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ON THE LIMIT OF SOME PENALIZED DEGENERATED PROBLEMS (L^1-DUAL) DATA

L. Aharouch, Y. Akdim and M. Rhoudaf

Abstract. This paper is concerned with the existence and uniqueness result of a solution for some degenerated bilateral problem by using the penalization methods.

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§1. Introduction

Let us consider an open bounded subset Ω of $\mathbb{R}^N$, $(N > 1)$ and a real number p such that $1 < p < \infty$. In [5] Dall'aglio and Orsina have considered the following sequence of equations,

$$\begin{cases} -\Delta_p(u_n) + |g(x,u_n)|^{n-1}g(x,u_n) = f \text{ in } \Omega, \\ u_n = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $g(x,s)$ is a Carathéodory function satisfying some suitable conditions and where Δ_p is the usual p-Laplacian Operators, that is $-\Delta_p(u) = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and have proved that the sequence of solutions of (1) converge to some element u which is exactly a solution of the following bilateral problem

$$\begin{cases} \langle -\Delta_p u, v-u\rangle \geq \langle f, v-u\rangle \quad \forall v \in K, \\ u \in K = \{v \in W_0^{1,p}(\Omega),\ q_- \leq v(x) \leq q_+ \ a.e. \text{ in } \Omega\}, \end{cases} \tag{2}$$

where $q_+(x) = \inf\{s > 0,\ g(x,s) \geq 1\}$ and $q_-(x) = \sup\{s < 0,\ g(x,s) \leq -1\}$, $f \in L^1(\Omega)$. Our purpose in this paper, is to study the existence and uniqueness of the following degenerated bilateral problem,

$$\begin{cases} u \in \mathcal{T}_0^{1,p}(\Omega,w),\ q_- \leq v(x) \leq q_+ \ a.e. \text{ in } \Omega, \\ \langle Au, T_k(u-v)\rangle \leq \langle \mu, T_k(u-v)\rangle \quad \forall v \in K, \end{cases} \tag{3}$$

where $\mathcal{T}_0^{1,p}(\Omega,w) = \{u \text{ measurable } : T_k(u) \in W_0^{1,p}(\Omega,w)\}$ and where $\mu = f - \operatorname{div} F$, $f \in L^1(\Omega)$, $F = \Pi_{i=1}^N L^p(\Omega,w^*)$. The family $w = \{w_i,\ 0 \leq i \leq N\}$ is a collection of weight functions defined on Ω which expressed the degeneracy of the Leray-Lions operators $Au =$

$-\operatorname{div}(a(x,\nabla u))$ and $w^* = \{w_i^{-\frac{1}{p-1}},\ 0 \le i \le N\}$. Note that T_k is the usual truncation operators. To this end, we have used a new penalization method. More precisely we approach our problem (3) by the following sequence of degenerated equations,

$$\begin{cases} Au_n + |g(x,u_n)|^{n-1} g(x,u_n)|G(x,u_n,\nabla u_n)| = f \text{ in } \Omega, \\ u_n \in W_0^{1,p}(\Omega,w), |g(x,u_n)|^n G(x,u_n,\nabla u_n) \in L^1(\Omega), \end{cases} \tag{4}$$

where $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $G : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ are two Carathédory functions, which satisfies some hypothesis (see below).

Our paper can be seen as a generalization of the previous work [5] and as a continuation of the work [1] where the non weighted case is treated in the first paper and some degenerated problem is studied in the second paper.

§2. Basic assumption and statement of result

Let Ω be a bounded open subset of $\mathbb{R}^N (N \ge 1)$. Let $1 < p < \infty$, and let $w = \{w_i(x); i = 0,\dots,N\}$ be a vector of weight functions, i.e., every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that for $0 \le i \le N$,

$$w_i \in L^1_{loc}(\Omega) \text{ and } w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega). \tag{5}$$

Finally, we define the weighted space as

$$\tau_0^{1,p}(\Omega,w) = \{u,\ T_k(u) \in W_0^{1,p}(\Omega,w),\ \forall k > 0\},$$

where T_k is usual truncation operator. Now, we state the following assumptions.

