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Best regularity for a Schrödinger type equation with non smooth data and interpolation spaces

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Abstract. Given a vector field $U(x)$ and a nonnegative potential $V(x)$ on a smooth open bounded set Ω of $\mathbb{R}^n$, we shall discuss some regularity results for the following equation

$$-\Delta\omega + U \cdot \nabla\omega + V\omega = f \qquad \text{in } \Omega \tag{0.1}$$

whenever δf is a bounded Radon measure with $\delta(x)$ is the distance between x and the boundary $\partial\Omega$.

§1. Introduction

To explain the origin of our study, let us recall some recent results concerning the very weak solution in the sense of Brezis concerning the Laplacian operator, (say $U = V = 0$ in the above equation)
and when f belongs to $L^1_+(\Omega, \delta)\backslash L^1(\Omega; \delta(1 + |\ln \delta|)))$ with $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, then (see [10])

$$\omega \notin W^1_0 L(\operatorname{Log} L) = \left\{v \in W^{1,1}_0(\Omega) : \nabla v \in L(\operatorname{Log} L)^n\right\},$$

and

$$\int_\Omega |\nabla\omega|\, |\operatorname{Log}\delta| dx = +\infty.$$

More, we have (see [11]) the

Theorem 1. *Let*

$$W_+ = \left\{\psi \in W^{2,n}(\Omega) \cap H^1_0(\Omega) : -\Delta\psi \geqslant 0\right\}$$

and

$$L_+ = \left\{f \in L^1_+(\Omega; \delta) : \exists\psi \in W_+ \ s.t \int_\Omega f(x)\psi(x)dx = +\infty\right\}.$$

Then the unique solution $u \in L^{n',\infty}(\Omega)$ of

$$\int_\Omega u\Delta\varphi = \int_\Omega f\varphi,\ \forall\, \varphi \in C^2_0(\overline{\Omega}) = \left\{\varphi \in C^2(\overline{\Omega}) : \varphi = 0 \ on\ \partial\Omega\right\}$$

verifies

$$\int_\Omega |\nabla u| dx = +\infty : u \notin W^{1,1}(\Omega).$$

But we know (see [1]), that

$$W^1(L(\mathrm{Log}\, L)) \subset L^{n'}(\mathrm{Log}\, L)^{\beta(n'-1)} \quad \forall \beta > 1,\ n' = \frac{n}{n-1}.$$

and this last set is included in the so called small Lebesgue spaces

$$L^{(n',1} \subset L^{(n',\alpha}, \quad 0 < \alpha < 1.$$

Nevertheless, we have shown in [6] that if f is in $L^1(\Omega; \delta(1 + |\mathrm{Log}\, \delta|)^\alpha)$, $\frac{1}{n'} < \alpha \leqslant 1$ then the unique solution u of the equation (0.1) belongs to $L^{(n',\theta}(\Omega)$ for some θ.
More precisely, we have shown in [4, 6] the following

Theorem 2. *Let Ω be a bounded open set of class C^2 of $\mathbb{R}^n$, $|\Omega| = 1$, $\alpha \geqslant \frac{1}{n'}$ where $n' = \frac{n}{n-1}$, $f \in L^1(\Omega; \delta)$. Consider $u \in L^{n',\infty}(\Omega)$, the* v.w.s. *of*

$$-\int_\Omega u \Delta\varphi dx = \int_\Omega f\varphi dx \qquad \forall \varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega. \tag{1.1}$$

Then,

1. *if $f \in L^1\big(\Omega; \delta(1 + |\mathrm{Log}\, \delta|)^\alpha\big)$, and $\alpha > \frac{1}{n'}$*

$$u \in L^{(n',n\alpha-n+1}(\Omega) = G\Gamma(n', 1; w_\alpha),\ w_\alpha(t) = t^{-1}(1 - \mathrm{Log}\, t)^{\alpha-1-\frac{1}{n'}}$$

and

$$\|u\|_{G\Gamma(n',1;w_\alpha)} \leqslant K_0 |f|_{L^1(\Omega;\delta(1+|\mathrm{Log}\,\delta|)^\alpha)} \tag{1.2}$$

