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Accurate least squares fitting with a general class of shape preserving bases

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Abstract. In this paper we consider the problem of least squares fitting with a very general class of bases with interest in Computer Aided Geometric Design and Approximation Theory. We compute a factorization of the collocation matrix A of these bases that allows us to obtain a QR decomposition of A. Then the triangular system corresponding to the matrix factor R is solved using a bidiagonal factorization of this matrix. Numerical experiments show the accuracy of this procedure.

Keywords: B-basis, Bidiagonal decompositions, Least Squares, Accurate computations.

AMS classification: 65D17, 65F05, 65D05, 41A05, 42A10.

§1. Introduction

The accurate computation with structured classes of matrices is an important issue in Numerical Linear Algebra and it is receiving increasing attention in the recent years (cf. [10, 23, 7]). For this purpose, a parametrization adapted to the structure of the considered matrices is needed. Let us recall that an algorithm can be performed with high relative accuracy (HRA) when it only uses products, quotients, additions of numbers with the same sign or subtractions of initial data (cf. [11]). Performing an algorithm with HRA is a very desirable goal because it implies that the relative errors of the computations are of the order of the machine precision, independently of the size of the condition number of the considered problem. Bidiagonal factorizations provide a parametrization that has played a crucial role to derive algorithms with HRA for some classes of totally positive (TP) matrices. In this case, the mentioned bidiagonal factorizations can be explicitly computed by means of an elimination process called Neville elimination (cf. [12]). When the bidiagonal factorization of the considered matrix is obtained with HRA, the computation of the inverse matrix, its eigenvalues and singular values, the solutions of some linear systems or the computation of its QR factorization can be also performed with HRA using the algorithms presented by Koev in [17] and [16]. Up to now, this has been achieved with some relevant subclasses of TP matrices with applications to Computer Aided Geometric Design (cf. [22, 6, 7, 23, 19]), to Finance (cf. [5]) or to Combinatorics (cf. [8]).

In Computer Aided Geometric Design shape preserving representations are associated with normalized totally positive (NTP) bases because parametric curves inherit the geometric properties of their control polygons with respect to these bases. Among all NTP bases of a given space of functions, there exists a unique normalized B-basis, which is the basis with optimal shape preserving properties (cf. [24], [4]). The Bernstein bases and the B-spline bases are the normalized B-bases of their corresponding spaces. The matrices considered in [6, 9] are collocation matrices of polynomial rational functions. However the rational model has several drawbacks (see [20]). Rational curves require additional parameters (weights), which do not have an evident geometric meaning and whose selection is often unclear. In addition, the behavior of rational bases with respect to differentiation and integration operations, is particularly unpleasant and the exact integration of rational curves is hard and requires (whenever possible) involved non rational forms. On the other hand, the rational model cannot encompass transcendental curves such as the helix or the cycloid, which are of interest in many applications. Furthermore the parametrization of conic sections does not correspond to the natural arc-length parametrization, so given uniform partitions in the parameter space we can get unevenly spaced points. Therefore, non-polynomial basis functions (such as trigonometric functions, hyperbolic functions or their mixtures with polynomials) are often used to represent some typical curves or surfaces without rational forms. In [19] algorithms for the computation of the bidiagonal decomposition of square collocation matrices of a very general class of non-polynomial bases with interest in Computer Aided Geometric Design and Approximation Theory are provided. The obtained algorithms are used in [19] to perform accurate algebraic computations, such as the calculation of their inverses, their eigenvalues or their singular values. In this paper, following the approach of [21] for a polynomial case, we generalize the mentioned bidiagonal factorizations to the case of rectangular collocation matrices. Using their *QR* decompositions, we focus on the problem of least squares fitting in the spaces generated by the general class of bases defined in [19]. By computing the bidiagonal decomposition of the coefficient matrix of the least squares problem, an algorithm for the computation of its *QR* decomposition is then applied. Finally, using the bidiagonal decomposition of the matrix factor R, a triangular system is solved.

