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A TRIAXIAL MODEL FOR THE ROTO-ORBITAL COUPLING IN A BINARY SYSTEM

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Abstract. We study the roto-orbital dynamics of a uniform sphere and a triaxial body by means of a model which defines a 2-DOF Hamiltonian system using variables referred to the total angular momentum. The validity and applicability of our model is been assessed numerically. We present a classification of some relative equilibria, finding constant radius solutions filling 4-D and lower dimensional tori. These families of relative equilibria include some of the classical ones reported in the literature and some new types showing the triaxiality influence on both. For a number of scenarios the relation between the triaxiality and the inclination connected with relative equilibria are discussed and a full analysis in in progress [2].

Keywords: Roto-orbital dynamics, rigid body, relative equilibria, triaxiality..

§1. Introduction

We study a 2-DOF Hamiltonian model for the roto-orbital dynamics of a general binary system made of two rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$, with masses m_1 and m_2 respectively. This problem is known as the full gravitational 2-body problem (FG2BP)[10] and usually is approximated by means of the MacCullagh's truncation [8], which is the second non vanishing term of the expansion of the potential energy. That is to say, the associated Hamiltonian with the FG2BP is obtained out of the sum of the rotational and orbital kinetic energies plus the potential energy, which is computed as a series expansion in Legendre polynomials. The first step in this expansion leaves us with a maximally super-integrable model, the Kepler plus the free rigid body. Nevertheless, accuracy increasing demands ask for a more realistic model. With this purpose, the usual procedure is adding the following term (MacCullagh's term) of the potential expansion, leading us to a non-integrable system in many degrees of freedom, which involves an extraordinary complexity. The main idea of this communication is to present a halfway model between these two extremes. The interest of our model is twofold. On the one hand, it allows us to identify special solutions that could become nominal trajectories in missions design whereas it alleviates usual heavy computations. On the other hand, it can be used to build a perturbation theory based on a new unperturbed part avoiding the degenerate character inherent to the classical superintegrable models. In other words, a first order perturbed solution based on this model might be accurate enough for tracking purposes. The benefits of a similar approach are now seen in areas such as the relative motion in formation flights [7].

In a series of previous works and with the same idea in mind, the authors have presented and analysed 1-DOF models [4, 3, 1]. In this work, we consider a 2-DOF model.

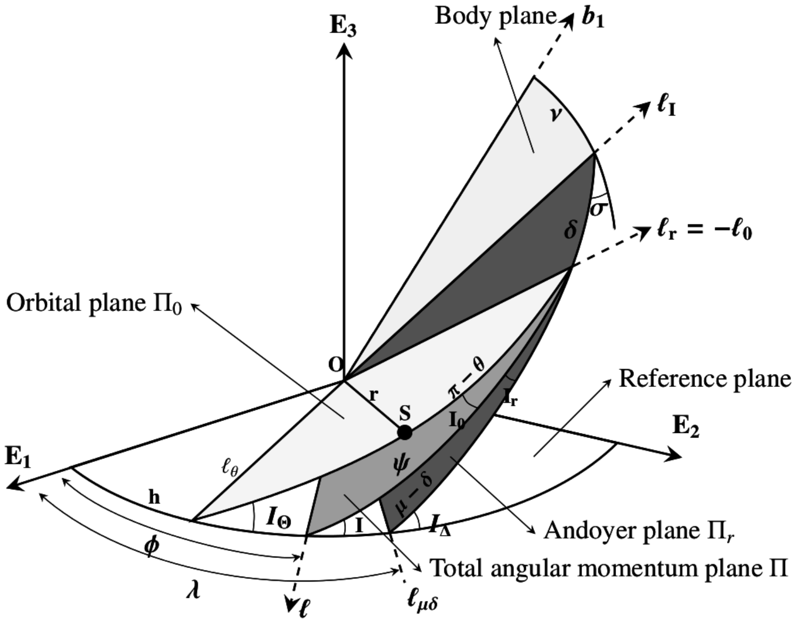

Figure 1: Geometry of the variables $(r, \phi, \psi, \theta, \delta, \nu, R, \Phi, \Psi, \Theta, \Delta, N)$. The variable r and the angles are explicitly given in the figure, while the associated momenta are included implicitly through the inclinations of the planes. The conjugate variable R remains unrepresented because of its pure dynamical sense. Note that this figure appeared first in [6]