2. *if $\alpha = \frac{1}{n'}$ then*

$$u \in L^{n'}(\Omega) \text{ and similar estimate as (1.2) holds.}$$

In a recent paper [5], we improve the inequality (1.2) namely for the dimension 2 by getting similar information for $\alpha \leqslant \frac{1}{2}$. Here, we want to extend those results replacing the Laplacian operator by a more general one as it is given in (0.1). Namely, we shall prove the following:

Theorem 3. *Let U be in $L^p(\Omega)^n$, $p > n$, $\mathrm{div}\,(U) = 0$ in $\mathcal{D}'(\Omega)$, $U \cdot \nu = 0$ on $\partial\Omega$, $V \in L^p(\Omega)$, $V \geqslant 0, \beta > \frac{n-1}{n}$, $f \in L^1(\Omega; \delta(1 + |\mathrm{Log}\, \delta|)^\beta)$, $\beta = \frac{n-1+\theta}{n}$, $\theta = n\beta - n + 1$.*
Then the unique solution $u \in L^{n',\infty}(\Omega)$ of

$$\int_\Omega u\Big[-\Delta\varphi - U \cdot \nabla\varphi + V\varphi\Big] dx = \int_\Omega f\varphi dx \quad \forall \varphi \in C_0^2(\overline{\Omega}) \tag{1.3}$$

belongs to $L^{(n',\theta}(\Omega)$ *and there exists a constant c (depending only of the data U V and* Ω*) such that*

$$\|u\|_{L^{(n',\theta}} \leqslant c \int_\Omega |f|\delta(1 + |\text{Log}\,|)^\beta dx.$$

When f is in $L(\text{Log}\,L)^\beta$, we may obtain a similar result concerning the gradient of u but under weaker assumptions on the operator, we will show for $\beta > \dfrac{1}{n'}$

$$\|\nabla u\|_{L^{(n',n\beta-n+1}} \leqslant c\|f\|_{L(\text{Log}\,L)^\beta}. \tag{1.4}$$

§2. Notation Primary results

For a measurable function $f : \Omega \to \mathbb{R}$, we set for $t \geqslant 0$

$$D_f(t) = \text{measure}\left\{x \in \Omega : |f(x)| > t\right\}$$

and f_* the decreasing rearrangement of $|f|$, for $s \in (0, |\Omega|)$

$$f_*(s) = \inf\left\{t : D_f(t) \leqslant s\right\},\ |\Omega| \text{ is the measure of } \Omega,$$

that we shall assume to be equal to 1 for simplicity.

If A_1 and A_2 are two quantities depending on some parameters, we shall write

$$A_1 \lesssim A_2 \text{ if there exists } c > 0 \text{ (independent of the parameters) such that } A_1 \leqslant cA_2$$

$$A_1 \simeq A_2 \text{ if and only if } A_1 \lesssim A_2 \text{ and } A_2 \lesssim A_1$$

We recall also the following definition of interpolation spaces. Let $(X_0, \|\cdot\|_0)$, $(X_1, \|\cdot\|_1)$ two Banach spaces contained continuously in a Hausdorff topological vector space (that is (X_0, X_1) is a compatible couple). For $g \in X_0 + X_1$, $t > 0$ one defines the so called K functional $K(g, t; X_0, X_1) \doteq K(g, t)$ by setting

$$K(g,t) = \inf_{g=g_0+g_1} (\|g_0\|_0 + t\|g_1\|_1). \tag{2.1}$$

For $0 \leqslant \theta \leqslant 1$, $1 \leqslant p \leqslant +\infty$, $\alpha \in \mathbb{R}$ we shall consider

$$(X_0, X_1)_{\theta,p;\alpha} = \left\{g \in X_0 + X_1,\ \|g\|_{\theta,p;\alpha} = \|t^{-\theta-\frac{1}{p}}(1 - \text{Log}\,t)^\alpha K(g,t)\|_{L^p(0,1)} \text{ is finite}\right\}.$$

Here $\|\cdot\|_V$ denotes the norm in a Banach space V. The weighted Lebesgue space $L^p(0, 1; \omega)$, $0 < p \leqslant +\infty$ is endowed with the usual norm or quasi norm, where ω is a weight function on $(0, 1)$, $L^p_+(0, 1, \omega) = \left\{f \in L^p(0, 1; \omega),\ f \geqslant 0\right\}$. Our definition of the interpolation space is different from the usual one (see [2, 13]) since we restrict the norms on the interval $(0, 1)$.

