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Relative entropy inequality for dissipative measure-valued solutions of compressible non-Newtonian system

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Abstract. The aim of the paper is to extend the result by Novotný and Nečasová [13] to the case of dissipative measure-valued solution and to derive a relative entropy inequality.

Keywords: Dissipative measure-valued, relative entropy inequality, non-Newtonian compressible fluids.

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§1. Setting of the problem

We consider a compressible non-Newtonian fluid of power-law type. The aim of the paper is to extend the result given by Novotný and Nečasová [13] to the more general case of measure-valued solutions and to derive a relative entropy inequality for this system.

Before stating the problem let us first explain the meaning of a measure-valued solution. It is a map which gives for every point of the domain a probability distribution of values such that the equation is satisfied only in an average sense. In a case that the probability distribution is reduced to a point mass almost everywhere in the domain then a measure valued solution is a weak solution of the problem, see e.g. the case of incompressible non-Newtonian case in work of Nečas et al. [12] or Bellout and Bloom [3]. The advantage of measure-valued solutions is the property that in many cases, the solutions can be obtained from weakly convergent sequences of approximate solutions. The measure-valued solutions for systems of hyperbolic conservations laws were initially introduced by DiPerna [4]. He used Young measures to pass to the limit in the artificial viscosity term. In the case of the incompressible Euler equations, DiPerna and Majda [5] proved global existence of measure-valued solutions for any initial data with finite energy. They introduced generalized Young measures to take into account oscillation and concentration phenomena. Thereafter the existence of measure-valued solutions was finally shown for further models of fluids, e.g. compressible Euler and Navier-Stokes equations [15]. The existence of measure-valued solutions for non-Newtonian fluids was proved by Novotný and Nečasová [13] . A generalization was given by Alibert and Bouchité, see [1]. More details can be found in [9], [10] and [16].

Recently, weak-strong uniqueness for generalized measure-valued solutions of isentropic Newtonian Euler equations were proved in [8]. Inspired by previous results, the concept of dissipative measure-valued solution was finally applied to the barotropic compressible Navier-Stokes system [6].

We consider the motion of the fluid in a bounded domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^{0,1}$ which is governed by the following system of equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho u) = 0 \quad \text{in} \quad (0,T)\times\Omega, \tag{1}$$

$$\partial_t(\varrho u) + \mathrm{div}_x(\varrho u \otimes u) + \nabla_x p = \mathrm{div}_x S \quad \text{in} \quad (0,T)\times\Omega, \tag{2}$$

where ϱ is the mass density and u is the velocity field, functions of the spatial position $x \in \mathbb{R}^3$ and the time $t \in \mathbb{R}$. The scalar function p is the pressure, and it is supposed to be given function of the density. In particular, we consider the isothermal case, namely $p = \lambda\varrho$, with $\lambda > 0$ a constant. The stress tensor is given by

$$S_{ij} = \beta\, \mathrm{div}_x u\, \delta_{ij} + 2\omega\, e_{i,j}(u), \quad \text{where} \tag{3}$$

$$\beta = \beta\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right), \quad \omega = \omega\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right), \quad \beta \geq -\frac{2}{3}\omega, \quad \omega \geq 0 \tag{4}$$

and $\widehat{u} = \sqrt{e_{i,j}(u)e_{i,j}(u)}$, $e_{i,j}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$.

For $T \in (0,\infty)$ we denote the time interval I, $I \equiv (0,T)$, and by $Q_T \equiv I\times\Omega$ the time-space cylinder. We consider the Dirichlet boundary conditions and initial data, respectively,

$$u = 0 \quad \text{in} \quad I\times\partial\Omega, \qquad u(0) = u_0, \qquad \varrho(0) = \varrho_0. \tag{5}$$

We consider also the following assumptions on the viscosity coefficients:

$$2\,\omega\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right)\left|\widehat{u}\right|^2 + \beta\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{div}_x u\ \mathrm{div}_x u \;\geq\; k_2\left|\widehat{u}\right|^{\gamma}, \tag{6}$$

$$2\,\omega\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right)e_{ij}(u) + \beta\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{div}_x u\ \delta_{ij} \;\leq\; k_1\left|\widehat{u}\right|^{\overline{\gamma}-1}, \tag{7}$$

for $i, j \in 1,2,3$ with $k_1, k_2 > 0$, $\gamma \leq \overline{\gamma} < \gamma+1$, $\gamma \geq 2$. Furthermore we assume the existence of a positive function $\vartheta(e_{ij})$ such that

$$\frac{\partial\vartheta}{\partial e_{ij}} = 2\,\omega\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right)e_{ij}(u) + \delta_{ij}\beta\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right)\mathrm{div}_x u. \tag{8}$$

We consider in this work a power-law type of fluids. For more details see [12].