The layout of the paper is as follows. Section 2 includes matrix notations basic concepts and tools. We also recall the Neville elimination procedure, which allows us to introduce the bidiagonal factorization of a square strictly totally positive matrix. Section 3 introduces the class of fg-Bernstein bases and recalls the bidiagonal factorization of the collocation matrices associated to these bases derived in [19]. In Section 4, we generalize these decompositions to the case of rectangular matrices. Then a procedure for computing the solution of the least squares problems in the space generated by fg-Bernstein bases is obtained. Finally, Section 5 shows numerical examples with accurate results obtained when we apply the explained procedure.

§2. Basic notations and auxiliary results

A matrix is *totally positive* (TP) if all its minors are nonnegative and *strictly totally positive* (STP) if they are positive (see [1]). A system of functions $(u_0, \ldots, u_n)$ defined on $I \subseteq \mathbb{R}$ is TP if all its collocation matrices

$$\Big(u_{j-1}(t_i)\Big)_{1\le i\le l+1;1\le j\le n+1}, \quad t_1 < \cdots < t_{l+1} \text{ in } I$$

are TP. A TP system of functions on I is *normalized* (NTP) if $\sum_{i=0}^{n} u_i(t) = 1$, for all $t \in I$. NTP bases are commonly used in Computer Aided Geometric Design due to their shape preserving properties (see [3], [24]).

Among all NTP bases of a space, we can find a unique normalized B-basis, which is the optimal shape preserving basis (cf. [4]). For instance, the Bernstein bases and the B-spline bases are the normalized B-bases of their corresponding spaces. The following characterization of a B-basis is a consequence of Corollary 3.10 of [4] and Proposition 3.11 of [4].

Theorem 1. *Let* $(u_0, \ldots, u_n)$ *be a TP basis of a space* $\mathcal{U}$*. Then* $(u_0, \ldots, u_n)$ *is a B-basis iff for any other TP basis* $(v_0, \ldots, v_n)$ *of* $\mathcal{U}$ *the matrix* K *of change of basis such that* $(v_0, \ldots, v_n) = (u_0, \ldots, u_n)K$ *is TP.*

Let us now recall some basic matrix notations and results on Neville elimination. Our notation follows the notation used in [12, 15]. Given $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$, let $Q_{k,n}$ be the set of increasing sequences of k positive integers less than or equal to n. If $\alpha, \beta \in Q_{k,n}$, we denote by $A[\alpha|\beta]$ the $k \times k$ submatrix of A containing rows of places α and columns of places β.

Neville elimination is a procedure to make zeros in a column of a matrix by adding to a given row an appropriate multiple of the previous one (see [12, 15]). For a given nonsingular matrix $A = (a_{i,j})_{1\le i,j\le n}$, let us present this elimination procedure for the case that no row exchanges are necessary. Neville elimination consists of at most $n-1$ successive major steps, resulting in the sequence of matrices:

$$A^{(1)} := A \to A^{(2)} \to \cdots \to A^{(n)} = U.$$

For $1 \le k \le n-1$, $A^{(k+1)} = (a_{i,j}^{(k+1)})_{1\le i,j\le n}$ is obtained from $A^{(k)} = (a_{i,j}^{(k)})_{1\le i,j\le n}$ by defining

$$a_{i,j}^{(k+1)} := a_{i,j}^{(k)} - \frac{a_{i,k}^{(k)}}{a_{i-1,k}^{(k)}} a_{i-1,j}^{(k)} \quad \text{if } a_{i-1,k}^{(k)} \neq 0, \quad k+1 \le i, j \le n,$$

so that $A^{(k+1)}$ has zeros below its main diagonal in the k first columns. Finally, U is an upper triangular matrix. The element $p_{i,j} := a_{i,j}^{(j)}$, $1 \le j \le i \le n$, is called the (i, j) pivot of the Neville elimination of A. The pivots $p_{i,i}$ are called diagonal pivots. The Neville elimination can be performed without row exchanges if all the pivots are nonzero and, in this case, Lemma 2.6 of [12] implies that $p_{i,1} = a_{i,1}$, $1 \le i \le n$, and

$$p_{i,j} = \frac{\det A[i-j+1, \ldots, i|1, \ldots, j]}{\det A[i-j+1, \ldots, i-1|1, \ldots, j-1]}, \quad 1 < j \le i \le n. \tag{1}$$