If we consider ordered couple, i.e. $X_1 \hookrightarrow X_0$ and $\alpha = 0$,

$$(X_0, X_1)_{\theta,p;0} = (X_0, X_1)_{\theta,p}$$

is the interpolation space as it is defined by J. Peetre (see [2, 13, 3]).

$$C^2_0(\overline{\Omega}) = \left\{\varphi : \overline{\Omega} \to \mathbb{R},\ \text{twicely differentiable and vanishing at the boundary}\right\}$$

$$W^1V = \left\{\varphi \in L^1_{loc}(\Omega) : \nabla\varphi \in V^n\right\}.$$

2.1. A few description of $G\Gamma(p, m; w_1, w_2)$

Definition 1 (of a Generalized Gamma space with double weights)**.** Let w_1, w_2 be two weights on $(0, 1)$, $m \in [1, +\infty]$, $1 \leqslant p < +\infty$. We assume the following conditions:

c1) There exists $K_{12} > 0$ such that $w_2(2t) \leqslant K_{12} w_2(t)$ $\forall\, t \in (0, 1/2)$. The space $L^p(0, 1; w_2)$ is continuously embedded in $L^1(0, 1)$.

c2) The function $\displaystyle\int_0^t w_2(\sigma)d\sigma$ belongs to $L^{\frac{m}{p}}(0, 1; w_1)$.

A generalized Gamma space with double weights is the set

$$G\Gamma(p, m; w_1, w_2) = \left\{ v : \Omega \to \mathbb{R} \text{ measurable } \int_0^t v_*^p(\sigma) w_2(\sigma) d\sigma \text{ is in } L^{\frac{m}{p}}(0, 1; w_1) \right\}.$$

A similar definition has been considered in [8]. They were interested in the embeddings between $G\Gamma$-spaces.

Properties. Let $G\Gamma(p, m; w_1, w_2)$ be a Generalized Gamma space with double weights and let us define for $v \in G\Gamma(p, m; w_1, w_2)$

$$\rho(v) = \left[\int_0^1 w_1(t) \left(\int_0^t v_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}}$$

with the obvious change for $m = +\infty$.
Then,

1. ρ is a quasinorm.
2. $G\Gamma(p, m; w_1, w_2)$ endowed with ρ is a quasi-Banach function space.
3. If $w_2 = 1$

$$G\Gamma(p, m; w_1, 1) = G\Gamma(p, m; w_1).$$

Example 1 (of weights)**.** Let $w_1(t) = (1 - \operatorname{Log} t)^\gamma$, $\quad w_2(t) = (1 - \operatorname{Log} t)^\beta$ wit $(\gamma, \beta) \in \mathbb{R}^2$. Then

$$w_2 \text{ satisfies condition c1) and } w_1 \text{ and } w_2 \text{ are in } L_{exp}^{\max(\gamma;\beta)}(]0, 1[)\,.$$

Definition 2 (of the small Lebesgue space)**.** The small Lebesgue space associated to the parameter $p \in]1, +\infty[$ and $\theta > 0$ is the set

$$L^{(p,\theta}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable such that} \right.$$

$$\left. \|f\|_{(p,\theta} = \int_0^1 (1 - \operatorname{Log} t)^{-\frac{\theta}{p} + \theta - 1} \left(\int_0^t f_*^p(\sigma) d\sigma \right)^{1/p} \frac{dt}{t} < +\infty \right\}.$$

Let us notice that the small Lebesgue space is a G-gamma space.

