R}^N$ by 0 and $\tilde{f} = \partial_{ij}^2 \tilde{F}_{ij} \in W_\ell^{-2,p}(\mathbb{R}^N)$. By Theorem 5, there exists $\tilde{z} \in W_\ell^{2,p}(\mathbb{R}^N)$ such that $\tilde{f} = \Delta^2 \tilde{z}$ in $\mathbb{R}^N$, and writing $z = \tilde{z}|_{\mathbb{R}_+^N}$, we have $f = \Delta^2 z$ in $\mathbb{R}_+^N$, with $z \in W_\ell^{2,p}(\mathbb{R}_+^N)$, $z|_\Gamma \in W_\ell^{2-1/p,p}(\Gamma)$ and $\partial_N z|_\Gamma \in W_\ell^{1-1/p,p}(\Gamma)$. Since $\mathscr{B}_{[2+\ell-N/p']} = \{0\}$, the compatibility condition (12) vanishes in Lemma 7 which asserts the existence of a solution $v \in W_\ell^{2,p}(\mathbb{R}_+^N)$ to the homogeneous problem

$$\Delta^2 v = 0 \quad \text{in } \mathbb{R}_+^N, \qquad v = z \text{ and } \partial_N v = \partial_N z \quad \text{on } \Gamma.$$

The function $u = z - v$ answers to problem $(P^\star)$ in this case.

Step 2. Assume that $2-\ell-N/p < 0$. We have shown that if $2+\ell-N/p' < 0$, the operator

$$\Delta^2 : \mathring{W}_\ell^{2,p}(\mathbb{R}_+^N)/\mathscr{B}_{[2-\ell-N/p]} \longrightarrow W_\ell^{-2,p}(\mathbb{R}_+^N)$$

is an isomorphism. Thus, by duality we deduce if $2-\ell-N/p < 0$, the isomorphism

$$\Delta^2 : \mathring{W}_\ell^{2,p}(\mathbb{R}_+^N) \longrightarrow W_\ell^{-2,p}(\mathbb{R}_+^N) \perp \mathscr{B}_{[2+\ell-N/p']}.$$

Step 3. Assume that $2+\ell-N/p' \ge 0$ and $2-\ell-N/p \ge 0$, which implies $\ell \in \{-1,0,1\}$. We begin to establish a preliminary result:

Lemma 10. *Let $\ell \in \{-1, 0\}$ and assume that $N/p \neq 1$ if $\ell = -1$. For any $f \in W^{0,p}_{\ell}(\mathbb{R}^N_+)$, there exists $z \in W^{4,p}_{\ell}(\mathbb{R}^N_+)$, such that $\Delta^2 z = f$.*

Then we can prove in a similar fashion to Step 1 the isomorphism result for $\ell \in \{-1, 0\}$, and we deduce the case $\ell = 1$ by duality from $\ell = -1$.

It remains to combine the three steps to obtain the isomorphism

$$\Delta^2 : \overset{\circ}{W}{}^{2,p}_{\ell}(\mathbb{R}^N_+)/\mathscr{B}_{[2-\ell-N/p]} \longrightarrow W^{-2,p}_{\ell}(\mathbb{R}^N_+) \perp \mathscr{B}_{[2+\ell-N/p']},$$

for any $\ell \in \mathbb{Z}$ verifying (5). This answers globally to problem $(P^{\star})$ and thus to general problem (P). □

To extend Theorem 8, we also have establish a global result for different types of data.

Theorem 11. *Let $\ell \in \mathbb{Z}$ and $m \in \mathbb{N}$. Under hypothesis (10), for any $f \in W^{m-2,p}_{m+\ell}(\mathbb{R}^N_+)$, $g_0 \in W^{m+2-1/p,p}_{m+\ell}(\Gamma)$ and $g_1 \in W^{m+1-1/p,p}_{m+\ell}(\Gamma)$ satisfying the compatibility condition (13), problem (P) has a solution $u \in W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+)$, unique up to an element of $\mathscr{B}_{[2-\ell-N/p]}$, with the estimate*

$$\inf_{q \in \mathscr{B}_{[2-\ell-N/p]}} \|u + q\|_{W^{m+2,p}_{m+\ell}(\mathbb{R}^N_+)} \leq C \left(\|f\|_{W^{m-2,p}_{m+\ell}(\mathbb{R}^N_+)} + \|g_0\|_{W^{m+2-1/p,p}_{m+\ell}(\Gamma)} + \|g_1\|_{W^{m+1-1/p,p}_{m+\ell}(\Gamma)} \right).$$

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