§2. Variables

The variables in which the problem is posed may have a significant impact on its treatment. Our choice is the use of the total angular momentum as the key object to define them, which application for the roto-translatory problem was first introduced in [6] as a result of the application of the elimination of the nodes in the n-body problem [5] to the roto-translatory model. Nevertheless, quoting Meyer [9] “there is a saying in celestial mechanics that no set of coordinates is good enough”. This claim highlights that in every choice of variables, a sacrifice must be done. More precisely, Cartesian variables have a simple formulation, but they do not take advance of the presence of symmetries. Conversely, by using variables referred to the total angular momentum, we incorporate the angles associated to the symmetries allowing for compact expressions and intuitive geometric insight of the relative equilibria. However, this is done at the expenses of having singularities, *i. e.* a global study of the system requires the use of several charts.

A complete set of canonical variables related with the angular momentum planes are used here denoted by $(r, \phi, \psi, \theta, \delta, \nu, R, \Phi, \Psi, \Theta, \Delta, N)$. We are not going to provide a complete derivation of them, which may be found in [6]. Instead and with the aim of fixing notation, we provide the geometric meaning of the angles by means of Figure 1 and briefly recall the definition of the canonical angles by following [4]: Let us consider the reference frame $S^* = (\boldsymbol{\ell},\ \mathbf{n}\times\boldsymbol{\ell},\ \mathbf{n})$, where $\boldsymbol{\ell}$ is the unitary vector defining node of the total angular momentum plane with the horizontal spatial plane and $\mathbf{n}$ is the unitary vector pointing in the direction of the total angular momentum. In addition, $S^E = \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ and $S^b = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are the spatial and body frames respectively, where $\mathbf{b}_i$ corresponds with the principal moment of inertia of $\mathcal{B}_1$. The orientation and center of mass of the body are referred to the new frame by means of $(r, \phi, \psi, \theta, \delta, \nu)$, see Figure 1. These angles are determined by the nodes; $\boldsymbol{\ell}_{\mu\delta}$ defined by the rotational angular momentum and the spatial plane, $\boldsymbol{\ell}_r = -\boldsymbol{\ell}_o$ given by the intersection of the total, rotational and orbital angular momentum planes and $\boldsymbol{\ell}_\theta$ generated by the orbital and spatial planes intersection. Precisely, we have that $\phi = (\widehat{\mathbf{E}_1, \boldsymbol{\ell}})$, $\psi = (\widehat{\boldsymbol{\ell}, \boldsymbol{\ell}_I})$, $\theta = (\widehat{\boldsymbol{\ell}_o, \mathbf{r}})$, $\delta = (\widehat{\boldsymbol{\ell}_r, \boldsymbol{\ell}_I})$ and $\nu = (\widehat{\boldsymbol{\ell}_I, \mathbf{b}_1})$. Moreover, there are three more auxiliary angles which are not among the canonical variables but we will use them later on; $\lambda = (\widehat{\mathbf{E}_1, \boldsymbol{\ell}})$, $\mu = (\widehat{\boldsymbol{\ell}_{\mu\delta}, \boldsymbol{\ell}_I})$, $\sigma = (\widehat{\Pi_r, \Pi_b})$ and $h = (\widehat{\mathbf{E}_1, \boldsymbol{\ell}_\theta})$. In addition, the conjugate momenta of the variables read as follows

$$R, \quad \Phi = \mathbf{G}\cdot\mathbf{E}_3, \quad \Psi = \mathbf{G}\cdot\mathbf{n} = G, \quad \Theta = G_o, \quad \Delta = G_r, \quad N = \mathbf{G}_r\cdot\mathbf{b}_3,$$