Definition 3 (of the Grand Lebesgue space)**.** The associate space of the small Lebesgue space is denoted by $L^{p),\theta}(\Omega)$ for $1 < p < +\infty$, $\theta > 0$ and is defined as

$$L^{p),\theta}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable such that } \|f\|_{p),\theta} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int_\Omega |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} \text{ is finite} \right\}.$$

Properties of small and Grand Lebesgue spaces.

1. They are rearrangement invariant Banach function spaces. One has the following equivalent norm :

$$\|u\|_{L^{(p,\theta}(\Omega)} = \inf_{u=\sum_k u_k} \left\{ \sum_k \inf_{0<\varepsilon<p'-1} \varepsilon^{-\frac{\theta}{(p'-\varepsilon)}} \left(\int_\Omega |u_k|^{(p'-\varepsilon)'} dx \right)^{\frac{1}{(p'-\varepsilon)'}} \right\}$$

$$\|u\|_{p),\theta} \approx \sup_{0<t<|\Omega|} (1 - \operatorname{Log} t)^{-\frac{\theta}{p}} \left(\int_t^{|\Omega|} u_*(s)^p ds \right)^{\frac{1}{p}}.$$

2. $\bigcup_{\varepsilon>0} L^{p+\varepsilon}(\Omega) \overset{\subset}{\neq} \bigcup_{\beta>1} L^p(\operatorname{Log} L)^{\frac{\beta\theta}{p'-1}}(\Omega) \overset{\subset}{\neq} L^{(p,\theta}(\Omega) \subset L^p(\operatorname{Log} L)^{\frac{\theta}{p'-1}}.$

3. $L^p(\Omega) \overset{\subset}{\neq} \dfrac{L^p}{\operatorname{Log}^\theta L}(\Omega) \overset{\subset}{\neq} L^{p),\theta}(\Omega) \overset{\subset}{\neq} \bigcap_{\alpha>1} \dfrac{L^p}{\operatorname{Log}^{\alpha\theta} L}(\Omega) \overset{\subset}{\neq} \bigcap_{0<\varepsilon<p-1} L^{p-\varepsilon}$

4. $\displaystyle\int_\Omega u \cdot v dx \leqslant \|u\|_{L^{(p',\theta}} \|v\|_{L^{p),\theta}}, \ \frac{1}{p} + \frac{1}{p'} = 1.$

$$VMO(\Omega) = \left\{ f \in L^1(\Omega) : \lim_{R\to 0} \sup_{r<R, x_0\in\Omega} \frac{1}{r^n} \int_{B(x_0,r)\cap\Omega} |f - f_r| dx = 0 \right\}$$

here $f_r = \dfrac{1}{|B(x_0;r)\cap\Omega|} \displaystyle\int_{B(x_0;r)\cap\Omega} f(x)dx.$

§3. Proof of Theorem 3

The proof of Theorem 3 follows the same scheme as in [6] by considering the following dual problem

Lemma 4. *For any* $g \in L_+^{n),\theta}(\Omega)$, $V \in L^{n),\theta}(\Omega)$ *and* $\theta > 0$ *the unique solution* $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *of*

$$-\Delta\varphi + U \cdot \nabla\varphi + V\varphi = g \text{ in } H^{-1}(\Omega) \tag{3.1}$$

satisfies $\varphi \in W^2 L^{n),\theta}(\Omega)$ *and there exists a constant* $c_n > 0$ *independent of* θ *such that*

$$\|\varphi\|_{W^2 L^{n),\theta}(\Omega)} \leqslant c_n \|g\|_{L^{n),\theta}(\Omega)}.$$

Here, we assume the same integrability for U as in Theorem 3.

Proof. The existence, uniqueness of φ is given in [4]. Indeed, we have for $n \geqslant 2$,

$$L^{n),\theta}(\Omega) \subset L^{n-\varepsilon}(\Omega), \ \forall\, 0 < \varepsilon < \frac{1}{2}.$$

Thus $g \in L^{\frac{n}{2},1}(\Omega) \subset H^{-1}(\Omega)$.
To obtain the $W^2 L^{n),\theta}$ regularity, we may assume first V and g bounded. Then following Proposition 11 of [4], we have $\varphi \in W^2 L^p(\Omega), \ p > n$.