§2. Preliminary results

We define $\phi(t) = e^t - t - 1$ and $\phi_2(t) = e^{t^2} - 1$ the Young functions and by $\psi(t) = (1+t)\ln(1+t) - t$ and $\psi_{1/2}(t)$ their complementary Young functions. The corresponding Orlicz spaces are denoted by $L_\phi(\Omega)$, $L_{\phi_2}(\Omega)$, $L_\psi(\Omega)$, $L_{\psi_{1/2}}(\Omega)$. These are Banach spaces equipped with a Luxembourg norm

$$\|u\|_{L_f(\Omega)} = \inf_h\left\{h > 0; \int_\Omega f\left(\frac{|u(x)|}{h}\right)dx \leq 1\right\} < +\infty,$$

where f stands for ϕ_1, ϕ_2, ψ, $\psi_{1/2}$. Let $C(\Omega)$ be the set of bounded continuous functions which are defined in Ω. We denote C_ψ, C_ϕ, $C_{\psi_{1/2}}$ and C_{ϕ_2} the closure of $C(\Omega)$ in $L_\psi(\Omega)$, $L_\phi(\Omega)$, $L_{\psi_{1/2}}(\Omega)$ and $L_{\phi_2}(\Omega)$, respectively. We have $\left(C_\phi(\Omega)\right)^* = L_\psi(\Omega)$, $\left(C_\psi(\Omega)\right)^* = L_\phi(\Omega)$, $\left(C_{\phi_2}(\Omega)\right)^* = L_{\psi_{1/2}}(\Omega)$, $\left(C_{\psi_{1/2}}(\Omega)\right)^* = L_{\phi_2}(\Omega)$, where C_ψ, C_ϕ, $C_{\psi_{1/2}}$ and C_{ϕ_2} are separable Banach spaces.

Definition 1. (Δ_2 - condition) A Young function Φ satisfies the $\triangle_2$-condition ($\Phi \in \triangle_2$) if and only if there exists $c > 0$ and $t_0 \geq 0$ such that $\Phi(2t) \leq c\Phi(t)$ for every $t > t_0$.

Definition 2. (Space of Radon measures) Let Ω be an open subset of $\mathbb{R}^N$ with Lebsgue measure. Consider the space $E = C_c(\Omega)$ the space of continuous functions with compact support in Ω, equipped with the norm $\|u\|_E = \sup_{x\in\Omega}|u(x)|$. The dual space of E, denoted by $M(\Omega)$, is called the space of Radon measures on Ω.

Since ψ, $\psi_{1/2}$ satisfy the $\triangle_2$-condition then the following identity hold: $C_\psi(\Omega) = L_\psi(\Omega)$, $C_{\psi_{1/2}}(\Omega) = L_{\phi_2}(\Omega)$. Let $L^\infty_w\left(Q_T, M\left(\mathbb{R}^{N^2}\right)\right)$ denote the spaces of all weakly measurable mappings from Q_T into $M\left(\mathbb{R}^{N^2}\right)$ with finite $L^\infty\left(Q_T, M\left(\mathbb{R}^{N^2}\right)\right)$ norm. We call $\nu \in L^\infty\left(Q_T, M\left(\mathbb{R}^{N^2}\right)\right)$ a weakly measurable map if and only if $(x,t) \to (\nu_{x,t}, g(x,t))$ is Lebesgue measurable in Q_T for every $g \in L^1\left(Q_T, C_c\left(\mathbb{R}^{N^2}\right)\right)$ where N is the dimension of space. We define by $L^p(\Omega)$, $W^{l,p}(\Omega)$ (resp. $W^{l,p}_0(\Omega)$), $1 \leq l, p < \infty$, the usual Lebesgue space and Sobolev spaces. By $W^{-l,p'}$ we denote the dual space to $W^{l,p}_0$. We define $V^k(\Omega)$ as $V^k(\Omega) = W^{k,2}(\Omega) \cap W^{1,2}_0(\Omega)$, $k \geq 2$.

Remark 1. For more details about Orlicz spaces see [11].