where $\mathbf{G}$ is the total angular momentum vector, $\mathbf{G}_r$ is the angular momentum of the secondary body in the body frame and $\mathbf{G}_o$ is the orbital angular momentum. Thus, we have the following interpretation of the momenta: (R) Radial velocity of the center of mass. (Φ) Third component of the total angular momentum in space frame. (Ψ) Magnitude of the total angular momentum. (Θ) Magnitude of the angular momentum of the center of mass. (Δ) Magnitude of the angular momentum of the rigid body. (N) Third component of the angular momentum of the rigid body in the body frame (principal axes of inertia).

§3. Hamiltonian formulation of the triaxial model

The formulation of the triaxial model follows the same derivation as the one made in Crespo *et al.* [4], which is based in six simplifying assumptions. More specifically, the following set of simplifications are assumed in order to define our modelization: *(i) Barycentric coordinates.* The inertial frame is chosen to be moving with the total center of mass. *(ii) Shape and mass distribution of $\mathcal{B}_2$.* The main body $\mathcal{B}_2$ (mass m_2) is endowed with spherical symmetry. *(iii) Size ratios.* Dimensions of the secondary body $\mathcal{B}_1$ are small when compared to the distance between the centers of mass of the two bodies. *(iv) Shape and mass distribution of $\mathcal{B}_1$.* The secondary body may be approximated by an homogeneous triaxial ellipsoid with total mass m_1. *(v) Eccentricity.* Only small eccentricity orbits are considered. *(vi) Resonances.* The case of spin-orbit resonances is not considered.

The Hamiltonian of the roto-orbital model is obtained from the mechanic energy function. Thus, denoting T_O, T_R the orbital and rotational kinetic energies and $\mathcal{P}$ the potential, the Hamiltonian function is defined in the cotangent bundle of the special Euclidean group $T^*SE(3)$

$$\mathcal{H} = T_O + T_R + \mathcal{P} = T_O + T_R - \frac{\mathcal{G}M}{r} + \mathcal{V} = \mathcal{H}_K + \mathcal{H}_R + \mathcal{V},$$

in other words, the potential is usually split in two parts: a term which depends only on $1/r$ and $\mathcal{V}$, called the perturbing potential, depending on the rest of the variables of the problem. As a result, we have that $\mathcal{H}_K = T_O - \mathcal{G}M/r$ is the Keplerian part of the system, where $\mathcal{G}$ is the gravitational constant and $\mathcal{H}_R = T_R$ is referred as the Euler system (or the free rigid body).More explicitly, we obtain the following expression for $\mathcal{H}$ in the $\mathcal{B}_1$-body frame

$$\mathcal{H}(\mathbf{r},\mathbf{A},\mathbf{p},\mathbf{\Pi}) = \frac{|\mathbf{p}|^2}{2m} + \frac{1}{2}\,\mathbf{\Pi}\cdot\mathbf{I}^{-1}\cdot\mathbf{\Pi} - \mathcal{G}\,m_2\int_{\mathcal{B}_1}\frac{dm_1(\mathbf{x_1})}{|\mathbf{r}-\mathbf{x_1}|},$$

where $m = m_1 m_2/(m_1+m_2)$, $\mathbf{r}$ is the vector joining the center of mass of the bodies, $\mathbf{A}$ is the rotation matrix transforming a vector in the body-fixed frame into the inertial frame and $\mathbf{p}$ and $\mathbf{\Pi}$ are the linear and angular momenta. In addition, the assumption *(iii)* allows us to consider the approximation of the gravitational potential $\mathcal{P}$ given by $-\mathcal{G}M/r$ and the MacCullagh's term [8]

$$U = -\frac{\kappa\, m}{2\,m_1\,r^3}\left[(A_3-A_2)(1-3\gamma_3^2)-(A_2-A_1)(1-3\gamma_1^2)\right], \tag{1}$$

where $\kappa = \mathcal{G}M$, $M = m_1+m_2$ is the total mass of the system, $A_1 \le A_2 \le A_3$ are the principal moments of inertia associated to the secondary body and $(\gamma_1,\gamma_2,\gamma_3)$ are the director cosines of $\mathbf{r}$.