Let us show that, we have $\varepsilon_0 > 0$ and a constant $c_0 > 0$ depending only on the data $U,\ V,\ \Omega$ such that $\forall \varepsilon \in [0, \varepsilon_0]$

$$||\varphi||_{W^2L^{n-\varepsilon}} \leqslant c_0 ||g||_{L^{n-\varepsilon}}. \tag{3.2}$$

Let $0 < \varepsilon < \dfrac{1}{2}$. Then from the equation satisfied by φ, one has :

$$||\Delta\varphi||_{L^{n-\varepsilon}} \leqslant ||U \cdot \nabla\, \varphi||_{L^{n-\varepsilon}} + ||V\, \varphi||_{L^{n-\varepsilon}} + ||g||_{L^{n-\varepsilon}}. \tag{3.3}$$

Since $\varphi \in L^\infty(\Omega)$ and

$$||\varphi||_{L^\infty} \leqslant c_n ||g||_{L^{\frac{n}{2},1}} \leqslant c_n ||g||_{L^{n),\theta}}. \tag{3.4}$$

So that

$$||V\, \varphi||_{L^{n-\varepsilon}} \leqslant c||V||_{L^{n-\varepsilon}} ||\varphi||_{L^\infty} \leqslant c||V||_{L^{n-\varepsilon}} ||g||_{L^{n),\theta}}. \tag{3.5}$$

By Hölder inequality, for $p > n$,

$$||U \cdot \nabla\, \varphi||_{L^{n-\varepsilon}} \leqslant ||U||_{L^p} ||\nabla\, \varphi||_{L^{\frac{p(n-\varepsilon)}{p-n+\varepsilon}}} \leqslant c||U||_{L^p} ||\nabla\, \varphi||_{L^{p(n)}} \text{ where } p(n) = \frac{pn}{p-n}. \tag{3.6}$$

We shall choose $\varepsilon_0 > 0 : (n-\varepsilon_0)^* > p(n)$ i.e $0 < \varepsilon < \min\left(\dfrac{1}{2}; \dfrac{n(p-n)}{2p-n}\right)$. In that case, we have the compact embedding $W^2L^{n-\varepsilon_0}(\Omega) \hookrightarrow W^1L^{p(n)}(\Omega)$. Therefore $\forall\, \eta > 0$, there exists $c_\eta > 0$ such that

$$||\nabla\, \varphi||_{L^{p(n)}} \leqslant \eta ||\varphi||_{W^2L^{n-\varepsilon_0}} + c_\eta ||\varphi||_{L^2}. \tag{3.7}$$

From Agmon-Douglis-Niremberg's theorem and Marcienkiewicz interpolation's theorem, one has a constant $c_n > 0$ such that

$$||\varphi||_{W^2L^{n-\varepsilon}} \leqslant c_n ||\Delta\, \varphi||_{L^{n-\varepsilon}} \qquad \forall\, \varphi \in W^2L^{n-\varepsilon}(\Omega) \cap H_0^1(\Omega) \text{ and } \forall\, \varepsilon \in [0, \varepsilon_0]. \tag{3.8}$$

Combining relations (3.3) to (3.8), we deduce for all $\eta > 0$, one has a constant $c_\eta > 0$, for all $\varepsilon \in [0, \varepsilon_0]$

$$||\varphi||_{W^2L^{n-\varepsilon}} \leqslant \eta ||U||_{L^p} ||\varphi||_{W^2L^{n-\varepsilon}} + c_\eta ||U||_{L^p} ||\varphi||_{L^\infty} + c' ||V||_{L^{n-\varepsilon}} ||g||_{L^{n),\theta}} + ||g||_{L^{n-\varepsilon}}. \tag{3.9}$$