Definition 3. (Measure-valued solution) Let (ϱ, u, ν) be such that

$$\varrho \in L^\infty(I, L_\psi), \quad u \in L^2(I, V^k(\Omega)) \cap L^\gamma(I, W^{1,\gamma}_0), \quad \nu \in L^\infty_w\left(Q_T, M\left(\mathbb{R}^{N^2}\right)\right).$$

Assume also that the functions σ_{ij}, $\beta(\widehat{\sigma}, \mathrm{Tr}\sigma, \det(\sigma))\,\mathrm{Tr}\sigma$, $\omega(\widehat{\sigma}, \mathrm{Tr}\sigma, \det(\sigma))\,\sigma_{ij}$, are ν-integrable in $\mathbb{R}^{N^2}$ ($\mathrm{Tr}\sigma = \sigma_{ii}$) such that

$$\int_{\mathbb{R}^{N^2}} \sigma_{ij}\,\mathrm{d}\nu_{t,x}(\sigma) = \frac{\partial u_i}{\partial x_j}, \quad a.e. \quad in \quad Q_T.$$

Then, we define a measured-valued solution for the system (1) - (8) in the sense of DiPerna [4], in the following way:

$$-\int_{Q_T} \varrho u_i \frac{\partial \varphi_i}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} \varrho u_i u_j \varphi_{i,j}\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_0} \varrho_0 u_0 \varphi_i(0)\,\mathrm{d}x - \lambda \int_{Q_T} \varrho\varphi_{i,i}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_{Q_T} \mathrm{d}x\,\mathrm{d}t \left(\int_{\mathbb{R}^{N^2}} \beta(\widehat{\sigma}, \mathrm{Tr}\sigma, \det(\sigma))\,\mathrm{Tr}\sigma\delta_{ij} + 2\omega(\widehat{\sigma}, \mathrm{Tr}\sigma, \det(\sigma))\,\sigma_{ij}\mathrm{d}\nu_{t,x}(\sigma) \right) \varphi_{i,j} = 0, \tag{9}$$

for all $\varphi \in C^\infty\left(\overline{Q_t}\right)$, $\varphi(t) \in W^{1,\gamma}_0(\Omega)$ for any $t \in I$ and $\varphi(T) = 0$.

Remark 2. In Definition 3 the Young measures are defined for the gradient of the velocity field. In the next sections the measures will be considered for the density and the velocity field.

Theorem 1. *Let $u_0 \in V^k(\Omega)$, $k \geq 3$, $\varrho_0 \in C^d(\overline{\Omega})$, $\varrho_0 > \varepsilon > 0$, $d = 1, 2, ...$ Let assumptions (6) - (8) be satisfied with $\gamma \geq 2$. Then, there exists (ϱ, u) and a family of a probability measure $\nu_{x,t}$ on $\mathbb{R}^{N^2}$ with the following properties:*
(i) $\nu \in L^\infty_w\left(Q_T, \mathbb{R}^{N^2}\right),\ \left\|\nu_{x,t}\right\|_{M\left(\mathbb{R}^{N^2}\right)} = 1,\ for\ a.e.\ (x,t) \in Q_T,$
(ii) $\operatorname{supp} \nu_{x,t} \subset \mathbb{R}^{N^2},\ for\ a.e.\ (x,t) \in Q_T,$
(iii) $u \in L^\gamma(I, W_0^{1,\gamma}(\Omega)) \cap L^\gamma\left(I, W_0^{1,\alpha}(\Omega)\right),\ \alpha\gamma > 3,\ \alpha < 1,$
(iv) $\varrho \in L^\infty(I, L_\psi(\Omega)) \cap L^2(I, W^{-1,2}(\Omega)),$
(v) $\varrho u \in L^\gamma(I, W^{-\alpha,\gamma'}(\Omega)),\ \alpha\gamma > 3,\ \alpha < 1,\ \frac{1}{\gamma} + \frac{1}{\gamma'} = 1,$
(vi) $\varrho u_i u_j \in L^\gamma(I, W^{-\alpha,\gamma'}(\Omega))$ *and such that (ϱ, u, ν) satisfies (9).*

To prove the existence of measure-valued solutions we introduce the following approximation scheme (multipolar fluids) introduced by Nečas and Šilhavý (see [14]),

$$\tau_{ij} = \sum_{s=0}^{k-1} \tau_{ij}^{(s,u)},$$

$$\tau_{ij}^{(s,u)} = \tau_{ij}^{(s,u,lin)} + S_{ij},$$

$$\text{with} \qquad \tau_{ij}^{(s,u,lin)} = (-1)^s \left(\mu_1^s \triangle^s \operatorname{div}_x u\, \delta_{ij} + 2\mu_2^s \triangle^s e_{ij}(u)\right).$$