The direction cosines appearing in (1) may be expressed in the body frame by means of the following composition of rotations:

$$\gamma = R_3(\nu)\,R_1(\sigma)\,R_3(\delta)\,R_1(\iota)\,R_3(\pi-\theta)\;\mathbf{e}_1$$

where $\gamma = (\gamma_1,\gamma_2,\gamma_3)$ and $\mathbf{e}_1 = (1,0,0)$. Finally, taking into account that $\gamma_1^2+\gamma_2^2+\gamma_3^2 = 1$ and after some calculations, we are allowed to express the MacCullagh's term (1) as follows

$$U = \frac{\kappa m}{32 m_1 r^3}\left[(2A_3-A_2-A_1)V_1+\frac{3}{2}(A_2-A_1)V_2\right], \tag{2}$$

where

$$\begin{aligned} V_1 \;=\;& -2(1-3c_\iota^2)(1-3c_\sigma^2)\\ & -3s_\sigma^2\Big[(1-c_\iota)^2C_{2,2,0}+(1+c_\iota)^2C_{-2,2,0}\Big]\\ & -6s_\iota^2\Big[s_\sigma^2C_{0,2,0}-(1-3c_\sigma^2)C_{2,0,0}\Big]\\ & +12c_\sigma s_\iota s_\sigma\Big[(1-c_\iota)C_{2,1,0}+2c_\iota C_{0,1,0}-(1+c_\iota)C_{-2,1,0}\Big] \end{aligned} \tag{3}$$

which is independent of ν, and V_2, the "triaxiality part" given by

$$\begin{aligned} V_2 \;=\;& -(1-c_\sigma)^2\Big[(1-c_\iota)^2C_{2,2,-2}+(1+c_\iota)^2C_{-2,2,-2}+2s_\iota^2C_{0,2,-2}\Big]\\ & -(1+c_\sigma)^2\Big[(1-c_\iota)^2C_{2,2,2}+(1+c_\iota)^2C_{-2,2,2}+2s_\iota^2C_{0,2,2}\Big]\\ & -6s_\iota^2s_\sigma^2\Big[C_{2,0,2}+C_{2,0,-2}\Big]+4s_\sigma^2(1-3c_\iota^2)C_{0,0,2}\\ & +4s_\iota s_\sigma(1-c_\sigma)\Big[(1-c_\iota)C_{2,1,-2}+2c_\iota C_{0,1,-2}-(1+c_\iota)C_{-2,1,-2}\Big]\\ & +4s_\iota s_\sigma(1+c_\sigma)\Big[-(1-c_\iota)C_{2,1,2}-2c_\iota C_{0,1,2}+(1+c_\iota)C_{-2,1,2}\Big], \end{aligned} \tag{4}$$

and the notation has been abbreviated by writing $C_{i,j,k} \equiv \cos(i\theta+j\delta+k\nu)$ and $c_x \equiv \cos x$ and $s_x \equiv \sin x$.

3.1. A model for roto-orbital dynamics.

Facing a non-integrable Hamiltonian system of 4-DOF requires the development of a perturbation theory. A usual way to proceed is to expand the Hamiltonian function in power series and truncate it at a certain order; this procedure gives in general an approximation. However a different approach to the problem is based in a simplification of the original Hamiltonian considering a related Hamiltonian of less degrees of freedom. In fact in this search for a simplified model a radial of 2 separable DOF has been proposed. Indeed, in [4], the authors proposed an axis-symmetric integrable model, whose accuracy was tested by comparing with the MacCullagh's truncation and showing a good performance in the numerical experiments. Here, we continue this previous study by investigating a triaxial case. One of our aims is to analyze the physical-parametric families of relative equilibria asociated. Keeping this motivation in mind, we propose our model following exactly the same procedure than in [4], except for the triaxial parameter. That is to say, we only take into account the first line of V_1 in (3) and in V_2 (4) the only terms that depends exclusively on ν . Then, the perturbing potential of the model is given by