Furthermore, the (i, j) multiplier of the Neville elimination of A is

$$m_{i,j} := \frac{a_{i,j}^{(j)}}{a_{i-1,j}^{(j)}} = \frac{p_{i,j}}{p_{i-1,j}}, \quad 1 \le j < i \le n. \tag{2}$$

Neville elimination has been used to characterize TP and STP matrices (see [12, 15]). From Theorem 4.1 of [12] and p. 116 of [15], a given matrix A is STP if and only if the Neville elimination of A and A^T can be performed without row exchanges, all the multipliers of the Neville elimination of A and A^T are positive and all the diagonal pivots of the Neville elimination of A are positive.

Bidiagonal factorizations have played a crucial role to derive, for TP matrices, algorithms with HRA (cf. [16]). According to the arguments of p.116 of [15], an STP matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ can be factorized, in a unique way under certain conditions, in the form

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{3}$$

where F_i and G_i are the lower and upper triangular bidiagonal matrices

$$F_i = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & m_{i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & m_{n+1,n+1-i} & 1 \end{pmatrix},$$

$$G_i^T = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & 0 & 1 & & & \\ & & & \hat{m}_{i+1,1} & 1 & & \\ & & & & \ddots & \ddots & \\ & & & & & \hat{m}_{n+1,n+1-i} & 1 \end{pmatrix}, \tag{4}$$

and $D = \operatorname{diag}(p_{1,1}, \dots, p_{n+1,n+1})$. The entries $m_{i,j}$ and $\hat{m}_{i,j}$ are the multipliers of the Neville elimination of A and A^T, respectively, and the diagonal entries $p_{i,i}$ are the diagonal pivots of the Neville elimination of A. In fact a unique bidiagonal factorization can be obtained for nonsingular TP matrices (see [14, 15]).

§3. The class of fg-Bernstein bases

Let us suppose that $I = [a, b]$ and $f, g : I \to \mathbb{R}$ are nonnegative continuous functions. For a given $n \in \mathbb{N}$, the corresponding fg-Bernstein basis of order n was defined in [19] as

$$(u_0^n, \dots, u_n^n), \quad u_k^n(t) := \binom{n}{k} f^k(t) g^{n-k}(t), \quad t \in [a, b], \quad k = 0, \dots, n. \tag{5}$$

The following result corresponds to Proposition 19 of [18] and characterizes when the fg-Bernstein basis defined in (5) is a B-basis.

Proposition 2. *The system given in* (5) *is a B-basis if and only if the function f/g defined on $I_0 := \{t \in I \mid g(t) \neq 0\}$ is increasing and satisfies*

$$\inf\left\{\frac{f(t)}{g(t)} \mid t \in I_0\right\} = 0, \quad \sup\left\{\frac{f(t)}{g(t)} \mid t \in I_0\right\} = +\infty. \tag{6}$$

Theorem 2 of [19] proves that, given nonnegative $f, g : I \to \mathbb{R}$ such that $f(t) \neq 0$, $g(t) \neq 0$, $\forall t \in (a, b)$ and f/g is a strictly increasing function, then

$$A := \left(\binom{n}{j-1} f^{j-1}(t_i) g^{n-j+1}(t_i)\right)_{1\le i,j\le n+1}, \quad a < t_1 < \cdots < t_{n+1} < b, \tag{7}$$

is STP. Moreover, in Theorem 3 of [19], the following bidiagonal decomposition (3) of the collocation matrices (7) was deduced