(H_1) – The expression

$$\|u\|_X = \left(\sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x)\, dx \right)^{\frac{1}{p}}, \tag{6}$$

is a norm defined on $X = W_0^{1,p}(\Omega,w)$ and is equivalent to the usual norm.

– There exist a weight function σ on Ω and a parameter q $(1 < q < \infty)$ such that

$$\sigma^{1-q'} \in L^1_{loc}(\Omega). \tag{7}$$

with $q' = \frac{q}{q-1}$ and such that the Hardy inequality

$$\left(\int_\Omega |u|^q \sigma(x)\, dx \right)^{\frac{1}{q}} \le C \left(\sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x)\, dx \right)^{\frac{1}{p}}, \tag{8}$$

holds for every $u \in X$ with a constant $C > 0$ independent of u. Moreover, the imbedding

$$X \hookrightarrow L^q(\Omega,\sigma) \tag{9}$$

determined by the inequality (8) is compact.

Let A be the nonlinear operator from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$ defined as

$$Au = -\operatorname{div}(a(x,\nabla u)),$$

where $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:

(H_2) There exist a positive function $k(x)$ in $L^{p'}(\Omega)$ and a positive constant α such that

$$|a_i(x,\xi)| \le w_i^{\frac{1}{p}}(x)[k(x)+\sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}], \text{ for } i=1,\dots,N, \tag{10}$$

$$[a(x,\xi)-a(x,\eta)](\xi-\eta) > 0 \text{ for all } \xi\neq\eta\in\mathbb{R}^N, \tag{11}$$

$$a(x,\xi)\xi \ge \alpha\sum_{i=1}^{N} w_i(x)|\xi_i|^p. \tag{12}$$

Let $g : \Omega\times\mathbb{R}\to\mathbb{R}$ and $G : \Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ two Carathéodory functions satisfying

(H_3)
$$g(x,s).s \ge 0, \tag{13}$$

$$|g(x,s)| \le b(|s|), \tag{14}$$

$$|G(x,s,\xi)| \le \tilde{b}(|s|)\left(c(x)+\sum_{i=1}^{N} w_i(x)|\xi_i|^p\right), \tag{15}$$

where b and $\tilde{b} : \mathbb{R}^+\to\mathbb{R}^+$ are two nonnegative increasing functions and $c(x)$ is a positive function which belongs to $L^1(\Omega)$. We suppose also that g and G satisfy the following conditions

$$\{v\in W_0^{1,p}(\Omega,w),\ G(x,v,\nabla v)=0\ a.e.\text{ in }\Omega\}\subset\{v\in W_0^{1,p}(\Omega,w),\ |g(x,v)|\le 1\ a.e.\text{ in }\Omega\} \tag{16}$$

and

$$\begin{cases} \text{for almost everywhere all } x\in\Omega\backslash\Omega_+^\infty, \text{ there exists } \varepsilon=\varepsilon(x) \text{ such that} \\ g(x,s)>1, \forall s\in]q_+(x),q_+(x)+\varepsilon[, \\ \text{for almost everywhere all } x\in\Omega\backslash\Omega_-^\infty, \text{ there exists } \varepsilon=\varepsilon(x) \text{ such that} \\ g(x,s)<-1, \forall s\in]q_-(x)-\varepsilon,q_-(x)[, \end{cases} \tag{17}$$

where

$$\Omega_+^\infty=\{x\in\Omega,\ q_+(x)=+\infty\}, \qquad \Omega_-^\infty=\{x\in\Omega,\ q_-(x)=-\infty\}.$$

Theorem 1. *Let $f\in L^1(\Omega)$, assume that (H_1)–(H_3), (16) and (17) hold and that the function $s\to g(x,s)$ is an increasing function for almost every $x\in\Omega$. Then the following problems,*

$$(P_n)\begin{cases} u_n\in\mathscr{T}_0^{1,p}(\Omega,w), \quad |g(x,u_n)|^n G(x,u_n,\nabla u_n)\in L^1(\Omega), \\ \displaystyle\int_\Omega |g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-v)\,dx \\ \quad \displaystyle+\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n-v)\,dx \le \int_\Omega fT_k(u_n-v)\,dx+\int_\Omega F\nabla T_k(u_n-v)\,dx \\ \forall v\in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)\ \forall k>0, \end{cases}$$

has at least one solution u_n.