$$\mathcal{V} = \frac{\kappa m}{32 m_1 r^3}\left[-2(2A_3 - A_2 - A_1)(1-3c_\iota^2)(1-3c_\sigma^2) + \frac{3}{2}(A_2 - A_1)4s_\sigma^2(1-3c_\iota^2)\cos(2\nu)\right],$$

which leads us to the final expression of the model Hamiltonian

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\kappa}{r} + \frac{q}{2}\left[\left(\frac{\sin^2(\nu)}{A_1} + \frac{\cos^2(\nu)}{A_2}\right)(\Delta^2 - N^2) + \frac{1}{A_3}N^2\right] \\ & - \frac{\kappa(1-3c_\iota^2)}{16 r^3}\left[(2A_3 - A_2 - A_1)(1-3c_\sigma^2) + 3(A_1 - A_2)s_\sigma^2\cos(2\nu)\right],\end{aligned} \tag{5}$$

where $q = m/m_1$. Furthermore, with the aim of alleviate formulas, we have considered the Hamiltonian per unit of mass by scaling the system and inertia momenta as follows:

$$\begin{aligned}& \mathcal{H}' = \mathcal{H}/m; \quad R' = R/m; \quad \Theta' = \Theta/m; \quad \Delta' = \Delta/m; \quad N' = N/m; \quad \Psi' = \Psi/m; \\ & \Phi' = \Phi/m; \quad A_1' = A_1/m_1; \quad A_2' = A_2/m_1; \quad A_3' = A_3/m_1.\end{aligned} \tag{6}$$

Nevertheless, for the sake of simplicity, we keep the original notation without primes on the variables. Then, the 2-DOF Hamiltonian system of differential equations associated with (5) is given by the following expressions:

$$\begin{aligned}\dot{r} &= R \\ \dot{R} &= \frac{\Theta^2}{r^3} - \frac{\kappa}{r^2} - \frac{3\kappa\left(1-3c_\iota^2\right)}{16 r^4}\left[\alpha\left(1-3c_\sigma^2\right) - 3(A_2 - A_1)(1-c_\sigma^2)\cos(2\nu)\right] \\ \dot{\nu} &= q\left[\frac{1}{A_3} - \left(\frac{\sin^2(\nu)}{A_1} + \frac{\cos^2(\nu)}{A_2}\right) + \frac{3\kappa}{8\Delta^2 q r^3}\left(1-3c_\iota^2\right)(\alpha - (A_2 - A_1)\cos(2\nu))\right]N \\ \dot{N} &= q(A_1 - A_2)\left(1-c_\sigma^2\right)\Delta^2\left[\frac{1}{2A_1A_2} - \frac{3\kappa\left(1-3c\iota^2\right)}{8\Delta^2 q r^3}\right]\sin(2\nu) \\ \dot{\theta} &= \frac{\Theta}{r^2} - \frac{3\kappa}{8r^3}\left(\frac{c_\iota}{\Delta} + \frac{c_\iota^2}{\Theta}\right)\left[\alpha\left(1-3c_\sigma^2\right) - 3(A_2 - A_1)(1-c_\sigma^2)\cos(2\nu)\right]\end{aligned}$$

$$\dot{\psi} = \frac{3\kappa\Psi}{8r^3\Delta\Theta}\, c_\iota \left[\alpha\left(1-3c_\sigma^2\right) - 3(A_2 - A_1)\left(1-c_\sigma^2\right)\cos(2\nu)\right]$$