Since we have

$$||g||_{L^{n),\theta}} \simeq \sup_{0<\varepsilon<\frac{n-1}{2}} \left(\varepsilon^\theta \int_\Omega |g|^{n-\varepsilon}(x)dx\right)^{\frac{1}{n-\varepsilon}};$$

we deduce from relation (3.9) :

$$||\varphi||_{W^2L^{n),\theta}} (1 - \eta ||U||_{L^p}) \leqslant c_\eta ||U||_{L^p} ||g||_{L^{n),\theta}} + c(1 + ||V||_{L^{n),\theta}}) ||g||_{L^{n),\theta}}.$$

Choosing $\eta ||U||_{L^p} \leqslant \dfrac{1}{2}$, we then have a constant c depending only on U and Ω.

$$||\varphi||_{W^2L^{n),\theta}} \leqslant c(1 + ||V||_{L^{n),\theta}}) ||g||_{L^{n),\theta}}. \tag{3.10}$$

We conclude by usual density argument, say

replacing g by $g_k(x) = \min(k; |g(x)|)\text{sign}(g(x))$, $V_k = \min(V; k)$.

the solution of φ_k of $\begin{cases} -\Delta\varphi_k + U \cdot \nabla\, \varphi_k + V_k\, \varphi = g_k \\ \varphi_k \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases}$ satisfies (3.10).

Let $k \to \infty$, the uniqueness of solution (1.3) gives the result. □

§4. Regularity for data in $L(\operatorname{Log} L)^\alpha$ for a full linear operator

In [6], we have shown the following

Theorem 5. *Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geqslant 3$ of class $C^{1,1}$, $A(x) = (a_{ij}(x))_{i,j}$, $x \in \Omega$ a bounded coercitive matrix. Assume that $a_{ij} \in VMO(\Omega)$ and let f be in $L(\operatorname{Log} L)^\alpha$, $\alpha > \dfrac{n-1}{n}$. Then, the weak solution of*

$$\begin{cases} \operatorname{div}(A(x)\nabla u) = f \text{ in } \Omega \\ u \in W_0^1 L^{n'}(\Omega) \end{cases}$$

satisfies

$$\|\nabla u\|_{L^{(n',n\alpha-n+1}} \leqslant c(n;\alpha)\|f\|_{L(\operatorname{Log} L)^\alpha}. \tag{4.1}$$

We want to extend the above result replacing the main operator by

$$\mathcal{L}u \doteq -\operatorname{div}(A(x)\nabla u) + B(x)\cdot\nabla u - \operatorname{div}(C(x)u) + V(x)u.$$

For this, we will assume that

H1. $C(x) = (c_i(x))_{i\in\{1,\dots,n\}}$, $B(x) = (b_i(x))$ are such that c_i, b_i are in $L^n(\Omega)$ for all i and $V \in L^{\frac{n}{2}}(\Omega)$, A is symmetric.

H2. There exists a constant $c_0 > 0$: $V - \operatorname{div}(C) \geqslant c_0 > 0$ in $\mathcal{D}'(\Omega)$

We recall the following results (see [9]).

Lemma 6. *Under the above assumptions on A, B, C and V, $F \in L^p(\Omega)^n$, $1 < p < n$. There exists an unique solution $u \in W_0^{1,p}(\Omega)$ of*

$$\mathcal{L}u = -\operatorname{div}(F) \text{ in } \mathcal{D}'(\Omega).$$

Moreover, there exists a constant $k(p) > 0$ (independent of f and u) such that

$$\|\nabla u\|_{L^{\frac{np}{n-p}}(\Omega)} \leqslant k(p)\|F\|_{L^p(\Omega)^n}. \tag{4.2}$$

Lemma 7 (see [7])**.** *Let $1 < p < n$, $f \in L^p(\Omega)$ and v the unique solution of $-\Delta v = f$ in $\mathcal{D}'(\Omega)$, $v \in W_0^{1,p}(\Omega)$. Then there exist a constant c_n independent of p, f and v such that*

$$\|\nabla v\|_{L^{\frac{np}{n-p}}(\Omega)} \leqslant \frac{c_n}{(p-1)^{\frac{n-1}{n}}}\|f\|_{L^p(\Omega)}. \tag{4.3}$$