$$A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n, \tag{8}$$

where F_i and G_i, $1 \le i \le n$, are the lower and upper triangular bidiagonal matrices of the form (4) and $D = \operatorname{diag}(p_{1,1}, \dots, p_{n+1,n+1})$. The entries $m_{i,j}$, $\hat{m}_{i,j}$ and $p_{i,i}$ are given by

$$\begin{aligned}
m_{i,j} &= \frac{g^{n-j+1}(t_i) g(t_{i-j})}{g^{n-j+2}(t_{i-1})} \frac{\prod_{k=1}^{j-1} (f(t_i)g(t_{i-k}) - f(t_{i-k})g(t_i))}{\prod_{k=2}^{j} (f(t_{i-1})g(t_{i-k}) - f(t_{i-k})g(t_{i-1}))},\\
\hat{m}_{i,j} &= \frac{n-i+2}{i-1} \frac{f(t_j)}{g(t_j)}, \quad 1 \le j < i \le n+1,\\
p_{i,i} &= \binom{n}{i-1} \frac{g^{n-i+1}(t_i)}{\prod_{k=1}^{i-1} g(t_k)} \prod_{k=1}^{i-1} (f(t_i)g(t_k) - f(t_k)g(t_i)), \quad 1 \le i \le n+1.
\end{aligned} \tag{9}$$

Let us observe that a sufficient condition to obtain the bidiagonal decomposition of A with HRA is that the expressions $f(t_i)$, $g(t_i)$ and $f(t_i)g(t_k) - f(t_k)g(t_i)$, for all $k < i$, can be computed with HRA.

There are many interesting choices of functions f and g satisfying conditions (6) and allowing us the definition of B-bases whose STP collocation matrices can be factorized as in (8). For example, if

$$f(t) := \frac{t-a}{b-a}, \quad g(t) := \frac{b-t}{b-a}, \quad t \in [a, b],$$

the basis (5) is the Bernstein basis of the space of polynomials of degree not greater than n on the compact interval $[a, b]$. Let us observe that, in this case, the computation of $f(t_i)$, $g(t_i)$ and $f(t_i)g(t_k) - f(t_k)g(t_i) = (t_i - t_k)/(b - a)$, $k < i$, can be performed with HRA because it only requires quotients and subtractions of the initial data. Therefore we can also guarantee that the bidiagonal decomposition (8) of the corresponding collocation matrices (7) can be obtained with HRA. We can also consider

$$f(t) := t^2, \quad g(t) := 1 - t^2, \quad t \in [0, 1].$$

Taking into account Proposition 2, we deduce that the system (5) is the normalized B-basis of the space $\langle 1, t^2, \dots, t^{2n} \rangle$ of even polynomials of degree less than or equal to $2n$ on $[0, 1]$. Let us also observe that the computation of $f(t_i)$, $g(t_i)$ and $f(t_i)g(t_k) - f(t_k)g(t_i) = t_i^2 - t_k^2 = (t_i + t_k)(t_i - t_k)$, $k < i$, requires additions, products and subtractions of the initial data, therefore it can be done with HRA. Again, we can guarantee that the bidiagonal decomposition (8) of the corresponding collocation matrices (7) can be obtained with HRA.

Another particular case can be given by considering the functions

$$f(t) := \sin^2(t/2) = (1 - \cos t)/2, \quad g(t) := \cos^2(t/2) = (1 + \cos t)/2, \quad t \in I = [0, \pi]. \tag{10}$$

In [24] it was proved that the system (5) is the normalized B-basis of the space of even trigonometric polynomials $\langle 1, \cos t, \cos 2t, \ldots, \cos nt\rangle$ on I. On the other hand, if we consider $0 < \Delta < \pi/2$ and

$$f(t) := \sin((\Delta + t)/2), \quad g(t) := \sin((\Delta - t)/2), \quad t \in I = [-\Delta, \Delta], \tag{11}$$

for a given $n = 2m$, the system (5) is a basis that coincides, up to a positive scaling, with the normalized B-basis of the space $\langle 1, \cos t, \sin t, \ldots, \cos mt, \sin mt\rangle$ of trigonometric polynomials of degree less than or equal to m on I (see Section 3 of [25]). Finally, for any $\Delta > 0$, we can also consider

$$f(t) := \sinh((\Delta + t)/2)), \quad g(t) := \sinh((\Delta - t)/2), \quad t \in I = [-\Delta, \Delta]. \tag{12}$$

For $n = 2m$, the system (5) is a B-basis of the space $\langle 1, e^t, e^{-t}, \ldots, e^{mt}, e^{-mt}\rangle$ of hyperbolic polynomials of degree less than or equal to m on I.