$$\begin{aligned}\dot{\delta} = q\Bigg[&\left(\frac{\sin^2(\nu)}{A_1} + \frac{\cos^2(\nu)}{A_2}\right) - \frac{3\kappa}{8\Delta^2 q r^3}\left(1-3c_\iota^2\right)c_\sigma^2\,(\alpha - (A_2 - A_1)\cos(2\nu))\\ &-\frac{3\kappa}{8\Delta^2 q r^3}\, c_\iota\left(c_\iota + \frac{\Delta}{\Theta}\right)\left(\alpha\left(1-3c_\sigma^2\right) - 3(A_2 - A_1)\left(1-c_\sigma^2\right)\cos(2\nu)\right)\Bigg]\Delta\end{aligned}$$

where $\alpha = 2A_3 - A_2 - A_1$ together with the integrals $\dot{\phi} = \dot{\Phi} = \dot{\Theta} = \dot{\Psi} = \dot{\Delta} = 0$. In other words the 2-DOF system is made of the (r, ν) subsystem and three quadrature associated to θ, ψ and δ.

Note that, in general, a 2-DOF system is not integrable. Thus, in the triaxial case, the analytical integration is not provided and the integrability of the system remains as an open question, which is not in the scope of the present paper.

§4. Numerical assessment of our model.

We assess the validity of our model by carrying out a simulation comparing our model versus the MacCullagh's approximation [8]. The expansion of the gravitational potential truncated to the third term known as the MacCullagh's term is commonly used as a good approximation to the potential because considering the next term lead to expressions with r^5 in the denominator. For situations where the term with r^5 is required a new model should be provided. However this is out the scope of this paper.

Numerical simulations have been carried out by using the Mathematica 11 software [11] running on the platform macOS Sierra, 3.1 GHz Intel Core i5 (64-bit), 8 GB RAM.

There are several details to bear in mind through this section in order to proceed with the numerical experiment. Firstly, in what follows it is convenient to use the triaxiality parameter defined in [3] $\rho = (A_2 - A_1)/(2A_3 - A_2 - A_1)$, noticing that due to the constrains of the principal moments of inertia $\rho \in (0, 1)$. Secondly, we have considered the Hamiltonian per unit of mass and the canonical and inertia momenta have been scaled, see (6). Furthermore, we have changed internally the units for longitudes by choosing the radius of the spherical body R_p as the new one. However, we set these units back to Km when we present our results. Regarding the initial conditions, the radius and angles (radians) are given directly. In our simulations we consider the scenario of a massive spherical primary body and an arbitrary triaxial secondary body. More precisely, the two bodies are described as follows. Main body $\mathcal{B}_2$: a sphere with radius 500 Km and mean density $d = 2.8\, g/\text{cm}^3$, and mass $m_2 = 1.47 \cdot 10^{21}$ Kg. Secondary body $\mathcal{B}_1$: an ellipsoid with mean density $d = 1.4\, g/\text{cm}^3$ while the principal axes and the triaxiality parameter are: $A_1 = 1.069 \cdot 10^{21}$, $A_2 = 1.18 \cdot 10^{21}$, $A_3 = 1.28 \cdot 10^{21}$, $\rho = 0.353$. Initial distance between the center of masses is 2060 Km and we also assume the secondary body in a slow rotating regime. Solutions are evaluated for three orbital periods, see Figure 2, where we show the evolution of variables which are not constant for the model we are presenting.

We would like to highlight that, after three orbital periods, the differences between the slow variables r, R, ψ, θ, N are always in the order of thousandth or less. For the case of the fast variables δ, ν, we have a competitive performance for 4 hours, which represent 1/4 orbital periods, see Figure 3.