The second law of thermodynamics requires additional stress tensors with the power on an elementary surface $\mathrm{d}S\, \tau^n_{ii_1...i_m j} \frac{\partial^m u_i}{\partial x_{i_1}...\partial x_{i_m}} n_j$, where n is the normal vector to the boundary defined almost everywhere on $\partial\Omega$. The higher stress tensors are defined as follows

$$\tau^n_{ii_1...i_m j} = \operatorname{Sym}\left(\sum_{r=m}^{k-1} (-1)^{r+m} \triangle^{r-m} \frac{\partial^m q^r_{ii_m}}{\partial x_{i_1}...\partial x_{i_{m-1}} \partial x_j}\right),$$

$$\text{where} \qquad q^s_{ij} = \mu_1^s \left(\frac{\partial u_l}{\partial x_l}\right) \delta_{ij} + 2\mu_2^s e_{ij}(u),$$

and symmetrization is taken with respect to $(i_1, ..., i_m)$. We assume that μ_1^s and μ_2^s are constants and satisfy the bounds

$$\mu_1^s \geq -\frac{2}{3}\mu_2^s, \quad \mu_2^s > 0, \quad 0 \leq s \leq k-2,$$

$$\mu_1^{k-1} > -\frac{2}{3}\mu_2^{k-1}, \quad \mu_2^{k-1} > 0.$$

We denote

$$((u,w)) = \int_\Omega \left(\sum_{s=0}^{k-1} \left(2\mu_2^s \frac{\partial^s e_{ij}(u)}{\partial x_{i_1}...\partial x_{i_s}} \frac{\partial^s e_{ij}(w)}{\partial x_{i_1}...\partial x_{i_s}} + \mu_1^s \frac{\partial^s e_{rr}(w)}{\partial x_{i_1}...\partial x_{i_s}} \frac{\partial^s e_{ll}(u)}{\partial x_{i_1}...\partial x_{i_s}}\right)\right) \mathrm{d}x.$$

Moreover we assume

$$\mu_1^s > -\frac{2}{3}\mu_2^s, \qquad (s = 0, ..., k-2)\,.$$

Under the previous assumptions System (1)-(2) can be rewritten in the following form:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho u_i)}{\partial x_i} = 0, \quad \text{in} \quad (0,T) \times \Omega, \tag{10}$$

$$\frac{\partial (\varrho u_i)}{\partial t} + \frac{\partial \left(\varrho u_i u_j\right)}{\partial x_j} - \frac{\partial \tau_{ij}(u)}{\partial x_j} = -\lambda \frac{\partial \varrho}{\partial x_i}, \quad \text{in} \quad (0,T) \times \Omega, \tag{11}$$

with the initial data $u(0) = u_0$, $\varrho(0) = \varrho_0$ and boundary conditions $u = 0$ on $\partial\Omega \times I$ and $[[u,w]] = 0$ on $\partial\Omega \times I$, where

$$[[u,w]] = \sum_{m=1}^{k-1} \int_{\partial\Omega} \tau^n_{ii_1\ldots i_m j} \frac{\partial w_i^m}{\partial x_{i_1} \ldots \partial x_{i_m}} n_j \, \mathrm{d}S. \tag{12}$$

Weak formulation of (11) reads: for all $\varphi \in L^2\left(I, V^k(\Omega) \cap W_0^{1,\gamma}(\Omega)\right)$,

$$\begin{aligned}
\int_{Q_T} \frac{\partial (\varrho u_i)}{\partial t} \varphi_i \, \mathrm{d}x \, \mathrm{d}t - \int_{Q_T} \varrho u_i u_j \varphi_{i,j} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T ((u,\varphi)) \, \mathrm{d}t \\
+ \int_{Q_T} \beta\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right) \mathrm{div}_x u \frac{\partial \varphi_i}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}t \\
+ 2 \int_{Q_T} \omega\left(\widehat{u}, \mathrm{div}_x u, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right) e_{ij}(u) \frac{\partial \varphi_i}{\partial x_j} \, \mathrm{d}x \, \mathrm{d}t - \lambda \int_{Q_T} \varrho \frac{\partial \varphi_i}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}t = 0.
\end{aligned} \tag{13}$$

Let us formulate the existence and uniqueness results for the approximation scheme:

Lemma 2. *Assume that* $u_0 \in V^k(\Omega)$ *and* $\varrho_0 \in C^d\left(\overline{\Omega}\right)$*, where* $\varrho_0 > \varepsilon > 0$ *and* $d = 1, 2, \ldots$ *Let assumptions (6) - (8) be satisfied with* $k \geq 3$ *and* $\gamma \geq 2$*. Then, there exists at least one solution* (ϱ, u) *of (10) - (12) satisfying (13) such that* $\varrho \in L^\infty(I, W^{p,q}(\Omega))$ *where* $p = \min(d, k-2)$, $1 \leq q \leq 6\,(N=3)$, $1 \leq q < \infty\,(N=2)$, $\frac{\partial \varrho}{\partial t} \in L^2\left(I, W^{p-1,q}(\Omega)\right)$, $u \in L^2\left(I, V^k(\Omega)\right) \cap L^\infty\left(I, W^{k,2}(\Omega)\right)$, $\frac{\partial u}{\partial t} \in L^2(Q_T)$, $u \in L^\gamma\left(I, W_0^{1,\gamma}(\Omega)\right)$*. Moreover, assuming that* $\vartheta(e_{ij})$ *satisfying (8) is continuously differentiable in* $\mathbb{R}^{N^2}$ *then there exists at most one solution of the problem (10) - (13).*

Proof. Applying the methods of characteristics to the continuity equations together with the Galerkin approach to the momentum equation, we get global existence of the approximate problem. For more details on the proof see [13]. □

In passing to the limit with higher viscosity, the most problematic point is to find a representation in terms of

$$\int_{Q_T} \beta\left(\widehat{u^\mu}, \mathrm{div}_x \widehat{u^\mu}, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right) u^\mu_{i,i} \varphi_{i,i} \, \mathrm{d}x \, \mathrm{d}t + 2 \int_{Q_T} \omega\left(\widehat{u^\mu}, \mathrm{div}_x \widehat{u^\mu}, \det\left(\frac{\partial u_i}{\partial x_j}\right)\right) u^\mu_{i,j} \varphi_{i,j} \, \mathrm{d}x \, \mathrm{d}t.$$

Let us follow the classical theory introduced by Ball [2].

We define for each $(x,t) \in Q_T$ a sequence $\nu^j_{x,t} \equiv \delta_{\nabla u^j(x,t)}$, where δ_x is the Dirac measure at the point $x \in \mathbb{R}^{N^2}$, $\left(\nabla u^\mu(x,t) \in \mathbb{R}^{N^2}\right)$ and we put $\nu^j \,:\, (x,t) \in Q_T \to \nu^j_{x,t}$. Since $\{\nu^j\}_j$ is uniformly bounded in $L^\infty_w\left(Q_T; M\left(\mathbb{R}^{N^2}\right)\right)$, thanks to the representation theorem, we get

$$\left[L^1\Big(Q_T;\, C_c\Big(\mathbb{R}^{N^2}\Big)\Big)\right]^* \approx L_w\Big(Q_T\,;\, M\Big(\mathbb{R}^{N^2}\Big)\Big).$$

In addition, thanks to the separability of ν^j, we have $\nu \in L^\infty_w\left(Q_T\,;\, M\left(\mathbb{R}^{N^2}\right)\right)$ such that

$$\nu^j \to \nu, \quad \text{weakly} - * \text{ in } L^\infty_w\left(Q_T\,;\, M\left(\mathbb{R}^{N^2}\right)\right).$$

Let us recall the following special case of the Ball theorem (see [2]).

Lemma 3. *Let $\nabla v^j \,:\, Q_T \to \mathbb{R}^{N^2}$ be uniformly bounded in $L^\gamma(Q_T)$ and let the continuous function $\tau \,:\, \mathbb{R}^{N^2} \to \mathbb{R}$ satisfy*

$$c\left|\widehat{\sigma}\right|^\gamma \le \tau\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\sigma\right) \le c\left(1 + \left|\widehat{\sigma}\right|\right)^{\overline{\gamma}-1},$$

where $\gamma > \overline{\gamma} - 1$ and

$$\sup_{j=1,2,\ldots} \int_{Q_T} \eta\left(\left|\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\sigma\right)\right|\right) \mathrm{d}x\,\mathrm{d}t < \infty,$$

with η a Young function. Then, $\left\|\nu_{x,t}\right\| = 1$, a.e. in $\mathbb{R}^{N^2}$ and

$$\tau\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\sigma\right) \to \left(\tau, \nu_{x,t}\right) = \int_{\mathbb{R}^{N^2}} \tau\left(\widehat{\sigma}, \mathrm{Tr}\sigma, \det\sigma\right) \mathrm{d}\nu_{x,t}(\sigma),$$