In the last three cases, taking into account that $f(t_i)g(t_k) - f(t_k)g(t_i)$ is equal to $(\cos(t_k) - \cos(t_i))/2$, for the functions f and g defined in (10), $\sin(\Delta)\sin((t_i - t_k)/2)$, for the functions f and g defined in (11) and $\sinh(\Delta)\sinh((t_i - t_k)/2)$, for the functions f and g defined in (12), the computation with HRA of the corresponding bidiagonal decomposition (8) should require the evaluation with HRA of the involved trigonometric or hyperbolic functions. Although this cannot be guaranteed, Section 5 and the numerical experiments in [19] show that accurate algebraic computations with the collocation matrices associated to these non-polynomial bases functions can be performed.

§4. Accurate least squares fitting with fg-Bernstein bases

Let us suppose that f and g are functions defined on $[a, b]$ such that $f(t) \neq 0$, $g(t) \neq 0$, $\forall t \in (a, b)$, and f/g is a strictly increasing function. Given a set of parameters $a < t_1 < \cdots < t_{l+1} < b$ and real values $p_1 < \cdots < p_{l+1}$, for some $n \leq l$, we want to compute a function

$$p(t) := \sum_{j=1}^{n+1} c_j \binom{n}{j-1} f^{j-1}(t) g^{n-j+1}(t), \quad t \in [a, b],$$

minimizing the sum of the squares of the deviations from the data $\sum_{i=1}^{l+1} |p_i - p(t_i)|^2$. In order to compute the coefficients of $p(t)$ with respect to the considered fg-Bernstein basis we have to solve, in the least square sense, the overdeterminated linear system $Ac = p$, where

$$A := \left(\tbinom{n}{j-1} f^{j-1}(t_i) g^{n-j+1}(t_i)\right)_{1\leq i\leq l+1; 1\leq j\leq n+1}$$

is the rectangular collocation matrix of the fg-Bernstein basis corresponding to the nodes $t_1 < \cdots < t_{l+1}$, $p = (p_1, \ldots, p_{l+1})^T$ is the data vector and $c = (c_1, \ldots, c_{n+1})^T$ is the vector with the coefficients we want to compute. Using Theorem 2 of [19], we can easily deduce that A is STP and so has maximal rank $n + 1$. Therefore this problem has a unique solution, which is given by the solution of the linear system

$$A^T A c = A^T p.$$

Solving the previous normal equations is a worse conditioned problem than computing the solution through the QR decomposition of the coefficient matrix A, which is the usual approach. In [16] an efficient algorithm for computing the QR decomposition of an STP matrix A is presented. In [17] the Matlab or Octave library `TNQR`, containing an implementation of the mentioned last algorithm, is available. Assuming that the bidiagonal factorization of A is known, `TNQR` computes the matrix Q and the bidiagonal factorization of the matrix R with HRA. Now, following the approach of [21], we shall describe how to solve our least squares problem by means of a bidiagonal decomposition for rectangular matrices that generalizes the bidiagonal factorization described, for the square case, in the previous section and the QR decomposition provided by `TNQR`.