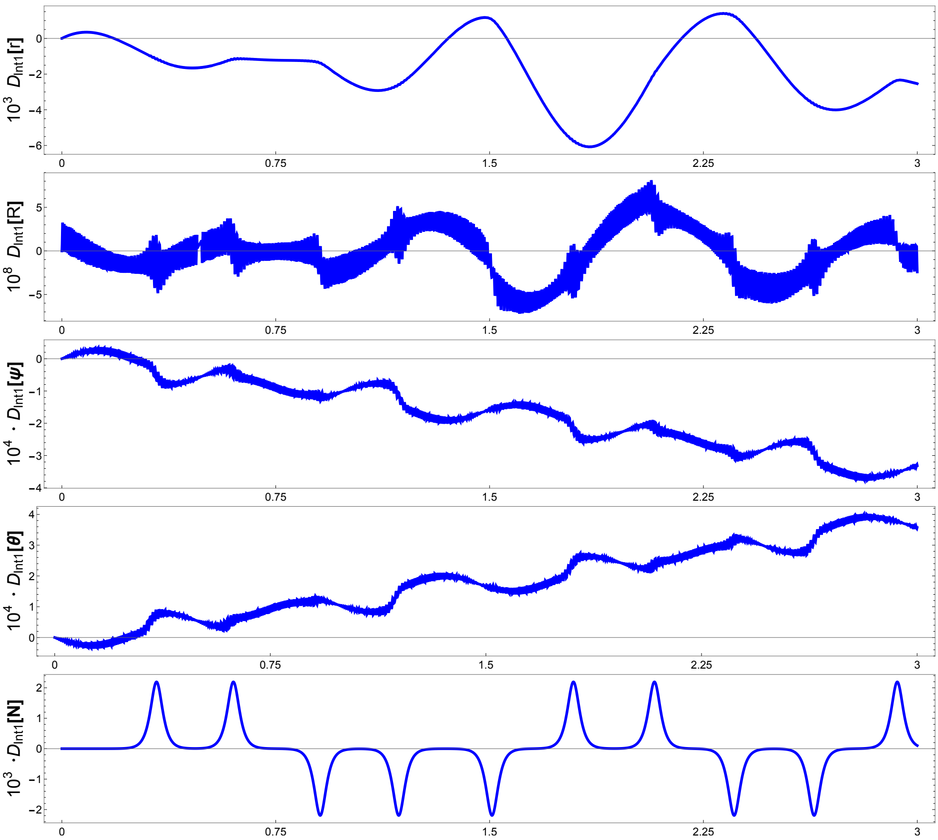

Figure 2: **Slow variables:** Differences between the 2-DOF model versus the MacCullagh's approximation. Abscissas are orbital periods and angles are given in radians. The orbital period is 16.5 hours and the rotation regime for each orbital period is 1-100.

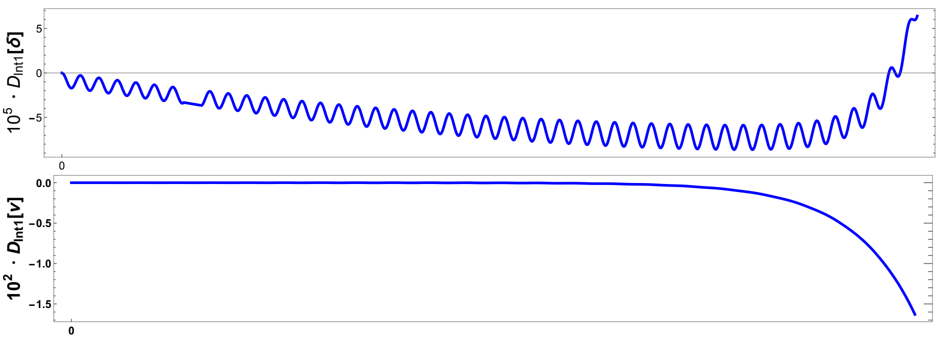

Figure 3: **Fast variables:** Differences between the 2-DOF model versus the MacCullagh's approximation. Abscissas represent 1/4 of the orbital period and angles are given in radians.

§5. Constant radius solutions. Some relative equilibria.

The system of differential equations defined by the Hamiltonian (5) is endowed with several distinguish and physical parameters. Thus, bifurcations occur in several directions in the parametric space [2].