*weakly - * in $L^\eta(Q_T)$.*

Proof. Applying Lemma 3 with $\eta(\xi) = \xi^{\gamma/(\overline{\gamma}-1)}$, we get

$$\int_{Q_T} \left[\beta\left(\widehat{u},\, \mathrm{div}\sigma,\, \det\sigma\right)\right]^{\gamma/(\overline{\gamma}-1)} \mathrm{d}x\,\mathrm{d}t \;\le\; \int_{Q_T} \left|\widehat{\sigma}^\gamma\right| \mathrm{d}x\,\mathrm{d}t \;\le\; \text{const},$$

which give us the measure-valued solution in the sense of DiPerna. □

§3. Relative entropy inequality

Let us denote by $\mathcal{P}\left([0,\infty) \times \mathbb{R}^N\right)$ the set of probability measures on $[0,\infty)\times\mathbb{R}^N$. We introduce the concept of *dissipative measure-valued solution* to the system (1) - (2) in the spirit of [8] and [6].

Definition 4. We say that a parameterized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,

$$\nu \in L^\infty_w\left((0,T)\times\Omega; \mathcal{P}\left([0,\infty)\times\mathbb{R}^N\right)\right), \quad \langle \nu_{t,x}; s\rangle \equiv \varrho, \quad \langle \nu_{t,x}; v\rangle \equiv u,$$

is a dissipative measure-valued solution of the compressible Navier-Stokes system (1) - (2) in $(0,T)\times\Omega$, with initial condition ν_0 and dissipation defect $\mathcal{D}$ with

$$\mathcal{D} \in L^\infty(0,T), \quad \mathcal{D} \ge 0,$$

if the following holds:
(i) ***Continuity equation***. There exist a measure $r^C \in L^1\left([0,T], M\left(\overline{\Omega}\right)\right)$ and $\chi \in L^1(0,T)$ such that for a.a. $\tau \in (0,T)$ and every $\psi \in C^1\left([0,T]\times\overline{\Omega}\right)$,

$$\left|\left\langle r^C(\tau);\nabla_x\psi\right\rangle\right| \le \chi(\tau)\,\mathcal{D}(\tau)\,\|\psi\|_{C^1\left(\overline{\Omega}\right)}$$

and

$$\int_\Omega \langle \nu_{t,x}; s\rangle\,\psi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \langle \nu_0; s\rangle\,\psi(0,\cdot)\,\mathrm{d}x = \int_0^\tau\int_\Omega \left[\langle \nu_{t,x}; s\rangle\,\partial_t\psi + \langle \nu_{t,x}; sv\rangle\cdot\nabla_x\psi\right]\mathrm{d}x\mathrm{d}t + \int_0^\tau \left\langle r^C;\nabla_x\psi\right\rangle\mathrm{d}t. \quad (14)$$

(ii) ***Momentum equation***.

$$u = \langle \nu_{t,x}; v\rangle \in L^2\left(0,T; W_0^{1,2}\left(\Omega;\mathbb{R}^N\right)\right),$$

and there exists a measure $r^M \in L^1\left([0,T], M\left(\overline{\Omega}\right)\right)$ and $\xi \in L^1(0,T)$ such that for a.a. $\tau \in (0,T)$ and every $\varphi \in C^1\left([0,T]\times\overline{\Omega};\mathbb{R}^N\right)$, $\varphi|_{\partial\Omega} = 0$,

$$\left|\left\langle r^M(\tau);\nabla_x\varphi\right\rangle\right| \le \xi(\tau)\,\mathcal{D}(\tau)\,\|\varphi\|_{C^1\left(\overline{\Omega}\right)},$$

and

$$\begin{aligned}\int_\Omega \langle \nu_{t,x}; sv\rangle\varphi(\tau,\cdot)\mathrm{d}x - \int_\Omega \langle \nu_0; sv\rangle\varphi(0,\cdot)\mathrm{d}x &= \int_0^\tau\int_\Omega \Big[\langle \nu_{t,x}; sv\rangle\partial_t\varphi + \langle \nu_{t,x}; s(v\otimes v)\rangle : \nabla_x\varphi + \langle \nu_{t,x}; p(s)\rangle\mathrm{div}_x\varphi\Big]\mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau\int_\Omega S\Big(\widehat{u},\mathrm{div}_x u,\det\Big(\frac{\partial u_i}{\partial x_j}\Big)\Big) : \nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \langle r^M;\nabla_x\varphi\rangle\mathrm{d}t. \quad (15)\end{aligned}$$