In order to compute the solution of the least squares problem, we define the $(l+1)\times(n+1)$ matrix M such that

$$\begin{aligned} M_{i,i} &:= p_{i,i}, \quad i = 1,\dots,n+1,\\ M_{i,j} &:= m_{i,j}, \quad j = 1,\dots,n+1; \quad i = j+1,\dots,l+1,\\ M_{i,j} &:= \hat{m}_{i,j}, \quad i = 1,\dots,n; \quad j = i+1,\dots,n+1, \end{aligned}$$

where the $m_{i,j}$, $\hat{m}_{i,j}$ and $p_{i,i}$ are obtained as in (9). Then, using `TNQR`, we can obtain the QR decomposition of A such that

$$A = Q\begin{pmatrix} R \\ 0 \end{pmatrix},$$

where $Q \in \mathbb{R}^{(l+1)\times(l+1)}$ is an orthogonal matrix and $R \in \mathbb{R}^{(n+1)\times(n+1)}$ is an upper triangular matrix with positive diagonal entries. Following Section 1.3.1 in [2], the solution of the least squares problem is obtained from

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = Q^T p, \quad Rc = d_1, \quad r = Q\begin{pmatrix} 0 \\ d_2 \end{pmatrix}, \tag{13}$$

where $d_1 \in \mathbb{R}^{n+1}$, $d_2 \in \mathbb{R}^{l-n}$ and $r = p(t) - Ac$. The matrices Q and R have an special structure described in [13]. In particular, R is nonsingular and TP. In order to obtain the solution of the upper triangular system $Rc = d_1$, we have used the routine `TNSolve` of [16], which uses the bidiagonal decomposition of the upper triangular TP matrix R.

§5. Numerical experiments

Now let us illustrate the accuracy of the method explained in the previous section for the computation of the solution of the least squares minimization problem with fg-Bernstein bases. For different choices of f and g, we have considered fg-Bernstein bases of order n defined on $[a,b]$ and computed with Matlab two approximations of the vector $c = (c_1,\dots,c_{n+1})$ such that the function

$$p(t) = \sum_{j=1}^{n+1} c_j \binom{n}{j-1} f^{j-1}(t) g^{n-j+1}(t), \quad t \in [a,b],$$

minimizes $\sum_{k=1}^{100}(p_k - p(t_k))^2$, where $p_1,\dots,p_{100}$ are given integer values and $t_1,\dots,t_{100}$, $l > n$, are equidistant parameters in (a,b). One approximation has been obtained using the procedure explained in the previous section and the other approximation has been obtained using

n+1	TNQR	$A \setminus p$	TNQR	$A \setminus \tilde{p}$
15	4.89151×10^{-15}	8.09049×10^{-13}	8.18912×10^{-15}	4.57057×10^{-13}
20	2.97354×10^{-15}	1.95465×10^{-12}	2.68153×10^{-15}	4.60592×10^{-12}
25	4.20615×10^{-15}	9.55201×10^{-10}	3.864845×10^{-15}	9.49291×10^{-10}
30	8.16195×10^{-16}	2.56043×10^{-8}	9.15474×10^{-16}	4.55599×10^{-8}

Table 1: Relative errors with $f(t) = (1+t)/2$, $g(t) = (1-t)/2$, $t \in [-1, 1]$.

n+1	TNQR	$A \setminus p$	TNQR	$A \setminus \tilde{p}$
15	8.4759×10^{-16}	1.31186×10^{-12}	1.80073×10^{-15}	4.36905×10^{-14}
20	1.74157×10^{-15}	3.8791×10^{-13}	1.77785×10^{-15}	1.39799×10^{-13}
25	7.41971×10^{-15}	4.14554×10^{-10}	1.92262×10^{-14}	2.00229×10^{-10}
30	2.36573×10^{-15}	1.67828×10^{-9}	1.10435×10^{-14}	1.18769×10^{-8}

Table 2: Relative errors with $f(t) = t^2$, $g(t) = 1 - t^2$, $t \in [0, 1]$.

the Matlab command \. We have also computed the solution of these least squares problems using the Mathematica command LeastSquares with a precision of 100 digits and considered this solution c as the exact solution of the problem. Let us recall that in general we cannot guarantee HRA. However the numerical experiments show great accuracy in all the considered cases.