Remark 1. Note that the existence solution u_n of the problem (P_n) is guaranteed by the Theorem 3.1 of [1].

Theorem 2. *Let u_n be a solution of the problem (P_n). Then there exists a function $u \in \tau_0^{1,p}(\Omega,w)$ such that*

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega,w),$$

which is the unique solution of the following bilateral degenerated problems

$$(P)\begin{cases} q_- \le u \le q_+ \ \text{ a.e. in } \Omega, \\ u \in \tau_0^{1,p}(\Omega,w), \\ \int_\Omega a(x,\nabla u)\nabla T_k(u-v)\,dx \le \int_\Omega f T_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-v)\,dx \\ \forall v \in K\cap L^\infty(\Omega)\ \forall k>0, \end{cases}$$

where $K = \{v \in W_0^{1,p}(\Omega,w), q_- \le u \le q_+ \text{ a.e. in } \Omega\}$.

Remark 2. If $G(x,s,0) \neq 0$, we can replace the condition (16) by the following condition

$$\{v \in \tau_0^{1,p}(\Omega,w), G(x,v,\nabla v) = 0 \ a.e.\} \subset \{v \in \tau_0^{1,p}(\Omega,w), |g(x,v)| \le 1 \ a.e.\}.$$

Remark 3. Remark that, since $a(x,\nabla u)$ (dosn't depend on u), we don't need to suppose the restriction $1 < q < p+p'$ on the Hardy parameter q (compare to [1]).

2.1. A priori estimates

Proposition 3. *Let u_n be a solution of the problem (P_n). The there exist some (various) positive constant c independent of n and k and some measurable function u such that*

i) $\alpha \int_\Omega \sum_{i=1}^N w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p dx \le kc.$

ii) $\int_\Omega |g(x,u_n)|^n |G(x,u_n,\nabla u_n)|\,dx \le c$

iii) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^{1,p}(\Omega,w)$,
$T_k(u_n) \to T_k(u)$ *strongly in* $L^q(\Omega,\sigma)$ *and a.e. in* Ω.

Proof. For i) and ii), it suffices to take $v=0$ as test function in (P_n) and using (12), (13) and Young's inequality. Now, since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega,w)$ we prove as in [1] that u_n converges almost everywhere to some measurable function u. □

2.2. Almost everywhere convergence of the gradient

Proposition 4. *Let u_n be a solution of the problem (P_n) and let u be the function of the Proposition 3. Then,*

i) $\nabla u_n \longrightarrow \nabla u$ *a.e. in* Ω *and* *ii)* $q_- \leq u \leq q_+$ *a.e. in* Ω

Proof. i) Put

$$I_n = \int_\Omega \{[a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\}^\theta \, dx = \int_\Omega A_n^\theta \, dx,$$

where $0 < \theta < 1$. By using the similar argument as in [4] we can show that $\lim_{n\to\infty} I_n = 0$. Then,

$$A_n^\theta \to 0 \text{ p.p. in } \Omega.$$

Following the same techniques as in [2] we can prove that

$$\nabla T_k(u_n) \to \nabla T_k(u) \; a.e. \text{ in } \Omega \quad \forall k > 0, \tag{18}$$

and then

$$\nabla u_n \to \nabla u \; a.e. \text{ in } \Omega. \tag{19}$$

ii) We show that $q_- \leq u \leq q_+$ a.e. in Ω.