(iii) ***Energy inequality***.

$$\begin{aligned}\int_\Omega \Big\langle \nu_{t,x}; (\frac{1}{2}s|u|^2 + P(s))\Big\rangle\,\mathrm{d}x + \int_0^\tau\int_\Omega S\Big(\widehat{u},\mathrm{div}_x u,\det\Big(\frac{\partial u_i}{\partial x_j}\Big)\Big) : \nabla_x u\,\mathrm{d}x\,\mathrm{d}t \\ + \mathcal{D}(\tau) \le \int_\Omega \Big\langle \nu_0; \Big(\frac{1}{2}s|u|^2 + P(s)\Big)\Big\rangle\mathrm{d}x,\ \textit{for a.e. } \tau \in (0,T), \quad (16)\end{aligned}$$

where $P(s) = (1+s)\ln(1+s) - s$. Moreover, the following version of Poincare's inequality holds

$$\int_0^\tau\int_\Omega \Big\langle \nu_{t,x}; |v-u|^2\Big\rangle\,\mathrm{d}x\mathrm{d}t \le C\,\mathcal{D}(\tau).$$

We introduce the relative entropy functional

$$\mathcal{E}(\varrho,u \mid r,U) = \int_\Omega \left[\frac{1}{2}\varrho\,|u-U|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r)\right]\mathrm{d}x,$$

with $P(\varrho) = (1+\varrho)\ln(1+\varrho) - \varrho$. It is shown in [7] that any finite energy weak solution (ϱ, u) to the compressible Newtonian barotropic Navier-Stokes system satisfies the relative entropy inequality for any pair (r, U) of sufficiently smooth test functions such that $r > 0$ and $U|_{\partial\Omega} = 0$. In the framework of dissipative measure-valued solution (in the spirit of [8] and [6]) we define the functional

$$\mathcal{E}_{mv}(\varrho, u, | \, r, U) \equiv \int_\Omega \left\langle \nu_{t,x}; \frac{1}{2} s\,|v - U|^2 + P(s) - P'(r)(s-r) - P(r) \right\rangle \mathrm{d}x.$$

Theorem 4. *Let* $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ *with*

$$\nu \in L^\infty_w\Big((0,T)\times\Omega; \mathcal{P}\Big([0,\infty)\times\mathbb{R}^N\Big)\Big), \quad \langle \nu_{t,x}; s\rangle \equiv \varrho, \quad \langle \nu_{t,x}; v\rangle \equiv u,$$

be a dissipative measure-valued solution to the compressible non-Newtonian system (1) - (2) with the initial condition ν_0 *and dissipation defect* $\mathcal{D}$*. Then,* (s, v) *satisfies the following relative entropy inequality*

$$\begin{aligned} \mathcal{E}_{mv} + \int_0^\tau \int_\Omega S\Big(\widehat{u}, \mathrm{div}_x u, \det\Big(\frac{\partial u_i}{\partial x_j}\Big)\Big) : \nabla(u - U)\,\mathrm{d}x\,\mathrm{d}t + \mathcal{D}(\tau) \\ \leq \int_\Omega \Big\langle \nu_{0,x}; \Big(\frac{1}{2} s|v - U(0,\cdot)|^2\Big) + P(s) - P'(r_0)(s-r_0) - P(r_0)\Big\rangle \mathrm{d}x \\ + \int_0^\tau \mathcal{R}(s, v, r, U)(t)\mathrm{d}t, \quad (17) \end{aligned}$$

for a.a. $\tau \in (0,T)$ *and any pair of test functions* (r, U) *such that* $U \in C^1\Big([0,T]\times\overline{\Omega}, \mathbb{R}^N\Big)$*,* $U|_{\partial\Omega} = 0$*,* $r \in C^\infty_c(\overline{Q_T})$*,* $r > 0$*, where*