Lemma 8. *Let $f \in L^p(\Omega)$, $1 < p < \dfrac{n(n-1)}{n^2-n-1}$, $p^* = \dfrac{pn}{n-p}$.*
Then, there exist a constant c_n' independent of p, f such that the unique solution $u \in W_0^{1,p}(\Omega)$ of $\mathcal{L}u = f$ in $\mathcal{D}'(\Omega)$ satisfies

$$\|\nabla u\|_{L^{p^*}} \leqslant \frac{c_n'}{(p-1)^{\frac{n-1}{n}}}\|f\|_{L^p(\Omega)}. \tag{4.4}$$

Proof. Let $r \in \left[\dfrac{n}{n-1} = n', \dfrac{n-1}{n-2} = (n-1)'\right]$ and $v \in W_0^{1,r}(\Omega) : -\Delta v \in L^m(\Omega)$ with $\dfrac{1}{m} = \dfrac{1}{r} + \dfrac{1}{n}$. From Lemma 6 for any solution $u \in W_0^{1,n'}(\Omega)$ of $\mathcal{L}u = -\Delta v$, one has

$$\|\nabla u\|_{L^{n'}(\Omega)} \leqslant k(n')\|\nabla v\|_{L^{n'}(\Omega)}. \tag{4.5}$$

and

$$\|\nabla u\|_{L^{(n-1)'}(\Omega)} \leqslant k((n-1)')\|\nabla v\|_{L^{(n-1)'}(\Omega)}. \tag{4.6}$$

Applying the Marcinkiewicz real interpolation method, we deduce that we have

$$\|\nabla u\|_{L^r(\Omega)} \leqslant \mathrm{Max}\,(k(n'); k(n-1)')\|\nabla v\|_{L^r(\Omega)}. \tag{4.7}$$

Taking $1 < p < \dfrac{n(n-1)}{n^2-n-1}$, we have $n' < p^* < (n-1)'$ and choosing v such that $-\Delta v = f \in L^p(\Omega)$, $v \in W^{1,p}(\Omega)$ then applying Lemma 7, relation (4.7) leads to :

$$\|\nabla u\|_{L^{p^*}(\Omega)} \leqslant \frac{c_n}{(p-1)^{\frac{n-1}{n}}}\|f\|_{L^p(\Omega)}. \quad \square$$

Theorem 9. *Let $f \in L(\mathrm{Log}\,L)^\alpha$, $\alpha > \dfrac{n-1}{n}$, u satisfying $\mathcal{L}u = f$ in $\mathcal{D}'(\Omega)$, $u \in W_0^1 L^{n',\infty}(\Omega)$. Then*

1. $|\nabla u| \in L^{(n',n\alpha-n+1}(\Omega)$.
2. $\|\nabla u\|_{L^{(n',n\alpha-n+1}(\Omega)} \leqslant c(n;\alpha)\|f\|_{L(\mathrm{Log}\,L)^\alpha}$.

Proof. Its follows the same arguments as in [6] using relation (4.4) and a suitable decomposition of f, whenever $f \geqslant 0$. We drop the details. $\square$

We may weaken hypothesis H2. on V and $C(x)$ by assuming

H3. $V - \mathrm{div}\,(C) \geqslant 0$ in $\mathcal{D}'(\Omega)$.
But we shall add an assumption as

H4. $V - \dfrac{1}{2}\mathrm{div}\,(C + B) \geqslant 0$ in $\mathcal{D}'(\Omega)$.