With the aim of simplifying this scenario and provide a geometric interpretation of our equilibria, we organize our families of relative equilibria according to the inclinations of pairs of fundamental planes (orbital, rotational and body planes) due to the fact that the associated momenta of (θ), (ψ), (δ) and (ν) are included through the inclinations of the planes. More precisely, we consider the relative inclination between orbital and rotational planes (ι) and the one determined by the rotational and body planes (σ). For that reason $\cos\iota$ and $\cos\sigma$ are the key objects to present the analysis of the relative equilibria and allowed us to classify the relative equilibria on the following families: critical inclination equilibria when $(1 - 3c_\iota^2) = 0$, body-inclined equilibria when $c_\sigma \neq 0$ and body-perpendicular equilibria when $c_\sigma = 0$. Each of these families of relative equilibria contains different orbits of constant radius filling different tori depending on the fixed angles. Note that $\rho = 1/3$ is equivalent to $A_3 - A_2 = A_2 - A_1$ leading to the maximum triaxiality case. Below we show two particular cases of these families of relative equilibria found in this problem [2].

Case 1: Body-Inclined equilibria $c_\sigma \neq 0$ with ν and ψ fixed.

This particular case shows a family of relative equilibria filling a 2-tori manifold $\mathbb{T}^2(\theta, \delta)$. On one hand the orbital variables behave as a keplerian "circular" orbit, however on the other hand the rotational part shows the triaxiality influence and introduce several novelties with respect to the classical scheme of the free rigid body. More precisely imposing the following initial conditions and relations between the momenta and physical parameters:

$$r = \frac{\Theta^2}{\kappa}, \qquad R = 0, \qquad c_\iota^2 = \frac{1}{3} - \frac{4q\Theta^6\Delta^2}{9\kappa^4 A_2 A_3}, \qquad \nu = 0, \pi, \qquad c_\sigma^2 = \frac{A_1 - 2A_2 + A_3}{3(A_3 - A_2)}$$

we get a relative equilibria with the following mean motions:

$$\dot\theta = \frac{\kappa^2}{\Theta^3} \qquad \dot\delta = q\Delta\left(\frac{-A_1 + 2(A_2 + A_3)}{3A_2 A_3}\right)$$

Note that the values $\nu = 0, \pi$ are related to well-known equilibria of the free rigid body. It is worth noticing that in general $c_\sigma \neq 0$ which is a notorious difference from the classical case. Nevertheless for the particular value $\rho = 1/3$ we get $c_\sigma = 0$ and therefore we recover the Euler equilibria and obtain a simplified form of the mean motion $\dot\delta = q\Delta/A_2$

Case 2: Body-perpendicular equilibria $c_\sigma = 0$ with ν and ψ fixed.

This case shows also a relative equilibria filling a 2-tori manifold $\mathbb{T}^2(\theta, \delta)$ where the orbital variables behave as a keplerian "circular". As it happens with the previous case the rotational part shows a triaxiality influence and introduce several novelties with respect to the classical scheme of the free rigid body. In particular imposing the following initial conditions and relations between the momenta and physical parameters:

$$r = \frac{\Theta^2}{\kappa}, \quad R = 0, \quad c_\iota^2 = \frac{1}{3} - \frac{4q\Theta^6\Delta^2}{9\kappa^4 A_1 A_2}, \quad \cos(2\nu) = \frac{2A_3 - A_1 - A_2}{3(A_2 - A_1)}, \quad c_\sigma = 0,$$

we get that a relative equilibria with the following mean motions:

$$\dot{\theta} = \frac{\kappa^2}{\Theta^3} \qquad \dot{\delta} = \frac{q\Delta}{3}\left(\frac{2A_1 + 2A_2 - A_3}{A_1 A_2}\right)$$

Note that for this relative equilibria being $c_\sigma = 0$ we get $\cos(2\nu) \neq 0$ which is also a difference from the classical. It is worth mentioning that for the particular value $\rho = 1/3$ we get $\cos(2\nu) = 1$ and $\dot{\delta} = \dfrac{q\Delta}{A_2}$ which is a particular relative equilibria of our model and it is work in progress [2].

Observe, on both cases shown, that conditions for periodic orbits are easily obtained since expression for mean motions are explicitly given.The reader should also take into account that bounds among the integrals and physical parameters have to be added to the formulas given above.

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