$$\begin{aligned} \mathcal{R}(s, v, r, U)(t) = & -\int_\Omega \Big(\langle \nu_{t,x}; sv\rangle\,\partial_t U + \langle \nu_{t,x}; sv\otimes v\rangle : \nabla_x U\Big)\mathrm{d}x \\ & - \int_\Omega \Big(\langle \nu_{t,x}; p(s)\rangle\,\mathrm{div}_x U\Big)\mathrm{d}x + \int_\Omega \Big(\langle \nu_{t,x}; s\rangle\, U\partial_t U + \langle \nu_{t,x}; sv\rangle\cdot U\cdot\nabla_x U\Big)\mathrm{d}x \\ & + \int_\Omega \Big[\Big\langle \nu_{t,x}; \Big(1 - \frac{s}{r}\Big)\Big\rangle\, p'(r)\partial_t r - \langle \nu_{t,x} : sv\rangle\cdot\frac{p'(r)}{r}\nabla_x r\Big]\mathrm{d}x \\ & - \langle r^M ; \nabla_x U\rangle + \int_\Omega \Big\langle r^C ; \frac{1}{2}\nabla_x |U|^2 - \nabla_x P'(r)\Big\rangle \mathrm{d}x. \quad (18) \end{aligned}$$

Proof. First using the continuity equation (14) with test function $\frac{1}{2}|U|^2$, we get

$$\begin{aligned} \int_\Omega \frac{1}{2}\langle \nu_{t,x}; s\rangle\,|U|^2(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \frac{1}{2}\langle \nu_0; s\rangle\,|U|^2(0,\cdot)\,\mathrm{d}x \\ = \int_0^\tau \int_\Omega \Big[\langle \nu_{t,x}; s\rangle\, U\partial_t U + \langle \nu_{t,x}; sv\rangle\cdot U\cdot\nabla_x U\Big]\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \langle r^C; \frac{1}{2}\nabla_x U\rangle\,\mathrm{d}t, \quad (19) \end{aligned}$$

provided $U \in C^1\Big([0,T]\times\overline{\Omega}; \mathbb{R}^N\Big)$.

Next, testing (14) by $P'(r)$, we get

$$\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; s\rangle\, P'(r)(\tau,\cdot)\,\mathrm{d}x \;-\; \int_\Omega \langle \nu_0; s\rangle\, P'(r)(0,\cdot)\,\mathrm{d}x \\
&\quad= \int_0^\tau \int_\Omega \Big[\langle \nu_{t,x}; s\rangle\, P''(r)\partial_t r \;+\; \langle \nu_{t,x}; sv\rangle\, P''(r)\cdot\nabla_x r\Big]\,\mathrm{d}x\,\mathrm{d}t \;+\; \int_0^\tau \langle r^C; \nabla_x P'(r)\rangle\,\mathrm{d}t \\
&\quad= \int_0^\tau \int_\Omega \Big[\langle \nu_{t,x}; s\rangle\, \frac{p'(r)}{r}\partial_t r \;+\; \langle \nu_{t,x}; sv\rangle \frac{p'(r)}{r}\cdot\nabla_x r\Big]\,\mathrm{d}x\,\mathrm{d}t \;+\; \int_0^\tau \langle r^C; \nabla_x P'(r)\rangle\,\mathrm{d}t,
\end{aligned} \tag{20}$$

provided $r \in C^1\left([0,T]\times\overline{\Omega};\mathbb{R}^N\right)$.

Furthermore using (15) tested by U we obtain

$$\begin{aligned}
&\int_\Omega \langle \nu_{t,x}; sv\rangle\, U(\tau,\cdot)\,\mathrm{d}x \;-\; \int_\Omega \langle \nu_0; sv\rangle\, U(0,\cdot)\,\mathrm{d}x \\
&\quad= \int_0^\tau \int_\Omega \Big[\langle \nu_{t,x}; sv\rangle\, \partial_t U \;+\; \langle \nu_{t,x}; s(v\otimes v)\rangle : \nabla_x U \;+\; \langle \nu_{t,x}; p(s)\rangle \mathrm{div}_x U\Big]\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad - \int_0^\tau \int_\Omega S\Big(\widehat{u},\, \mathrm{div}_x u,\, \det\Big(\frac{\partial u_i}{\partial x_j}\Big)\Big) : \nabla_x U\,\mathrm{d}x\,\mathrm{d}t \;+\; \int_0^\tau \langle r^M; \nabla_x U\rangle\,\mathrm{d}t,
\end{aligned} \tag{21}$$

for any $U \in C^1\left([0,T]\times\overline{\Omega};\mathbb{R}^N\right)$, $U|_{\partial\Omega} = 0$. Summing up (19) - (21) and (16), we get (17) - (18). □

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