To simplify, we suppose that $G(x,s,0) = 0$. We have for all $h > 0$,

$$\int_\Omega |g(x,T_h(u_n))|^n |G(x,T_h(u_n),\nabla T_h(u_n))| \, dx \leq C,$$

which implies that

$$\int_{\{|g(x,T_h(u_n))|>k\}} |g(x,T_h(u_n))|^n |G(x,T_h(u_n),\nabla T_h(u_n))| \, dx \leq C$$

and

$$\int_{\{|g(x,T_h(u_n))|>k\}} |G(x,T_h(u_n),\nabla T_h(u_n))| \, dx \leq \frac{C}{k^n},$$

where $k > 1$. Letting $n \to \infty$, we get

$$\int_{\{|g(x,T_h(u))|>k\}} |G(x,T_h(u),\nabla T_h(u))| \, dx = 0.$$

Hence, by (16), we have

$$|g(x,T_h(u))| \leq 1 \;\; a.e. \text{ in } \Omega,$$

then $q_- \leq T_h(u) \leq q_+$ a.e. in Ω, and letting $h \to \infty$, we deduce the result. □

2.3. Strong convergence of truncations

Proposition 5. *Let u_n be a solution of the problem (P_n) and let u be the function of the Proposition 3. Then,*

$$T_k(u_n) \longrightarrow T_k(u) \;\; \textit{strongly in } W_0^{1,p}(\Omega,w).$$

Proof. Let $k>0$, and let $\phi(t)=te^{\gamma t^2}$ with $\gamma>(\frac{\tilde{b}(k)}{2\alpha})^2$ such that

$$\phi'(s)-2\frac{\tilde{b}(k)}{\alpha}|\phi(s)|\geq\frac{1}{2} \tag{20}$$

holds for all $s\in\mathbb{R}$. Here, we define $w_n=T_{2k}\big(u_n-T_h(u_n)+T_k(u_n)-T_k(u)\big)$ and $v_{n,l}=T_l(u_n)-\phi(w_n)$, where $h>k>0$. The use of $v_{n,l}$ as test function in (P_n), we obtain, for all $\eta>0$,

$$\begin{aligned}\langle A(u_n),T_\eta(u_n-T_l(u_n)+\phi(w_n))\rangle&\\ +\int_\Omega|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_\eta(u_n-T_l(u_n)+\phi(w_n))\,dx&\\ \leq\int_\Omega fT_\eta(u_n-T_l(u_n)+\phi(w_n))\,dx.&\end{aligned} \tag{21}$$

Note that, for the first term of (21) is inspired from the step 2 of the proof of Theorem 3.1 of [1], for that we obtain

$$\lim_{n\to\infty}\int_\Omega[a(x,\nabla T_k(u_n))-a(x,\nabla T_k(u))][\nabla T_k(u_n)-\nabla T_k(u)]\,dx=0.$$

The proof is achieved by a standard argument. □

2.4. Proof of Theorem 2

Let θ be a real number such that $0<\theta<1$, let h be a positive real number, let v be a function in $K\cap L^\infty(\Omega)$, and choose $T_h(u_n)-T_k(u_n-\theta v)$ as test function in (P_n). We obtain

$$\begin{aligned}\langle A(u_n),T_k(u_n-T_h(u_n)+T_k(u_n-\theta v))\rangle&\\ +\int_\Omega|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-T_h(u_n)+T_k(u_n-\theta v))\,dx&\\ \leq\int_\Omega fT_k(u_n-T_h(u_n)+T_k(u_n-\theta v))dx+\int_\Omega F\nabla T_k(u_n-T_h(u_n)+T_k(u_n-\theta v))dx.&\end{aligned}$$

and since $\int_{|u_n-T_h(u_n)+T_k(u_n-\theta v)|\leq k}a(x,\nabla u_n)\nabla(u_n-T_h(u_n))\,dx\geq 0$, and from Young's inequality, we get

$$\begin{aligned}\int_{|u_n-T_h(u_n)+T_k(u_n-\theta v)|\leq k}a(x,\nabla u_n)\nabla T_k(u_n-\theta v)\,dx&\\ +\int_\Omega|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-T_h(u_n)+T_k(u_n-\theta v))\,dx&\\ \leq\int_\Omega fT_k(u_n-T_h(u_n)+T_k(u_n-\theta v))\,dx+\int_{|u_n-T_h(u_n)+T_k(u_n-\theta v)|\leq k}F\nabla T_k(u_n-\theta v)\,dx&\\ +\int_{|u_n|\geq h}|F|^{p'}.w^{1-p'}\,dx.&\end{aligned}$$