Hypothesis H4. ensures that for all $T \in H^{-1}(\Omega)$ the problem $\mathcal{L}u = T$ in $\mathcal{D}'(\Omega)$ (resp. $\mathcal{L}^*u = T$) possesses an unique solution $u \in H_0^1(\Omega)$, $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. As a by product of such result and Lemma 6 one has :

Lemma 10. *Let $r \in \left[\dfrac{2n}{n+2}, \dfrac{2n}{n-2}\right]$, $n \geqslant 3$, $F \in L^r(\Omega)^n$. Then, there exists an unique $u \in W_0^{1,r}(\Omega)$ of $\mathcal{L}u = -\mathrm{div}\,(F)$ in $\mathcal{D}'(\Omega)$. Moreover,*

$$\exists c(r) > 0 : \|\nabla u\|_{L^r} \leqslant c(r)\|F\|_{L^r}. \tag{4.8}$$

Proof. Let $F \in L^r(\Omega)^n$, $r \in \left[2, \dfrac{2n}{n-2}\right]$. Since $F \in L^2(\Omega)^n$, we may use hypothesis H4. to deduce that the problem $\mathcal{L}u = -\mathrm{div}\,(F)$ has an unique solution $u \in H_0^1(\Omega)$. Let $F_0 \in L^{2^*}(\Omega)^n$ such $-\mathrm{div}\,(F_0) = u$ and

$$\|F_0\|_{L^{2^*}} \leqslant c\|u\|_{L^2} \leqslant c\|F\|_{L^2} \leqslant c\|F\|_{L^r}. \tag{4.9}$$

We may write the equation $\mathcal{L}u = -\mathrm{div}\,(F)$ as

$$-\mathrm{div}\,(A(x)\nabla u) + B(x)\nabla u - \mathrm{div}\,(C(x)u) + (V+1)u = -\mathrm{div}\,(F_0 + F), \quad F_0 + F \in L^r(\Omega)^n$$

One has $V + 1 - \mathrm{div}\,(C) \geqslant 1 > 0$.
Applying Lemma 6, we deduce that $u \in W_0^{1,r}(\Omega)$ and

$$\|\nabla u\|_{L^r} \leqslant c(r)\|F_0 + F\|_{L^r} \leqslant c(r)\|F\|_{L^r}. \tag{4.10}$$

For $r \in \left[\frac{2n}{n+2}, 2\right]$, we argue by duality to conclude that one has an unique function $u \in W_0^{1,r}(\Omega)$ such that $\mathcal{L}u = -\mathrm{div}\,(F)$ in $\mathcal{D}'(\Omega)$

$$\|\nabla u\|_{L^r} \leqslant c(r)\|F\|_{L^r}. \quad \square \tag{4.11}$$

Thank to the above Lemma, we have:

Lemma 11. *Let $r \in [n', (n-1)']$ then there exists a constant $k(n) > 0$*

$$\|\nabla u\|_{L^r} \leqslant k(n)\|F\|_{L^r}$$

whenever u satisfies: $\mathcal{L}u = -\mathrm{div}\,(F)$ in $\mathcal{D}'(\Omega)$.

We conclude as before to derive the following:

Lemma 12. *Let $f \in L^p(\Omega)$, $1 < p < \frac{n(n-1)}{n^2 - n - 1}$, $p^* = \frac{pn}{n-p} = -p(n)$. Then the unique solution u of $\mathcal{L}u = f$, $u \in W_0^{1,p}(\Omega)$ satisfies*

$$\|\nabla u\|_{L^{p^*}} \leqslant \frac{c_n}{(p-1)^{\frac{n-1}{n}}}\|f\|_{L^p(\Omega)}.$$

Theorem 13. *Assume H1. H3. and H4. Then for $f \in L(\mathrm{Log}\,L)^\alpha$, $\alpha > \frac{n-1}{n}$, $n \geqslant 3$. There exists an unique solution $u \in L^{(n', n\alpha - n + 1}(\Omega)$ satisfying $\mathcal{L}u = f$ in $\mathcal{D}'(\Omega)$. Moreover, there exists a constant $c(n;\alpha) > 0$ such that:*

$$\|\nabla u\|_{L^{(n', n\alpha-n+1}(\Omega)} \leqslant c(n;\alpha)\|f\|_{L(\mathrm{Log}\,L)^\alpha}.$$

Proof. The proof follows the same argument as in [6]. $\square$

Recent developments concerning equation (2.1) but with singular potential as Colomb's potential is given in [12].

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