We have computed the relative error of every approximation $\tilde{c} = (\tilde{c}_1, \ldots, \tilde{c}_{n+1})$ of the solution c of the least squares problems by means of the formula

$$e = \frac{\|c - \tilde{c}\|_2}{\|c\|_2}.$$

We have considered $p_k := k \times (-1)^k$, $k = 1, \ldots, 100$ and also $\tilde{p}_k := k$, $k = 1, \ldots, 50$, $\tilde{p}_k := -k$, $k = 51, \ldots, 100$. The obtained errors are included in Table 1 (for the choice $f(t) = (1+t)/2$, $g(t) = (1-t)/2$, $t \in [-1, 1]$), in Table 2 (for the choice $f(t) = t^2$, $g(t) = 1 - t^2$, $t \in [0, 1]$), in Table 3 (for the choice $f(t) = (1 - \cos t)/2$, $g(t) = (1 + \cos t)/2$, $t \in [0, \pi]$), in Table 4 (for the choice $f(t) = \sin((1+t)/2)$, $g(t) = \sin((1-t)/2)$, $t \in [-1, 1]$) and, finally, in Table 5 (for the choice $f(t) = \sinh((1+t)/2)$, $g(t) = \sinh((1-t)/2)$, $t \in [-1, 1]$). The computed results confirm the accuracy of the proposed method that, clearly, keeps the accuracy when the dimension of the problem increases.

In conclusion, we have presented a method for solving least squares problems with collocation matrices of fg-Bernstein bases that can be performed, in some cases, with HRA. We think that the proposed method exploits the structural properties of totally positive matrices and this could explain the great accuracy, even though HRA cannot be guaranteed, providing results much more accurate than those obtained by Matlab using the standard method for the resolution of least squares problems.

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n+1	TNQR	$A \setminus p$	TNQR	$A \setminus \tilde{p}$
15	2.95136×10^{-14}	1.99703×10^{-12}	1.2823×10^{-14}	4.93654×10^{-13}
20	1.81561×10^{-14}	6.64461×10^{-11}	1.7008×10^{-15}	1.64829×10^{-12}
25	4.61364×10^{-14}	1.72592×10^{-9}	1.15137×10^{-14}	7.41444×10^{-10}
30	7.88557×10^{-14}	4.34384×10^{-8}	4.88251×10^{-15}	6.41024×10^{-9}

Table 3: Relative errors with $f(t) = (1 - \cos t)/2$, $g(t) = (1 + \cos t)/2$, $t \in [0, \pi]$.

n+1	TNQR	$A \setminus p$	TNQR	$A \setminus \tilde{p}$
15	1.23029×10^{-15}	2.63437×10^{-12}	1.05317×10^{-14}	3.25159×10^{-12}
20	4.79405×10^{-15}	4.12448×10^{-11}	1.34693×10^{-15}	3.39137×10^{-11}
25	6.14711×10^{-16}	1.05147×10^{-9}	1.57822×10^{-14}	3.70748×10^{-10}
30	6.47177×10^{-15}	6.98518×10^{-8}	9.14979×10^{-15}	1.6299×10^{-7}

Table 4: Relative errors with $f(t) = \sin((1 + t)/2)$, $g(t) = \sin((1 - t)/2)$, $t \in [-1, 1]$.

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n+1	TNQR	$A \setminus p$	TNQR	$A \setminus \tilde{p}$
15	4.66821×10^{-15}	5.93865×10^{-13}	3.33749×10^{-15}	2.32175×10^{-14}
20	1.8994×10^{-15}	1.84095×10^{-11}	2.55009×10^{-15}	7.65487×10^{-12}
25	2.29248×10^{-15}	3.38356×10^{-10}	4.82278×10^{-14}	6.31661×10^{-10}
30	5.00391×10^{-15}	4.98638×10^{-9}	1.89331×10^{-14}	3.22028×10^{-9}

Table 5: Relative errors with $f(t) = \sinh((1+t)/2)$, $g(t) = \sinh((1-t)/2)$, $t \in [-1, 1]$.

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