In virtue of Lebesgue's theorem and by letting $h \to \infty$, we obtain

$$\begin{aligned}&\int_\Omega a(x,\nabla u_n)\nabla T_k(u_n-\theta v)\,dx+\int_\Omega |g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-\theta v)\,dx\\&\quad\leq\int_\Omega fT_k(u_n-\theta v)\,dx+\int_\Omega F\nabla T_k(u_n-\theta v)\,dx.\end{aligned} \tag{22}$$

For the second term, we can write by using again the sign condition (13)

$$\begin{aligned}&\int_\Omega |g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-\theta v)\,dx\\&\quad\geq\int_{\{0\leq u_n\leq\theta v\}}|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-\theta v)\,dx\\&\qquad+\int_{\{\theta v\leq u_n\leq 0\}}|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-\theta v)\;dx.\end{aligned} \tag{23}$$

Let us define, for every x such that $v(x)>0$, $\delta_1(x)=\sup\limits_{0\leq s\leq\theta v(x)} g(x,s)$. It is easy to see that $0\leq\delta_1(x)<1$ a.e. in Ω. We have then using (15)

$$\begin{aligned}&\int_{\{0\leq u_n\leq\theta v\}}|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-\theta v)\;dx\\&\quad\leq k\int_{\{|u_n|\leq\|v\|_\infty\}}\tilde b(\|v\|_\infty)(\delta_1(x))^n\left(c(x)+\sum_{i=1}^N w_i(x)\left|\frac{\partial T_{\|v\|_\infty}(u_n)}{\partial x_i}\right|^p\right)\,dx\\&\quad\leq k\tilde b(\|v\|_\infty)\int_\Omega(\delta_1(x))^n\left(c(x)+\sum_{i=1}^N w_i(x)\left|\frac{\partial T_{\|v\|_\infty}(u_n)}{\partial x_i}\right|^p\right)\,dx.\end{aligned} \tag{24}$$

Consequently, by using the Vitaly's theorem, we deduce that

$$\lim_{n\to\infty}\int_{\{0\leq u_n\leq\theta v\}}|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-\theta v)\;dx=0.$$

By the same method, we prove that

$$\begin{aligned}&\int_{\{\theta v\leq u_n\leq 0\}}|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-\theta v)\;dx\\&\qquad\leq k\tilde b(\|v\|_\infty)\int_\Omega(\delta_2(x))^n\left(c(x)+\sum_{i=1}^N w_i(x)\left|\frac{\partial T_{\|v\|_\infty}(u_n)}{\partial x_i}\right|^p\right)\,dx,\end{aligned} \tag{25}$$

where $-1<\delta_2(x)=\inf\limits_{\theta v\leq s\leq 0}g(x,s)\leq 0$. Hence

$$\lim_{n\to\infty}\int_{\{\theta v\leq u_n\leq 0\}}|g(x,u_n)|^{n-1}g(x,u_n)|G(x,u_n,\nabla u_n)|T_k(u_n-\theta v)\;dx=0.$$

On the other hand, by using the Fatou's lemma and summing up, we obtain

$$\int_\Omega a(x,\nabla u)\nabla T_k(u-\theta v)\,dx\leq\int_\Omega fT_k(u-\theta v)\,dx+\int_\Omega F\nabla T_k(u_n-\theta v)\;dx,$$

for every $v\in K\cap L^\infty(\Omega)$ and for every $\theta\in(0,1)$. The result is proved by letting θ tend to 1.

Remark 4. The proof of the uniqueness solution u of the problem (P) is similar to the one used in the proof of the analogous statement in [3] for the non weighted case.

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L. Aharouch, Y. Akdim and M. Rhoudaf
Département de Mathématiques et Informatique
Faculté des Sciences Dhar-Mahraz
B.P. 1796 Atlas Fès, Maroc
`laharouch@fsdmfes.ac.ma`, `akdimyoussef@yahoo.fr` and
`mrhoudaf@fsdmfes.ac.ma`